TU Chemnitz:
2008 Summer School on Applied Analysis

Four lectures on

Theory and numerical analysis of Volterra functional equations

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Lecture I: Theory of Volterra functional equations

Classical and delay Volterra integral operators:

\begin{itemize}
  \item \((\mathcal{V}u)(t) := \int_0^t K_0(t,s)u(s) \, ds, \quad t \in I\)
  \item \((\mathcal{V}_\theta u)(t) := \int_0^{\theta(t)} K_1(t,s)u(s) \, ds, \quad t \in I\)
  \item \((\mathcal{W}_\theta u)(t) := \int_{\theta(t)}^t K(t,s)u(s) \, ds, \quad t \in I\)
\end{itemize}

Here, \( t \in I := [0,T] \), and the delay function (or: lag function) \( \theta \) has the form

\[ \theta(t) := t - \tau(t). \]

We refer to \( \tau \) as the delay.

\begin{itemize}
  \item **Non-vanishing delay:** \( \tau(t) \geq \tau_0 > 0 \) \((t \in I)\)
  \item **Vanishing delay:** \( \tau(0) = 0, \quad \tau(t) > 0 \) \((t > 0)\)
\end{itemize}
Volterra functional equations

- Volterra functional integral equations (VFIEs):
  \[ u(t) = g(t) + (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t), \quad t \in I \]

- Volterra functional integro-differential equations (VFIDEs):
  \[ u'(t) = a(t)u(t) + b(t)u(\theta(t)) + g(t) + (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t), \quad t \in I \]

  \[ \leftarrow \text{Special case: Delay differential equation (DDE)}: \]
  \[ u'(t) = a(t)u(t) + b(t)u(\theta(t)) + g(t), \quad t \in I \]

- First-kind VFIE:
  \[ (\mathcal{W}_\theta u)(t) = g(t), \quad t \in I \]

  \[ \leftarrow \theta(t) = qt (0 < q < 1): \text{Volterra (1897)} \]
Vito Volterra (1860 - 1940)
DDEs; Effect of delay on solutions

Exercise
The solutions of the ODE

\[ u'(t) = au(t), \quad t \geq 0; \quad \text{Re}(a) < 0, \]

satisfies

\[ \lim_{t \to \infty} u(t) = 0. \]

What is the asymptotic behaviour, as \( t \to \infty \), of the solutions to the DDEs

\[ u'(t) = bu(t - \tau), \quad t \geq 0, \quad \text{Re}(b) < 0, \]

with \( \tau > 0 \) and \( u(t) = 1 \) if \( t \leq 0 \), and

\[ u'(t) = bu/qt, \quad t \geq 0, \quad \text{Re}(b) < 0, \]

with \( 0 < q < 1 \) and \( u(0) = u_0 \) ?
Illustration:

\[ u'(t) = bu(qt), \quad u(0) = 1; \quad b < 0, \quad 0 < q < 1. \]

The solution is given by

\[
\begin{align*}
    u(t) &= \sum_{j=0}^{\infty} \frac{q^j(j-1)/2}{j!} (bt)^j, \quad t \geq 0 : \\
\end{align*}
\]

It is an entire function of order zero.

*(Comparison: For \( q = 1 \), the solution is an entire function of order one: \( u(t) = \exp(bt) \).)*

Example: \( b = -1, \quad q = 0.95 \):
Properties of solutions of VFIEs and VFIDES
(Representation / regularity)

- Classical VIES:
  \[ u(t) = g(t) + \int_0^t K(t,s)u(s) \, ds, \quad t \in I := [0, T] \]

**Theorem** (Volterra, 1896)
If \( K \in C(D) \) (\( D := \{(t, s) : 0 \leq s \leq t \leq T\} \)), then for any \( g \in C(I) \) the VIE has a unique solution \( u \in C(I) \). This solution is given by
  \[ u(t) = g(t) + \int_0^t R(t,s)g(s) \, ds, \quad t \in I, \]
where \( R \in C(D) \) denotes the **resolvent kernel** of \( K \):

\[ R(t,s) := \sum_{j=1}^{\infty} K_j(t,s), \quad (t,s) \in D \]

(*Neumann series of \( K \)). The **iterated kernels** \( K_j \) of \( K \) are defined by \( K_1(t,s) := K(t,s) \) and

\[ K_{j+1}(t,s) := \int_s^t K(t,v)K_j(v,s) \, dv \quad (j \geq 1). \]

Moreover,

\[ K \in C^d(D) \quad \text{and} \quad g \in C^d(I) \quad \Rightarrow \quad u \in C^d(I). \]
• VIES: with delay function $\theta(t) = qt$ ($0 < q < 1$)

$$u(t) = g(t) + \int_0^{qt} K(t, s)u(s)\, ds, \quad t \in I := [0, T]$$

**Theorem** (Andreoli (1914); Chambers (1990))

If $K \in C(D_\theta)$ ($D_\theta := \{(t, s) : 0 \leq s \leq \theta(t) (t \in I)\}$), then for any $g \in C(I)$ the VIE has a unique solution $u \in C(I)$. This solution is given by

$$u(t) = g(t) + \sum_{j=1}^{\infty} \int_0^{q^j t} K_j(t, s)g(s)\, ds, \quad t \in I.$$ 

The *iterated kernels* $K_j$ of $K$ are defined by $K_1(t, s) := K(t, s)$ and

$$K_{j+1}(t, s) := \int_{q^{-j}s}^{qt} K(t, v)K_j(v, s)\, dv \quad (j \geq 1).$$

For $0 < q < 1$ the kernel $K$ does not have a *Neumann series*!

However, as for classical VIEs,

$K \in C^d(D_\theta)$ and $g \in C^d(I) \Rightarrow u \in C^d(I)$. 

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Classical VIDEs:

\[ u'(t) = a(t)u(t) + g(t) + \int_0^t K(t,s)u(s)\,ds, \quad t \in I \]

\[ \Rightarrow \text{VIDE is equivalent to VIE} \]

\[ u(t) = g_0(t) + \int_0^t H(t,s)u(s)\,ds, \]

with

\[ g_0(t) := u_0 + \int_0^t g(s)\,ds \]

and

\[ H(t,s) := a(s) + \int_s^t K(v,s)\,dv. \]

**Theorem:** (Grossman & Miller (1970))

If \( a \in C(I) \) and \( K \in C(D) \), then for any \( g \in C(I) \) and any \( u_0 \in \mathbb{R} \) the VIDE has a unique solution \( u \in C(I) \) satisfying \( u(0) = u_0 \). This solution is given by

\[ u(t) = r(t,0)u_0 + \int_0^t r(t,s)g(s)\,ds, \quad t \in I, \]

where the (differential) resolvent kernel \( r(t,s) \) depends on \( a \) and \( K \).

Moreover,

\[ a, \, g \in C^d(I) \quad \text{and} \quad K \in C^d(D) \quad \Rightarrow \quad u \in C^{d+1}(I). \]
• VFIDEs with delay function $\theta(t) = qt \ (0 < q < 1)$

$$u'(t) = b(t)u(\theta(t)) + g(t) + \int_0^{\theta(t)} K(t, s)u(s) \, ds,$$

with $t \in I := [0, T]$ and $u(0) = u_0$.

**Theorem:** (Brunner & Hu (2007))
Assume that $b, g \in C^d(I)$ and $K \in C^d(D_\theta)$. Then for each $q \in (0, 1)$ and given $u_0$ the VFIDE has a unique solution $u \in C^{d+1}(I)$ that satisfies $u(0) = u_0$.

This solution has the representation

$$u(t) = \left(1 + \sum_{j=1}^{\infty} \tilde{H}_j(t, s) \, ds\right)u_0 + \int_0^t g(s) \, ds$$

$$+ \sum_{j=1}^{\infty} \int_0^{\theta(t)} \tilde{H}_j(t, s)g(s) \, ds, \quad t \in I.$$ 

where

$$\tilde{H}_j(t, s) := \int_s^{\theta_j(t)} H_j(t, v) \, dv \ (j \geq 1).$$

Here, the $H_j(t, s)$ are the iterated kernels of

$$H_1(t, s) := b(\theta^{-1}(s))\theta'(\theta^{-1}(s)) + \int_{\theta^{-1}(s)}^t K(v, s) \, dv.$$
• **Summary:**
If \( \theta(t) = qt \) \((0 < q < 1)\), or if \( \theta \) is **nonlinear** and satisfies
(i) \( \theta(0) = 0 \), \( \theta \) is strictly increasing on \( I \), and
(ii) \( \theta(t) \leq q_1 t \) for some \( q_1 \in (0, 1) \), then
**Smooth data** \( \Rightarrow \) **Solution of VFE is** (globally) smooth on \([0, T]\).

• **VFEs with non-vanishing delays**
**Assume:** the delay function \( \theta(t) = t - \tau(t) \) satisfies:
(D1) \( \tau(t) \geq \tau_0 > 0 \) for \( t \in I := [t_0, T] \)
(D2) \( \theta \) is strictly increasing on \( I \);
(D3) \( \tau \in C^d(I) \) for some \( d \geq 0 \).

**Definition:**
The points \( \{\xi_\mu\} \) generated by
\[
\theta(\xi_\mu) = \xi_{\mu-1}, \quad \mu \geq 1 \quad (\xi_0 := t_0),
\]
are called the **primary discontinuity points** (or: **breaking points**) induced by the delay function \( \theta \). \( \leftrightarrow \) **By (D1):** \( \xi_\mu - \xi_{\mu-1} \geq \tau_0 \) \((\mu \geq 1)\).

\( \leftrightarrow \) **Assume:** \( T \) such that
\[
T = \xi_{M+1} \quad \text{for some} \quad M \geq 1.
\]
‘Method of steps’:

Solve VFE on $I^{(\mu)} := [\xi_\mu, \xi_{\mu+1}]$ ($\mu = 0, \ldots, M$).

**Illustration:**

\[ u'(t) = au(t) + bu(\theta(t)) + \int_0^{\theta(t)} K(t, s)u(s) \, ds, \]
\[ t \in [0, T], \text{ with } u(t) = \phi(t) \text{ for } t \leq 0. \]

For $t \in I^{(0)} := [0, \xi_1]$; ($\Rightarrow \theta(t) \in I^{(-1)} := [\theta(0), 0]$):

\[ u'(t) = au(t) + \Phi_0(t). \]

where

\[ \Phi_0(t) := b\phi(\theta(t)) + \int_0^{\theta(t)} K(t, s)\phi(s) \, ds \]

is known.

For $t \in I^{(\mu)} := [\xi_\mu, \xi_{\mu+1}]$, ($\mu = 1, \ldots, M$):

\[ u'(t) = au(t) + \Phi_{\mu}(t). \]

with known

\[ \Phi_{\mu}(t) := bu(\theta(t)) + \int_0^{\theta(t)} K(t, s)u(s) \, ds. \]

**Question:** Regularity of solution $u(t)$ at $\xi_\mu$?
Representation of solutions:

Let \( \theta(t) = t - \tau(t), \; \tau(t) \geq \tau_0 > 0 \). For continuous data, the solution of the VFIDE

\[
\begin{align*}
    u'(t) &= a(t)u(t) + b(t)u(\theta(t)) + g(t) \\
    + &\left( \mathcal{V}u \right)(t) + \left( \mathcal{V}_\theta u \right)(t), \quad t \in I
\end{align*}
\]

(with \( u(t) = \phi(t), \; t \leq t_0 \)) on \( I^{(\mu)} := [\xi_\mu, \xi_{\mu+1}] \)

is given by

\[ u(t) = r_1(r, \xi_\mu)u(\xi_\mu) + \int_{\xi_\mu}^{t} r_1(t, s)g(s) \, ds \]

\[ + F_\mu(t) + \Phi_\mu(t), \quad t \in I^{(\mu)}. \]

Here,

\[
\begin{align*}
    F_\mu(t) &:= \sum_{\nu=0}^{\mu-1} \int_{\xi_\nu}^{\xi_{\nu+1}} r_{\mu,\nu}(t, s)g(s) \, ds \\
    \Phi_\mu(t) &:= \sum_{\nu=0}^{\mu-1} p_{\mu,\nu}(t)u(\xi_\nu) + G^{(1)}_{\mu}(t; \phi)
\end{align*}
\]

and

\[ \Phi_\mu(t) := \sum_{\nu=0}^{\mu-1} \int_{\xi_\nu}^{r_{\theta^{\mu-\nu}}(t)} r_{\mu,\nu}(t, s)g(s) \, ds + G^{(2)}_{\mu}(t; \phi). \]
Non-vanishing $\tau(t)$: Regularity results

- **VFIDEs (and DDEs):**

  $$u'(t) = au(t) + bu(\theta(t)) + (Vu)(t) + (V_\theta u)(t) :$$

  **Theorem:** (*Smoothing of solutions*)
  Assume that the given functions are arbitrarily *smooth*, and $V_\theta \neq 0$. Then at $t = \xi_\mu$,

  $$u \in C^\mu \text{ but } u \notin C^{\mu+1} \ (\mu = 0, \ldots, M).$$

  If $b \equiv 0$, then at $t = \xi_\mu$ ($\mu = 0, \ldots, M$),

  $$u \in C^{2\mu} \ ('super-smoothing') .$$

  **Neutral VFIDE:**

  $$u'(t) = au(t) + bu(\theta(t)) + cu'(\theta(t))$$

  $$+ (Vu)(t) + (V_\theta u)(t) :$$

  **Theorem:** (*Non-smoothing of solutions*)
  If given functions are smooth and $c \neq 0$, then at $t = \xi_\mu$ ($\mu = 0, 1, \ldots, M$),

  $$u \in C^0 \text{ but } u \notin C^1 :$$

  there is no smoothing at $t = \xi_\mu$ as $\mu$ increases.
VFIEs with non-vanishing delay: 

\[ u(t) = g(t) + (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t), \]

\( t \in (0, T], \) with \( u(t) = \phi(t) \) for \( t \leq 0. \)

**Theorem:** *(Smoothing of solutions)*

For smooth data and \( \mathcal{V}_\theta \neq 0, \) the solution satisfies

\[ u \in C^{\mu-1} \quad \text{but} \quad u \notin C^{\mu} \]

for \( \mu = 1, \ldots, M. \) At \( t = \xi_0 = 0 \) the solution is in general **discontinuous**; that is, \( u \) has a **finite jump** at \( t = \xi_0 \) (except for specially chosen initial functions \( \phi \)).

(Note:)

\[ u(0^-) = \phi(0), \quad u(0^+) = g(0) - \int_{\theta(0)}^{0} K_1(0,s)\phi(s) \, ds \]

**Exercise:** Regularity of solution of

\[ u(t) = g(t) + b(t)u(\theta(t)) + (\mathcal{W}_\theta u)(t), \]

where

\[ (\mathcal{W}_\theta u)(t) := \int_{\theta(t)}^{t} K(t,s)u(s) \, ds \]
• State-dependent delays

Example:
Mathematical model of population whose life span depends on the size of the population (crowding effects) (Bélair, 1990):

\[ u(t) = \int_{t-\tau(y(t))}^{t} k(t-s)G(u(s)) \, ds, \quad t > 0, \]

with \( u(t) = \phi(t) \) for \( t \leq 0 \).

Survey of state-dependent DDEs:


The theory and numerical analysis of state-dependent VFEs and VFIDEs remain to be established!
Lecture I: Basic references


Lecture II:
Collocation in piecewise polynomial spaces

Mesh (or: grid) on $I := [t_0, T]$:

$$I_h := \{t_n : t_0 < t_1 < \cdots < t_N = T\},$$

with

$$e_n := (t_n, t_{n+1}), \quad h_n := t_{n+1} - t_n;$$

$h := \max \{h_n : 0 \leq n \leq N - 1\}$ is called the mesh diameter.

**Definition:** For given integers $r \geq 1$, $-1 \leq d < r$,

$$S_r^{(d)}(I_h) := \{v \in C^d(I) : v|_{e_n} \in \pi_r \quad (0 \leq n \leq N - 1)\}$$

denotes the space of piecewise polynomials (with respect to the given mesh $I_h$) of degree $r$; if $d \geq 0$ these functions are globally in $C^d(I)$.

$$\dim S_r^{(d)}(I_h) = N(r - d) + (d + 1).$$

For $d = -1$,

$$S_r^{(-1)}(I_h) := \{v : v|_{e_n} \in \pi_r \quad (0 \leq n \leq N - 1)\}.$$
Illustration:
Approximation of the solution of the ODE
\[ u'(t) = f(t, u(t)), \quad t \in [0, T]; \quad u(0) = u_0, \]
by collocation in \( S^{(0)}_m(I_h) \) \((r = m, d = 0)\).
Since \( \dim S^{(0)}_m(I_h) = Nm + 1 \), choose
\[
X_h := \{ t_n + c_i h_n : 0 < c_1 < \cdots < c_m \leq 1 \ (0 \leq n \leq N - 1) \}
\]
as collocation points \((\Rightarrow |X_h| = Nm)\),
\(\hookrightarrow\) Find \( u_h \in S^{(0)}_m(I_h) \) satisfying the ODE on the finite subset \( X_h \) of \([0, T]\):
\[
 u_h'(t) = f(t, u_h(t)) \quad \text{for all} \quad t \in X_h,
\]
with \( u_h(0) = u_0 \).

Remark:
For \( k \)th-order ODEs \((k \geq 2)\),
\[
 u^{(k)}(t) = f(t, u(t), \ldots, u^{(k-1)}(t)),
\]
choose the collocation space
\[
S^{(d)}_{m+d}(I_h) \quad \text{with} \quad d := k - 1,
\]
and the same set of collocation points \( X_h \)
(since
\[
\dim S^{(d)}_{m+d}(I_h) = Nm + d + 1 = Nm + k.
\]
Questions:

• Collocation for ODEs (and VEs) in smoother piecewise polynomial spaces:
  \[ S^{(d)}_r(I_h) \quad \text{with} \quad d \geq 1 \quad (d < r) \quad ? \]

• Computational form of collocation equation ?

• Global order of convergence (on I):
  \[ \| u - u_h \|_\infty \leq Ch^p : p = ? \]

• Local order of convergence (on \( I_h \)):
  \[ \max \{ |u(t) - u_h(t)| : t \in I_h \} \leq Ch^{p^*} : p^* > p ? \]
  \( \hookrightarrow \) Local superconvergence on \( I_h \) ?

• Do the above optimal orders remain true for VFEs ?
Computational form of collocation equation: Let
\[ L_j(v) := \prod_{k \neq j} \frac{v - c_k}{c_j - c_k}, \quad v \in [0, 1] \quad (j = 1, \ldots, m) \]
denote the Lagrange canonical polynomials with respect to the collocation parameters \( \{c_i\} \).
Setting \( Y_{n,j} := u'_h(t_n + c_j h_n) \) and
\[ u'_h(t_n + vh_n) = \sum_{j=1}^{m} L_j(v) Y_{n,j}, \quad v \in (0, 1], \]
we obtain the local representation of the collocation solution \( u_h \in S^{(0)}_m(I_h) \) on the subinterval \( [t_n, t_{n+1}] \):
\[ u_h(t_n + vh_n) = u_h(t_n) + h_n \sum_{j=1}^{m} \beta_j(v) Y_{n,j}, \quad v \in [0, 1], \]
with
\[ \beta_j(v) := \int_0^v L_j(s) \, ds. \]
Computation of \( \{Y_{n,j}\} (0 \leq n \leq N - 1) \):
\[ Y_{n,i} = f \left( t_n + c_i h_n, y_n + h_n \sum_{j=1}^{m} a_{i,j} Y_{n,j} \right) (i = 1, \ldots, m) \]
where \( y_n := u_h(t_n) \) and \( a_{i,j} := \beta_j(c_i) \).
The pair of equations (for $0 \leq n \leq N - 1$):

$$u_h(t_n + vh_n) = u_h(t_n) + h_n \sum_{j=1}^{m} \beta_j(v)Y_{n,j}, \; v \in [0, 1]$$

(local representation of the collocation solution $u_h \in S_m^0(I_h)$ on the subinterval $[t_n, t_{n+1}]$)

and

$$Y_{n,i} = f(t_n + c_i h_n, y_n + h \sum_{j=1}^{m} a_{i,j}Y_{n,j}) (i = 1, \ldots, m)$$

(collocation equations for $t = t_n + c_i h_n$)

represent an $m$-stage continuous implicit Runge-Kutta method for solving the ODE initial-value problem

$$u'(t) = f(t, u(t)), \quad t \in [0, T]; \quad u(0) = u_0.$$ 

For arbitrary $\{c_i\}$ (and $u \in C^d(I)$ with $d \geq m + 1$):

$$\|u^{(k)} - u_h^{(k)}\|_\infty \leq Ch^m \quad (k = 0, 1).$$

Question:
Choice of collocation parameters $\{c_i\}$?
Convergence results for ODEs: \( u_h \in S_m^0(I_h) \).

- If \( u \in C^{m+1}(I) \):
  \[
  \|u^{(k)} - u_h^{(k)}\|_\infty \leq Ch^m \quad (k = 0, 1)
  \]
  for arbitrary \( \{c_i\} \).

Let

\[
J_\nu := \int_0^1 s^\nu \prod_{i=1}^m (s - c_i) \, ds \quad (\nu \in \mathbb{N}).
\]

- If \( u \in C^{m+2}(I) \) and \( J_0 = 0 \):
  \[
  \|u - u_h\|_\infty \leq Ch^{m+1}.
  \]

- Let \( u \in C^{m+\kappa+1}(I) \) \( (\kappa \leq m) \).
  If \( J_\nu = 0, \nu = 0, \ldots, \kappa - 1 \), and \( J_\kappa \neq 0 \):
  \[
  \max\{|u(t) - u_h(t)| : t \in I_h\} \leq Ch^{m+\kappa}.
  \]

\( \kappa = m \): \( \Rightarrow \) \( \{c_i\} \) are the Gauss points

\[
\max\{|u(t) - u_h(t)| : t \in I_h\} \leq Ch^{2m}.
\]

The underlying method is the \( m \)-stage continuous implicit Runge-Kutta-Gauss method.
Why $O(h^{2m})$-convergence on $I_h$?

Illustration:

$u'(t) = a(t)u(t) + g(t), \ t \in I; \ u(0) = u_0$

Collocation equation: $u_h \in S_m^{-1}(I_h)$:

$u'_h(t) = a(t)u_h(t) + g(t) - \delta_h(t), \ t \in I; \ u_h(0) = u_0,$

where the defect function $\delta_h$ vanishes at the collocation points $t_n + c_i h_n$:

$\delta_h(t) = 0$ for all $t \in X_h$.

$\Rightarrow$ Collocation error $e_h := u - u_h$ satisfies

$e'_h(t) = a(t)e_h(t) + \delta_h(t), \ t \in I; \ e_h(0) = 0.$

Thus, setting $r(t,s) := \exp \left( \int_s^t a(v) \, dv \right)$:

$e_h(t) = \int_0^t r(t,s)\delta_h(s) \, ds, \ t \in I.$

$\Leftarrow$ For $t = t_n \in I_h$:

$e_h(t_n) = \sum_{\ell=0}^{n-1} h_\ell \int_0^1 r(t_n, t_\ell + sh_\ell)\delta_h(t_\ell + sh_\ell) \, ds.$

$\Leftarrow$ Recall that

$\delta_h(t_\ell + c_i h_\ell) = 0$ for $i = 1, \ldots, m; \ 0 \leq \ell \leq N - 1.$
m-point interpolatory quadrature formula:
Abscissas \{d_j\} with \(0 \leq d_1 < \cdots < d_m \leq 1\):

\[
\int_0^1 \phi(t_n + s h_n) \, ds = \sum_{j=1}^{m} w_j \phi(t_n + d_j h_n) + E_n(\phi),
\]

with quadrature weights

\[
w_j := \int_0^1 L_j(s) \, ds \quad (j = 1, \ldots, m)
\]

Quadrature error \(E_n(\phi)\).

- For arbitrary abscissas \(\{d_j\}\) (and \(\phi \in C^m\)):

\[
|E_n(\phi)| \leq Q_m h_n^m.
\]

- If the \(\{d_j\}\) satisfy

\[
J_\nu := \int_0^1 s^\nu \prod_{i=1}^{m} (s - d_j) \, ds \quad (\nu = 0, \ldots, \kappa - 1)
\]

and \(J_\kappa \neq 0\) \((1 \leq \kappa \leq m)\), then

\[
|E_n(\phi)| \leq Q_m h_n^{m+\kappa},
\]

provided that \(\phi \in C^{m+\kappa}\).

\(\kappa = m\): The \(\{d_j\}\) are the Gauss (-Legendre) points (zeros of \(P_m(2s - 1)\)).

\(\kappa = m - 1\) and \(d_m = 1\):
The \(\{d_j\}\) are the Radau II points.

\(\kappa = m - 2\) and \(d_1 = 0, d_m = 1\) \((m \geq 2)\):
The \(\{d_j\}\) are the Lobatto points.
The **collocation error** \( e_h := u - u_h \) satisfies
\[
e_h(t) = \int_0^t r(t, s)\delta_h(s) \, ds, \quad t \in I.
\]

⇒ For \( t = t_n \in I_h \):
\[
e_h(t_n) = \sum_{\ell=0}^{n-1} h_\ell \int_0^1 r(t_n, t_\ell + sh_\ell)\delta_h(t_\ell + sh_\ell) \, ds.
\]

Set
\[
\phi_n(t_\ell + sh_\ell) := r(t_n, t_\ell + sh_\ell)\delta_h(t_\ell + sh_\ell).
\]

Since \( \delta_h(t) = 0 \) for \( t = t_\ell + c_j h_\ell \in X_h \) ⇒ choose the **collocation parameters** as **quadrature abscissas** ( \( d_j = c_j \) ) :
\[
e_h(t_n) = \sum_{\ell=0}^{n-1} h_\ell \int_0^1 \phi_n(t_\ell + sh_\ell) \, ds = 0 + \sum_{\ell=0}^{n-1} h_\ell E_{n,\ell}
\]
for \( n = 1, \ldots, N \). This implies that
\[
|e_h(t_n)| \leq \sum_{\ell=0}^{n-1} h_\ell |E_{n,\ell}| \leq Q_m h^{m+\kappa} \sum_{\ell=0}^{n-1} h_\ell \leq C_m h^{m+\kappa},
\]
with \( C_m := Q_m T \). The **optimal order** (on the mesh \( I_h \)) is attained when \( \kappa = m_i \) ⇔ the \( \{c_i\} \) are the **Gauss points**.

However, we then only have
\[
\max\{|u'(t) - u'_h(t)| : t \in I_h \setminus \{0\}\} \leq C'_m h^m!
\]
ODEs: Collocation in *smoother* piecewise polynomial spaces?

- \( u_h \in S^{(m-1)}_m(I_h) \quad (d = m - 1) \):
  \[ \rightarrow \quad u_h \text{ is divergent (as } h \rightarrow 0 \text{) when } m \geq 4 \]
  (Loscalzo & Talbot, 1967)

- \( u_h \in S^{(2)}_4(I_h), \quad 0 < c_1 < c_2 = 1 \):
  \( u_h \) is **divergent** if
  \[
  \frac{1 - c_1}{c_1} > 1.
  \]

- \( u_h \in S^{(2)}_m(I_h) \quad (m \geq 4) \):
  \( u_h \) is **divergent** if the \( \{c_i\} \) are the **Radau II points**.

  (Complete convergence / divergence analysis for ODEs: M"ulthei, 1979)

**Remark:**
The *natural* (and *optimal*) piecewise polynomial spaces for (first-order) ODEs and VIDEs are the spaces \( S^{(0)}_m(I_h) \) with \( m \geq 1 \).
For VIEs the natural spaces are \( S^{(-1)}_{m-1}(I_h) \).
Notes

1. Higher-order ODEs:

\[ u^{(k)}(t) = f(t, u(t), \ldots, u^{(k-1)}(t)) \quad (k \geq 2) : \]

\( \leftrightarrow \) Collocation in \( S^{(d)}_{m+d}(I_h) \) with \( d := k - 1 \) and collocation points

\[ X_h = \{ t_n + c_i h : 0 < c_1 < \cdots < c_m \leq 1 \} \]

\( \Rightarrow \) Continuous Runge-Kutta-Nyström methods.

2. The collocation solutions \( u_h \in S^0_m(I_h) \) for the ODE

\[ u'(t) = au(t), \quad t \in I; \quad u_0 = u_0, \]

and \( v_h \in S^{(-1)}_{m-1}(I_h) \) for the ‘integrated ODE’

\[ u(t) = u_0 + \int_0^t au(s) \, ds, \quad t \in I \]

(Volterra integral equation), using the same set \( X_h \) of collocation points, are identical only if \( c_m = 1 \). In particular:

\[ u_h(t_n) = v_h(t_n) \quad (1 \leq n \leq N) \Leftrightarrow c_m = 1. \]

\( \leftrightarrow \) For the Gauss points: \( u_h(t_n) \neq v_h(t_n) ! \)
Observations:

- Assume that the solution of a given functional (differential or integral) equation admits a ‘resolvent representation’ of the form

\[ u(t) = r(t, 0)u(0) + \int_0^t r(t, s)g(s)\, ds, \quad t \in \mathbb{I}. \]

or

\[ u(t) = g(t) + \int_0^t R(t, s)g(s)\, ds, \quad t \in \mathbb{I}. \]

Then the collocation solution (or a closely related ‘iterated collocation solution’) in the ‘natural’ piecewise polynomial space for the given VFE has the same superconvergence orders as the one for ODEs.

This is true for classical Volterra integral and integro-differential equations, and for delay differential and Volterra functional equations with non-vanishing delays (but not for VFEs with vanishing delays like \( \theta(t) = qt, \ 0 < q < 1 \)).

- The attainable order of superconvergence is governed by the regularity of the solution \( u \) and the choice of the collocation parameters.
Volterra integro-differential equations:

\[ u'(t) = a(t)u(t) + g(t) + \int_0^t K(t,s)u(s) \, ds, \quad t \in I, \]

with continuous \( a, g \) and \( K \). For given initial value \( u(0) = u_0 \) the (unique) solution \( u \in C^1(I) \) is given by

\[ u(t) = r(t,0)u_0 + \int_0^t r(t,s)g(s) \, ds, \quad t \in I, \]

where the (differential) \textbf{resolvent kernel} \( r(t,s) \) is defined by the \textit{resolvent equation}

\[ \frac{\partial r}{\partial s} = -r(t,s)a(s) - \int_s^t r(t,v)K(v,s) \, dv, \]

\((0 \leq s \leq t \leq T)\), with \( r(s,s) = 1, \ s \in I \).

\( \hookrightarrow \) Collocation in \( S_m^0(I_h) \): the \textbf{collocation error} \( e_h := u - u_h \) has the representation

\[ e_h(t) = \int_0^t r(t,s)\delta_h(s) \, ds, \quad t \in I. \]

Thus: for the \textbf{Gauss points} \( \{c_i\} \),

\[ \max\{|e_h(t)| : t \in I_h\} \leq C_m h^{2m}, \]

as for \textit{ODEs}!
Volterra integral equations:

\[ u(t) = g(t) + \int_0^t K(t, s)u(s) \, ds, \quad t \in I. \]

For continuous \( g \) and \( K \) the (unique) solution \( u \in C(I) \) is given by

\[ u(t) = g(t) + \int_0^t R(t, s)g(s) \, ds, \quad t \in I, \]

where \( R(t, s) \) is the \textbf{resolvent kernel} of \( K \):

\[ R(t, s) := \sum_{j=1}^{\infty} K_j(t, s) \quad (\text{Neumann series}), \]

with \textit{iterated kernels} \( K_1 := K \) and

\[ K_{j+1}(t, s) = \int_s^t K(t, v)K_j(v, s) \, dv \quad (j \geq 1). \]

Collocation: \( u_h \in S_{m-1}^{-1}(I_h) \), and corresponding \textbf{iterated} collocation solution

\[ u_h^{it}(t) := g(t) + \int_0^t K(t, s)u_h(s) \, ds, \quad t \in I : \]

resulting errors \( e_h := u - u_h \) and \( e_h^{it} := u - u_h^{it} \) have the representations

\[ e_h(t) = \delta_h(t) + \int_0^t R(t, s)\delta_h(s) \, ds, \quad t \in I \]

and

\[ e_h^{it}(t) = e_h(t) - \delta_h(t), \quad t \in I : \]
\[ e^{it}(t) = \int_0^t R(t, s)\delta_h(s) \, ds, \quad t \in I. \]

Collocation at **Gauss points**: 

\[ \Rightarrow \text{Iterated collocation error at the mesh points} \]

\[ t = t_n \ (1 \leq n \leq N) \text{ satisfies} \]

\[ \max\{|e^{it}(t)| : t \in I_h \setminus \{0\}\} \leq C_m h^{2m}. \]

**But:**

\[ \max\{|e_h(t)| : t \in I_h \setminus \{0\}\} \leq C_m h^m \]

only!

**Note:**

If \( c_m = 1 \), then

\[ u^{it}_h(t_n) = u_h(t_n), \quad n = 1, \ldots, N, \]

and thus \( e^{it}_h(t_n) = e_h(t_n). \) \[ \Rightarrow \]

\[ \max\{|e_h(t_n)| : 1 \leq n \leq N\} \leq C_m h^{2m-1} \]

if the collocation parameters \( \{c_i\} \) are the **Radau II points** \( (\kappa = m - 1: \text{zeros of } (P_m - P_{m-1})(2s - 1)). \)
Lecture II: Basic references


E. Hairer, Ch. Lubich & G. Wanner, *Geometric Numerical Integration* (2nd ed.), Springer-Verlag, 2006. (Section II.1)


Illustration: Collocation for DDE

\[ u'(t) = f(t, u(t), u(\theta(t))), \quad t \in [0, T], \]

with \( \theta(t) = t - \tau \) (\( \tau > 0 \)); \( u(t) = \phi(t), \quad t \leq 0. \)

Primary discontinuity points: \( \xi_\mu = \mu \cdot \tau \) (\( \mu \geq 0 \))

\[ \rightarrow \text{Assume: } T = \xi_{M+1} \text{ for some } M \geq 1, \text{ and} \]

let \( I(\mu) := [\xi_\mu, \xi_{\mu+1}] \) (\( 0 \leq \mu \leq M \)).

Collocation for DDE in \( S_m(0)(I_h) \), with constrained mesh \( I_h \),

\[ I_h := \bigcup_{\mu=0}^{M} I(\mu)_h \]

(containing the points \( \{\xi_\mu\} \)). \( I_h \) is defined by the local meshes

\[ I(\mu)_h := \{t_n(\mu) : \xi_\mu = t_0(\mu) < t_1(\mu) < \cdots < t_N(\mu) = \xi_{\mu+1}\}. \]

\[ \rightarrow \text{Local representation of } u_h \text{ on } [t_n(\mu), t_{n+1}(\mu)]: \]

for \( t = t_n(\mu) + v h_n(\mu), \quad v \in [0, 1]; \quad h_n(\mu) := t_{n+1}(\mu) - t_n(\mu) : \)

\[ u_h(t) = y_n(\mu) + h_n(\mu) \sum_{j=1}^{m} \beta_j(v) Y_n(\mu), \]

with \( y_n(\mu) := u_h(t_n(\mu)), \quad Y_n(\mu) := u'_h(t_n(\mu) + c_j h_n(\mu)). \)
Collocation points:

\[ X_h := \bigcup_{\mu=0}^{M} X_h^{(\mu)} , \]

with

\[ X_h^{(\mu)} := \{ t_n^{(\mu)} + c_i h_n^{(\mu)} : i = 1, \ldots, m; 0 \leq n \leq N - 1 \} \]

and prescribed \( 0 < c_1 < \cdots < c_m \leq 1 \).

For \( \mu = 0, \ldots, M \): generate \( u_h \in S_m^{(0)}(I_h) \) by

\[ u_h'(t) = f(t, u_h(t), u_h(\theta(t))), \quad t \in X_h^{(\mu)}, \]

with known \( u_h(\xi_{\mu}) \) and (when \( \mu = 0 \))

\[ u(\theta(t_n^{(0)} + c_i h_n^{(0)})) = \phi(\theta(t_n^{(0)} + c_i h_n^{(0)})). \]

Choose \( \theta \)-invariant mesh:

\[ \theta(I_h^{(\mu)}) = I_h^{(\mu-1)} \quad \text{for} \quad \mu = 1, \ldots, M. \]

Note that here we have

\[ \theta(t_n^{(\mu)} + c_i h_n^{(\mu)}) = t_n^{(\mu-1)} + c_i h_n^{(\mu-1)} \quad (\mu \geq 1), \]

since \( \theta \) is \textbf{linear}.

If \( \theta \) is \textbf{nonlinear}:

\[ \theta(t_n^{(\mu)} + c_i h_n^{(\mu)}) = t_n^{(\mu-1)} + \tilde{c}_i h_n^{(\mu-1)} \]

for some \( \tilde{c}_i \in (0, 1] \).
Collocation solution $u_h \in S_m^{(0)}(I_h)$:

$$u'_h(t) = f(t, u_h(t), u_h(t - \tau)), \ t \in X_h,$$

with $u_h(t) := \phi(t)$ if $t \in [-\tau, 0]$.

Using the local representation of $u_h$ on $[t_n^{(\mu)}, t_{n+1}^{(\mu)}]$,

$$u_h(t_n^{(\mu)} + vh_n^{(\mu)}) = y_n^{(\mu)} + h_n^{(\mu)} \sum_{j=1}^{m} \beta_j(v)Y_{n,j}^{(\mu)}, \ v \in [0, 1]$$

we obtain (setting $t_{n,i}^{(\mu)} := t_n^{(\mu)} + c_i h_n^{(\mu)}$)

$$Y_{n,j}^{(\mu)} = f\left(t_{n,i}^{(\mu)}, y_n^{(\mu)} + h_n^{(\mu)} \sum_{j=1}^{m} a_{i,j}Y_{n,i}^{(\mu)}, \Phi_{n,i}^{(\mu)}\right),$$

with

$$\Phi_{n,i}^{(\mu)} := u_h(t_{n,i}^{(\mu)} - \tau) = \theta(t_{n,i}^{(\mu)} + c_i h_n^{(\mu)})$$

If the mesh $I_h$ is $\theta$-invariant:

$$u_h(t_{n,i}^{(\mu)} - \tau) = u_h(t_{n,i}^{(\mu-1)} + c_i h_n^{(\mu-1)}) = u_h(t_{n,i}^{(\mu-1)}),$$

when $\theta$ is linear. For nonlinear $\theta$ we have

$$u_h(\theta(t_{n,i}^{(\mu)})) = u_h(t_{n,i}^{(\mu-1)} + \tilde{c}_i h_n^{(\mu-1)}).$$

$\Rightarrow$ m-stage continuous implicit Runge-Kutta method for the DDE

$$u'(t) = f(t, u(t), u(t - \tau)), \ t \in I.$$. 
Optimal convergence estimates

Assume that the delay function \( \theta(t) = t - \tau(t) \) satisfies:

(D1) \( \tau(t) \geq \tau_0 > 0 \) for \( t \in I := [t_0, T] \)

(D2) \( \theta \) is strictly increasing on \( I \);

(D3) \( \tau \in C^d(I) \) for some \( d \geq 0 \).

**Theorem:** (Bellen (1984))

Suppose that the mesh \( I_h \) in \( S^{(0)}_m(I_h) \) is \( \theta \)-invariant, and let the collocation parameters \( \{c_i\} \) satisfy

\[
J_\nu := \int_0^1 s^\nu \prod_{i=1}^m (s - c_i) \, ds = 0,
\]

\( \nu = 0, \ldots, \kappa - 1 \), for some \( \kappa \) with \( 1 \leq \kappa \leq m \).

If the given functions in the DDE (including \( \theta \)) are sufficiently smooth, then:

(a) \( \|u - u_h\|_\infty \leq C_m h^{m+1} \).

(b) \( \max\{|u(t) - u_h(t)| : t \in I_h\} \leq C_* h^{m+\kappa} \).

Here, \( h := \max_\mu \{h(\mu)\} \).
Summary: Constrained and $\theta$-invariant meshes

- **Primary discontinuity points** (or: *breaking points*) $\{\xi_\mu\}$ induced by the delay function $\theta$:
  \[
  \theta(\xi_\mu) = \xi_{\mu-1} \quad (\mu \geq 1; \; \xi_0 := 0),
  \]
  with $\xi_\mu - \xi_{\mu-1} \geq \tau_0 > 0$ for all $\mu \geq 1$.
  
  $\leftarrow$ **Assume:** $T = \xi_{M+1}$ for some $M \geq 1$.

- **Definition:**
  A mesh $I_h$ on $I := [0, T]$ is called a **constrained mesh** if it contains the *primary discontinuity points* $\{\xi_\mu\}$ induced by $\theta$; i.e.,
  \[
  I_h := \bigcup_{\mu=0}^{M} I_{h}^{(\mu)}
  \]
  is defined by the **local meshes**
  
  $I_{h}^{(\mu)} := \{t_{n}^{(\mu)} : \xi_\mu = t_{0}^{(\mu)} < t_{1}^{(\mu)} < \cdots < t_{N}^{(\mu)} = \xi_{\mu+1}\}$.

- **Definition:**
  A **constrained** mesh $I_h$ is said to be **$\theta$-invariant** if
  \[
  \theta : \quad I_{h}^{(\mu)} \longrightarrow I_{h}^{(\mu-1)} \quad \text{for} \quad \mu = 1, \ldots, M;
  \]
  that is, if
  \[
  \theta(t_{n}^{(\mu)}) = t_{n}^{(\mu-1)} \quad (n = 0, 1, \ldots, N)
  \]
  for $\mu = 1, \ldots, M$. 

Let $\theta(t) = t - \tau(t)$, $\tau(t) \geq \tau_0 > 0$. For $t \in [\xi_{\mu}, \xi_{\mu+1}]$ the collocation error $e_h := u - u_h$ associated with the collocation equation

$$u'_h(t) = a(t)u_h(t) + b(t)u(\theta(t)) + g(t)$$

$$(\mathcal{V}u_h)(t) + (\mathcal{V}_\theta u)_h(t) - \delta_h(t), \quad t \in I,$$

with $\delta_h(t) = 0$ for $t \in X_h$, has the representation

$$e_h(t) = r_1(t, \xi_{\mu})e_h(\xi_{\mu}) + \int_{\xi_{\mu}}^{t} r_1(t, s)d_h(s) ds$$

$$+ F_\mu(t) + \Phi_\mu(t), \quad t \in I^{(\mu)}.$$

Here,

$$F_\mu(t) := \sum_{\nu=0}^{\mu-1} \int_{\xi_{\nu}}^{\xi_{\nu+1}} r_{\mu,\nu}(t, s)d_h(s) ds +$$

$$\sum_{\nu=0}^{\mu-1} p_{\mu,\nu}(t)e_h(\xi_{\nu}) + G^{(1)}_\mu(t; \phi) \quad \text{and}$$

$$\Phi_\mu(t) := \sum_{\nu=0}^{\mu-1} \int_{\xi_{\nu}}^{\theta^{\mu-\nu}(t)} r_{\mu,\nu}(t, s)d_h(s) ds + G^{(2)}_\mu(t; \phi),$$

with

$$d_h(t) := \int_{0}^{t} \delta_h(s) ds.$$
Equation for collocation error:

\[ e_h(t) = r_1(t, \xi_\mu) e_h(\xi_\mu) + \int_{\xi_\mu}^{t} r_1(t, s) d_h(s) \, ds \]

\[ + F_\mu(t) + \Phi_\mu(t), \quad t \in I(\mu), \]

with

\[ F_\mu(t) := \sum_{\nu=0}^{\mu-1} \int_{\xi_\nu}^{\xi_{\nu+1}} r_{\mu, \nu}(t, s) d_h(s) \, ds + \]

\[ \sum_{\nu=0}^{\mu-1} p_{\mu, \nu}(t) e_h(\xi_\nu) + G_\mu^{(1)}(t; \phi) \quad \text{and} \]

\[ \Phi_\mu(t) := \sum_{\nu=0}^{\mu-1} \int_{\xi_\nu}^{t} r_{\mu, \nu}(t, s) d_h(s) \, ds + G_\mu^{(2)}(t; \phi). \]

\[ t = t_n^{(\mu)}: \]

If the mesh \( I_h \) is \( \theta \)-invariant, then

\[ \theta^{\mu-\nu}(t_n^{(\mu)}) = t_n^{(\nu)} \quad (\nu = 0, \ldots, \mu). \]

Hence, we can estimate the integrals by employing the techniques used for non-delay VIDEs (and ODEs).

An analogous error representation holds for VFIEs.
Superconvergence results for VFIDEs and VFIEs with non-vanishing delays:

**Theorem:** (Bellen (1984); Brunner (2004))

Let the delay function $\theta(t) = t - \tau(t)$ satisfy

(D1) $\tau(t) \geq \tau_0 > 0$ for $t \in I := [t_0, T]$

(D2) $\theta$ is strictly increasing on $I$;

(D3) $\tau \in C^d(I)$, with sufficiently large $d$.

Then:

For sufficiently smooth data (including the initial function $\phi$), the collocation solutions in $S_m^0(I_h)$ (for VFIDEs) or in $S_m^{-1}(I_h)$ (for VFIEs) possess the **same optimal orders of local superconvergence** on $I_h$ as the ones for classical VIDEs and VIEs with similarly smooth data if, and only if, the underlying mesh $I_h$ is $\theta$-invariant.

For example, if the $\{c_i\}$ are the Gauss points:

$$\max\{|u(t) - u_h(t)| : t \in I_h\} \leq C^*_m h^{2m}$$

for VFIDEs, and

$$\max\{|u(t) - u_{ht}(t)| : t \in I_h \setminus \{0\}\} \leq C^*_m h^{2m}$$

for VFIEs.
Remarks:

- **Fully discretised** collocation equations:
  The integrals occurring in the collocation equations for VFIDEs and VFIEs,
  \begin{align*}
  \int_0^1 K(t^{(\mu)}_{n,i}, t^{(\mu)}_{\ell} + \text{sh}^{(\mu)}_{\ell}) \beta_j(s) \, ds
  \end{align*}
  and
  \begin{align*}
  \int_0^1 K(t^{(\mu)}_{n,i}, t^{(\mu)}_{\ell} + \text{sh}^{(\mu)}_{\ell}) L_j(s) \, ds,
  \end{align*}
  can in general not be found analytically and thus have to be **approximated** by appropriate **quadrature formulas**.
  Use **m-point interpolatory quadrature** with **abscissas** given by the **collocation points**.  
  Order of quadrature error is (at least) equal to the local order of the **exact** collocation solution.

- **Non-monotonic delay functions**:  
  See monograph by **Bellen & Zennaro (2003)**; also: **Brunner & Maset (2008)**.
Lecture III: Basic references


Lecture IV:
VFES with vanishing delays

Volterra functional equations (on $I := [0, T]$):

- $u'(t) = a(t)u(t) + b(t)u(\theta(t)) + (\mathcal{V}_\theta u)(t)$,
- $u(t) = g(t) + (\mathcal{V}_\theta u)(t)$;
- $u(t) = g(t) + b(t)u(\theta(t)) + (\mathcal{V}_\theta u)(t)$;

Volterra integral operators ($C(I) \to C(I)$)

$$
(\mathcal{V}u)(t) := \int_0^t K_0(t,s)u(s)\,ds \\
(\mathcal{V}_\theta u)(t) := \int_0^{\theta(t)} K_1(t,s)u(s)\,ds.
$$

Also:

$$
(\mathcal{W}_\theta u)(t) := \int_{\theta(t)}^t K(t,s)u(s)\,ds.
$$

Assume that the delay function $\theta = \theta(t)$ satisfies:

(D1) $\theta(0) = 0$; $\theta(t) \leq q_1 t$ for some $q_1 \in (0, 1)$;

(D2) $\theta$ is strictly increasing in $I$;

(D3) $\theta \in C^d(I)$ for some $d \geq 1$.

Pantograph equation:

$$
u'(t) = au(t) + bu(qt), \ t \in I : \\
\theta(t) = qt = t - (1 - q)t, \ 0 < q < 1.$$

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**Illustration:** Pantograph DDE: $\theta(t) = qt$

$u'(t) = au(t) + bu(qt), \quad t \in [0, T] \quad (0 < q < 1)$, with $u(0) = u_0$.

Collocation solution $u_h \in S_m^{(0)}(I_h)$, with uniform mesh $I_h$:

$u'_h(t) = au_h(t) + bu_h(qt), \quad t \in X_h; \quad u_h(0) = u_0$.

Define:

$q^I := \left\lceil \frac{q}{1 - q} c_1 \right\rceil, \quad q^{II} := \left\lceil \frac{q}{1 - q} c_m \right\rceil$.

For the collocation points $t = t_n + c_i h \in e_n$, the **images** $q(t_n + c_i h)$ satisfy

- **Phase I:** $0 \leq n < q^I$
  
  $q(t_n + c_i h) \in (t_n, t_{n+1})$ for all $i = 1, \ldots, m$.

- **Phase II:** $q^I \leq n < q^{II}$
  
  $q(t_n + c_i h) \leq t_n$ for some $i < m$.

- **Phase III:** $q^{II} \leq n \leq N - 1$
  
  $q(t_n + c_i h) \leq t_n$ for all $i = 1, \ldots, m$. 
Continuous implicit Runge-Kutta method:

\[ u_h \in S_m^{(0)}(I_h), \text{ with uniform } I_h: \]

\[ u_h(t_n + vh) = u_h(t_n) + h \sum_{j=1}^{m} \beta_j(v) Y_{n,j}; \quad v \in [0, 1]. \]

Let \( Y_n := (Y_{n,1}, \ldots, Y_{n,m})^T \in \mathbb{R}^m. \)

- **Phase I:** \( 0 \leq n < q^I \)
  \[ \Rightarrow \text{ Linear algebraic systems for } Y_n: \]
  \[ [I_m - h(\mathcal{A}_n + \mathcal{B}_n^I(q))] Y_n = r_n^I. \]

- **Phase II:** \( q^I \leq n < q^{II} \)
  \[ [I_m - h(\mathcal{A}_n + \mathcal{B}_n^{II}(q))] Y_n = r_n^{II} + \tilde{\mathcal{B}}_n^{II}(q) Y_{n-1}. \]

- **Phase III:** \( q^{II} \leq n \leq N - 1 \)
  \[ [I_m - h\mathcal{A}_n] Y_n = r_n^{III} + \mathcal{B}_n^{III}(q) Y_{\tilde{n}}, \]
  for some \( \tilde{n} < n. \) Here, \( I_m \) denotes the identity matrix in \( \mathbb{R}^{m \times m}, \) and \( \mathcal{A}_n \) is the Runge-Kutta matrix corresponding to the ODE part in the pantograph DDE.
Collocation solutions for VFIDEs

\[ u'(t) = a(t)u(t) + b(t)u(\theta(t)) \]
\[ + (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t), \quad t \in I, \]

with delay function \( \theta \) satisfying

(D1) \( \theta(0) = 0; \quad \theta(t) \leq q_1 t \) for some \( q_1 \in (0, 1) \);
(D2) \( \theta \) is strictly increasing in \( I \);
(D3) \( \theta \in C^d(I) \) for some \( d \geq 0 \).

Let \( I_h := \{ t_n : 0 = t_0 < t_1 < \cdots < t_N = T \} \), with

\[ e_n := (t_n, t_{n+1}], \quad h_n := t_{n+1} - t_n, \quad h := \max \{ h_n \}. \]

- **Collocation space:**

\[ S_m^{(0)}(I_h) := \{ v \in C(I) : v|_{e_n} \in \pi_m \ (0 \leq n \leq N-1) \}, \]

with \( \Rightarrow \) \( \dim (S_m^{(0)}(I_h)) = Nm + 1 \).

- **Collocation equation:** Find \( u_h \in S_m^{(0)}(I_h) \) so that for all \( t \in X_h \),

\[ u_h'(t) = a(t)u_h(t) + b(t)u_h(\theta(t)) \]
\[ + (\mathcal{V}u_h)(t) + (\mathcal{V}_\theta u_h)(t), \]

with \( u_h(0) = u_0. \)
Structure of algebraic equations for $Y_n$ in the local representation of $u_h$,

$$u_h(t) = u_h(t_n) + h \sum_{j=1}^{m} \beta_j(v) Y_{n,j}, \quad v \in [0, 1] :$$

- **Phase I:** $0 \leq n < q^I$

  $$[I_m - h(A_n + B_{n}^I(q)) - h^2(C_n + C_{n}^I(q))] Y_n = r_{n}^I.$$

- **Phase II:** $q^I \leq n < q^{II}$

  $$[I_m - h(A_n + B_{n}^{II}(q)) - h^2(C_n + C_{n}^{II}(q))] Y_n$$

  $$= r_{n}^{II} + h[\tilde{B}_{n}^{II}(q) + h\tilde{C}_{n}^{II}(q)] Y_{n-1}.$$

- **Phase III:** $q^{II} \leq n \leq N - 1$

  $$[I_m - h(A_n + hC_n)] Y_n$$

  $$= r_{n}^{III} + h[\tilde{B}_{n}^{III}(q) + h\tilde{C}_{n}^{III}(q)] Y_{\tilde{n}},$$

  for some $\tilde{n} < n$. 
Collocation solutions for VFIEs

\[ u(t) = g(t) + (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t), \quad t \in [0, T], \]

with

\[ (\mathcal{V}u)(t) := \int_0^t K_0(t, s)u(s) \, ds \]

and

\[ (\mathcal{V}_\theta u)(t) := \int_0^{\theta(t)} K_1(t, s)u(s) \, ds. \]

Let \( I_h := \{t_n : 0 = t_0 < t_1 < \cdots < t_N = T\} \), with

\[ e_n := (t_n, t_{n+1}], \quad h_n := t_{n+1} - t_n, \quad h := \max\{h_n\}. \]

- **Collocation space:**

\[ S^{(-1)}_{m-1}(I_h) := \{v : v|_{e_n} \in \pi_{m-1} (0 \leq n \leq N)\}. \]

- **Collocation points:**

\[ X_h := \{t_n + c_k h_n : 0 \leq n \leq N - 1\}, \]

with \( 0 \leq c_1 < \cdots < c_m \leq 1 \).

- **Collocation equation:** Find \( u_h \in S^{(-1)}_{m-1}(I_h) \) so that

\[ u_h(t) = g(t) + (\mathcal{V}u_h)(t) + (\mathcal{V}_\theta u_h)(t), \quad t \in X_h. \]

\[ \Leftarrow \text{ Iterated collocation solution:} \]

\[ u_h^{it}(t) := g(t) + (\mathcal{V}u_h)(t) + (\mathcal{V}_\theta u_h)(t), \quad t \in I. \]

**Note** that \( u_h^{it}(t) = u_h(t) \) for all \( t \in X_h \).
**VFEs with vanishing delays:**

**Global (super-) convergence on uniform** $I_h$:

- $u_h \in S^{(-1)}_{m-1}(I_h)$ for the **VFIE**

\[ u(t) = g(t) + (Vu)(t) + (V_{\theta}u)(t), \quad t \in I : \]

**Theorem:** (B. & Hu (2005))

(i) For general $\{c_k\}$:

\[ \|u - u_h\|_\infty \leq C m h^m. \]

(ii) If the $\{c_k\}$ are the $m$ Gauss points in $(0, 1)$:

\[ \|u - u_{h_{it}}\|_\infty \leq \tilde{C} m h^{m+1}. \]

- $u_h \in S^{(0)}_m(I_h)$ for the **VFIDE**

\[ u'(t) = au(t) + bu(\theta(t)) + (Vu)(t) + (V_{\theta}u)(t), \quad t \in I : \]

**Theorem:** (B. & Hu (2007))

(i) For general $\{c_k\}$:

\[ \|u - u_h\|_\infty \leq C m h^m. \]

(ii) For the Gauss points $\{c_k\}$:

\[ \|u - u_h\|_\infty \leq \tilde{C} m h^{m+1}. \]
VIDEs with vanishing delays:

**Local superconvergence on uniform $I_h$**

Collocation solution $u_h \in S^{(d)}_{m+d}(I_h)$ ($d := k-1$), with uniform mesh $I_h$, for

$$
u^{(k)}(t) = a(t)u(t) + b(t)u(\theta(t))$$

$$+ (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t) \quad t \in I := [0, T],$$

with delay function $\theta(t) = qt$ ($0 < q < 1$).

**Theorem:** (B. & Hu (2007) ($k = 1$); B. (2008)) If the $\{c_j\}$ are the Gauss points:

$$\max_{t \in I_h} |u^{(j)}(t) - u_h^{(j)}(t)| \leq C^*_m(q) \begin{cases} h^{2m} & \text{if } m = 1, 2 \\ h^{m+2} & \text{if } m > 2, \end{cases}$$

for $j = 0, \ldots, k-1$ and all $q \in (0, 1)$.

**Special case:** Pantograph DDE:

$$u'(t) = a(t)u(t) + b(t)u(qt) \quad (0 < q < 1).$$

The proofs of the optimal local superconvergence results for VFIDEs and VFIEs are based on the representations of the solutions $e_h := u - u_h$ of the error equations.
VFIEs with vanishing delays:

Local superconvergence on uniform $I_h$

Collocation solution $u_h \in S^{(-1)}_{m-1}(I_h)$ and corresponding \textit{iterated} collocation solution $u^i_{h}$ for

$$u(t) = g(t) + (\nabla u)(t) + (\nabla \theta u)(t), \quad t \in [0, T],$$

with $\theta(t) = qt \quad (0 < q < 1)$

$\rightarrow$ Observation: (B. (1997))

$$u(t) = u_0 + \int_0^{qt} (b/q)u(s) \, ds, \quad t \geq 0, \quad 0 < q < 1 :$$

If collocation is at \textit{Gauss points}, then $u^i_{h}(h)$ is \textbf{not} the $(m, m)$-\textit{Padé approximant} to $u(h)$:

$$|u(h) - u_h(h)| = O(h^{p^*}) \quad \text{with} \quad p^* < 2m + 1.$$ 

**Theorem:** (B. & Hu (2005))

If the \{c_k\} are the \textit{Gauss points} and $m \geq 2$:

$$\max_{t \in I_h} |u(t) - u^i_{h}(t)| \leq C^*_m(q) \left\{ \begin{array}{ll} h^{m+2} & \Leftrightarrow \frac{q}{m} = 1/2 \quad \text{and} \quad m \text{ even}, \\ h^{m+1} & \text{otherwise.} \end{array} \right.$$ 

**Comparison:** For $q = 1$ (classical VIE):

$$\max\{|u(t) - u^i_{h}(t)| : t \in I_h \setminus \{0\}\} \leq C^*_m h^{2m}.$$
Open Problem:

**Superconvergence analysis** of iterated collocation solution $u_h^{it}$ corresponding to $u_h \in S_{m-1}^{(-1)}(I_h)$ (on uniform mesh $I_h$) for theVFIEs

\[ u(t) = g(t) + b(t)u(\theta(t)) + (\mathcal{V}_{\theta}u)(t), \quad t \in [0, T], \]

and

\[ u(t) = g(t) + b(t)u(\theta(t)) + (\mathcal{W}_{\theta}u)(t), \quad t \in [0, T], \]

with

\[ (\mathcal{W}_{\theta}u)(t) := \int_{\theta(t)}^{t} K(t, s)u(s) \, ds, \]

and $\theta(t) = qt$ ($0 < q < 1$) ?

$\leftrightarrow$ **Ill-posed problem** (Denisov & Lorenzi (1997))

**Special case:**

\[ u(t) = g(t) + b(t)u(\theta(t)), \quad t \in [0, T] \]

(Liu (1995): $m = 1$).
Representation of collocation errors: VFIDEs

The collocation error \( e_h := u - u_h \) for

\[
 u^{(k)}(t) = a(t)u(t) + b(t)u(\theta(t)) + (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t) \quad (k \geq 1),
\]

with vanishing delay function \( \theta(t) \) (e.g. \( \theta(t) = qt \)) satisfies the VFIDE

\[
 e_h^{(k)}(t) = a(t)e_h(t) + b(t)e_h(\theta(t)) + \delta_h(t) + (\mathcal{V}e_h)(t) + (\mathcal{V}_\theta e_h)(t), \quad t \in [0, T],
\]

with \( e_h^{(j)}(0) = 0, \quad j = 0, \ldots, k - 1 \). The defect function \( \delta_h(t) \) is piecewise smooth and vanishes on \( X_h \).

For \( a(t) \equiv 0, \mathcal{V} = 0 \) the solution of the error equation is given by

\[
 e_h(t) = d_h(t) + \sum_{j=1}^{\infty} \int_0^t \mathcal{H}_{k,j}(t,s)d_h(s) \, ds, \quad t \in [0, T],
\]

where the kernels \( \mathcal{H}_{k,j} \) are smooth and

\[
 d_h(t) := \int_0^t \frac{(t-s)^{k-1}}{(k-1)!}\delta_h(s) \, ds.
\]

For \( t = t_n \) (uniform mesh), \( \theta_j(t_n) = t_{q_n,j} + \gamma_{n,j}h \), where

\[
 q_{n,j} := \lfloor \theta_j(t_n)/h \rfloor \in \mathbb{N}, \quad \gamma_{n,j} := \theta_j(t_n)/h - q_{n,j} \in [0, 1).
\]
For $\theta(t) = qt$, $t = t_n = nh$ ($1 \leq n \leq N$):

$$e_h(t_n) = d_h(t_n) + \sum_{j=1}^{\infty} \int_0^{qt_n} H_{k,j}(t_n, s)d_h(s)\,ds,$$

with

$$d_h(t) := \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} \delta_h(s)\,ds \quad \text{if} \quad k \geq 1,$$

and

$$d_h(t) := \delta_h(t) \quad \text{if} \quad k = 0.$$

(Recall that $\delta_h(t) = 0$ for $t \notin X_h$.)

Since $\theta(t_n) = t_{q_{n,j}} + \gamma_{n,j}h$, we have

$$\int_0^{qt_n} H_{k,j}(t_n, s)d_h(s)\,ds = \int_0^{t_{q_{n,j}}} H_{k,j}(t_n, s)d_h(s)\,ds$$

$$+ h \int_0^{\gamma_{n,j}} H(t_n, t_{q_{n,j}} + sh)d_h(t_{q_{n,j}} + sh)\,ds.$$

(etc.)
Collocation on (quasi-) geometric meshes

I. Non-vanishing delay techniques:
On $[0, t_0]$ (with suitably small $t_0 = t_0(q; N) > 0$), assume given initial approximation to $u(t)$.

Choose geometric macro-mesh on $[t_0, T]$ given by

$$\{\xi_\mu := q^{\kappa-\mu}T : 0 \leq \mu \leq \kappa\}, \quad \kappa = \kappa(q; N),$$

with appropriate $\kappa$ such that $\xi_0 := t_0 \to 0$ as $N \to \infty$. ➞ Local (uniform) meshes:

$$I_h^{(\mu)} := \{t_n^{(\mu)} : \xi_\mu = t_0^{(\mu)} < t_1^{(\mu)} < \ldots < t_N^{(\mu)} = \xi_{\mu+1}\}.$$

⇒ Collocation solution $u_h \in S_m^{(0)}(I_h)$ (at Gauss points, and on the global $\theta$-invariant mesh

$$I_h := \bigcup_{\mu=0}^{\kappa-1} I_h^{(\mu)}$$

for the VFIDE

$$u'(t) = a(t)u(t) + b(t)u(\theta(t)) + (Vu)(t) + (V_{\theta}u)(t) :$$

$$\Rightarrow \max_{t \in I_h} |u(t) - u_h(t)| \leq C_m^*(q)N^{-2m}.$$

II. Vanishing delay techniques:  
(Brunner, Hu & Lin (2001), B. & Hu (2007))

Global geometric mesh on $[0, T]$:

$$I_h := \{ t_n = t_n^{(N)} := d^{N-n}T : 0 \leq n \leq N \},$$

with suitably chosen $d = d(q; m, N) \in (0, 1)$.

Collocation in $S_m^{(0)}(I_h)$ for VFIDE

$$u'(t) = a(t)u(t) + b(t)u(\theta(t)) + (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t),$$

using the Gauss points, yields

$$\max_{t \in I_h} |u(t) - u_h(t)| \leq C^*_m(q)N^{-(2m-\varepsilon_N)},$$

where $\varepsilon_N \to 0$, as $N \to \infty$.

**Question:**

Numerical comparison of collocation solutions on quasi-geometric meshes (approach of Bellen et al.) and on geometric meshes (approach of Brunner & Hu)?

**Remark:**

The variable stepsize code RADAR5 (Guglielmi & Hairer (2001, 2005)), when applied to pantograph-type DDEs, appears to generate meshes $I_h$ with stepsizes $\{h_n\}$ that show exponential-like growth (Guglielmi (2006)).
Multiple vanishing delays:
The attainable order of local superconvergence at $t = t_1 = h$ for the double pantograph equation,

$$u'(t) = au(t) + b_1 u(q_1 t) + b_2 u(q_2 t), \quad t \in [0, T],$$

where $0 < q_1 < q_2 < 1$, is discussed in Zhao, Xu & Qiao (2005); see also Qiu, Mitsui & Kuang (1999) and Liu & Li (2004).

- **Optimal superconvergence** of $u_h \in S_m^0(I_h)$ on uniform meshes $I_h$ for the multiple delay VFIDE

$$u'(t) = a(t) u(t) + \sum_{j=1}^r b_j(t) u(\theta_j(t)) + \sum_{j=1}^r (V_{\theta_j} u)(t), \quad t \in [0, T],$$

where $\theta_j(t) = q_j t, \quad 0 < q_1 < \cdots < q_r < 1$.

**Theorem:** (B. (2008))
Collocation at Gauss points leads to

$$\max\{|u(t) - u_h(t)| : t \in I_h\} \leq C^*_m(q) h^{m+2}$$

for any $q := (q_1, \ldots, q_r)$ ($r \geq 2$) and all $m \geq 2$. 
Optimal orders of superconvergence of $u_h \in S_{m-1}^{-1}(I_h)$ and corresponding $u^{it}_h$, on uniform meshes, for VFIEs with multiple vanishing delays,

$$u(t) = g(t) + \sum_{j=1}^{r} (V_{\theta_j}u)(t), \quad t \in [0, T],$$

where $\theta_j(t) = q_j t$, $0 < q_1 < \cdots < q_r < 1$.

**Theorem:** (B., 2008)

Local superconvergence for $u_h$ or $u^{it}$ with $p^* = m + 2$ ($m \geq 2$) is not possible. If the $\{c_i\}$ are the Gauss points, then the optimal local order of convergence on uniform meshes $I_h$ is described by

$$\max\{|u(t) - u^{it}_h(t)| \ t \in I_h \ \backslash \ \{0\}\} \leq C^*_m(q) h^{m+1}$$

for all $q := (q_1, \ldots, q_r)$. It coincides with the optimal global order of superconvergence of $u^{it}_h$ on $I$. 
'Integral-algebraic' VFEs  
(VFEs with *non-local constraints*)

**Illustration:**

\[ u'(t) = F(t, u(t), u(\theta(t)), w(t), w(\theta(t))), \quad t \in [0, T], \]

\[ 0 = g(t) + \int_{\theta(t)}^{t} k(t - s) G(s, u(s), w(s)) \, ds, \]

with delay function \( \theta(t) \) satisfying \( \theta(0) = 0 \) (etc.).

(Collocation for delay DAEs with non-vanishing delays and *local* (algebraic) constraints was studied by *Hauber (1997).*

→ **Open problem:** For \( \theta(t) = qt \) \((0 < q < 1)\), the convergence analysis \((h \to 0)\) of collocation solutions \( u_h \in S_{m-1}(I_h) \) (with uniform \( I_h \)) for the **first-kind** VFIE \((\leftrightarrow \text{Volterra (1897)!})\)

\[ 0 = g(t) + \int_{qt}^{t} K(t, s) u(s) \, ds, \quad t \in [0, T], \]

where \( g(0) = 0, \; g \in C^1(I); \; |K(t, t)| \geq \kappa_0 > 0 \) and \( K \in C^1(D_\theta), \) is open.

- \( q = 0 \):

\[ \|u - u_h\|_{\infty} \to 0 \quad \Leftrightarrow \quad \prod_{i=1}^{m} \frac{1 - c_i}{c_i} \leq 1. \]
Lecture IV: **Basic references**


V. Concluding Remarks:  
Current and future research work

- DDEs and VFEs with non-monotonic (vanishing) delay functions

Illustration:

$$\theta(t) = q_1 t + (q_2 - q_1) t \sin^2(\omega t) \quad t \geq 0,$$
with $0 < q_1 < q_2 < 1$, $\omega \geq 1$.

Brunner & Maset (2008); B. & Guglielmi (2008)

- VFEs with weakly singular kernels

VFIDEs and VFIEs corresponding to delay integral operators of the form

$$(\mathcal{V}_{\theta, \alpha} u)(t) := \int_0^{\theta(t)} (t - s)^{-\alpha} K_1(t, s) u(s) \, ds$$
and

$$(\mathcal{W}_{\theta, \alpha} u)(t) := \int_{\theta(t)}^t (t - s)^{-\alpha} K(t, s) u(s) \, ds,$$
with $0 < \alpha < 1$.

- Current work with Q.-Y. Hu (collocation) and D. Schötzau (discontinuous Galerkin method).
• Analysis of asymptotic stability (and contractivity) of collocation solutions on uniform meshes for VFIDEs with vanishing delays (e.g. for $\theta(t) = qt$ $(0 < q < 1)$?

The solutions of the pantograph DDE

$$u'(t) = au(t) + bu(qt), \quad t \geq 0,$$

satisfy

$$\lim_{t \to \infty} u(t) = 0 \quad \text{if}$$

$$\text{Re}(a) < 0 \quad \text{and} \quad |b| < |a|. \quad (1)$$

⇒ **Open Problem 1:**
For which $\{c_i\}$ does the collocation solution $u_h \in S_{m}^{(0)}(I_h)$, with uniform $I_h$, satisfy

$$\lim_{t \to \infty} u_h(t) = 0 \ ?$$

**Special case:** $m = 1, \ q = 1/2 : \ c_1 \in [1/2, 1]$ (Buhmann, Nørsett & Iserles (1994); Liu, Wang & Hu (2005)).

⇒ **Open Problem 2:**
Assume (1). For which (continuous) $k_0$ and $k_1$ are the solutions of the VFIDE

$$u'(t) = au(t) + bu(qt) + \int_{0}^{t} k_0(t - s)u(s) \, ds,$$

$$+ \int_{0}^{qt} k_1(t - s)u(s) \, ds, \quad t \geq 0,$$

asymptotically stable?
• DEs and VFEs with advanced arguments

**Illustration:**

\[ u'(t) = au(t) + b u(qt), \quad t \geq 0, \quad q > 1 : \]

\[ \rightarrow \text{Ill-posed problem!} \]

\[ \rightarrow \text{Numerical analysis remains open.} \]

**Application:**

*Modelling of cell growth*: steady-state distribution of population of cells that grow and divide (each mother cell divides into \( q > 1 \) daughter cells of same size).

See, e.g., Hall & Wake (1989,1990+), Wall (2007); *also*: Marshall, van Brunt & Wake (2004) and references.

• Design of VFE software

\[ \rightarrow \text{Extension of RADAR5 to VFIDEs (and VFIEs)?} \]

(See www.unige.ch/~hairer for details of RADAR5.)
Collocation for VFEs with state-dependent delays

Illustration:
Population growth with ‘crowding effects’ (Bélair (1991)):

\[ u(t) = \int_{t-\tau(u(t))}^{t} P(t - s)G(u(s)) \, ds, \quad t > 0, \]

\[ \leftarrow \text{Attainable order of (super-) convergence of iterated collocation solution corresponding to collocation solution } u_{h} \in S_{m-1}^{(-1)}(I_{h})? \]

Current work with Stefano Maset (Trieste)

Partial VFIEs

Illustration:
Time-stepping for (semi-discretised) system corresponding to the partial VFIDE

\[ u_{t} - \Delta u = \int_{0}^{t} k(t - s)G(u(s), \cdot, u(\theta(s), \cdot)) \, ds, \]

with \( x \in \Omega \subset \mathbb{R}^{d} \) (\( d = 1, 2 \)), \( u(t, 0) = u_{0}(x) \) (plus homogeneous BCs), \( \theta(0) = 0 \), and

\[ G(u, w) = au^{p} + bw^{r}, \quad p > 1, \quad r > 1. \]