1. The Economics of Insurance

Insurance is designed to protect against serious financial reversals that result from random evens intruding on the plan of individuals.

Limitations on Insurance Protection

- It is restricted to reducing those consequences of random events that can be measured in monetary terms.

- Insurance does not directly reduce the probability of Loss.

Several examples of Situations where random events may cause financial losses

- The destruction of property by fire or storm

- A damage award imposed by a court as a result of a negligent act.

- Prolonged illness may strike at an unexpected time and result in financial loss.

- The death of a young adult may occur while long-term commitments to family or business remain unfilled.

- The individual survives to an advanced age, resources for meeting the costs of living may be depleted.

An *Insurance System* is a mechanism for reducing the adverse financial impact of random events that prevent the fulfillment of reasonable expectations.

Distinctions between Insurance and Related Systems

- Banking Institute

  Banking institutions do not make payments based on the size of a financial loss occurring from an event outside the control of the person suffering the loss.

- Gambling

  The typical gambling arrangement is established by defining payoff rules about the occurrence of a contrived event, and risk is voluntarily sought by the participants.

- The definition of an insurance system is purposefully board. It encompasses systems that cover losses in both property and human-life values.

- The economic justification for an insurance system is that it contributes to general welfare by improving the prospect that plan will not be frustrated by random events.

1.1. Utility Theory

Utility Theory An elaborate theory was developed to provides insight into decision making in the face of uncertainty.

Expected value principle Define the value of an economic project with a random outcome to be its expected value.

Fair or Actuarial value of the prospect The expected value of random prospects with monetary payments in economics.

Many decision makers do not adopt the expected value principle. For them, their wealth level and other aspects of the distribution of outcomes influence their decisions.

Example In all cases, it is assumed that the probability of an accident is 0.01 and the probability of no accident is 0.99. Three cases are considered according to the amount of loss arising from an accident; the expected loss is tabulated for each.

Case	Possible Losses		Expected Losses
1	0	1	0.01
2	0	1 000	10.00
3	0	1 000 000	10 000.00

For case 1 and 2, the loss 1 or 1000 might be of little concern to the decision maker who then might be unwilling to pay more than the expected loss to obtain insurance. However, the loss of 1,000,000 in case 3 which may exceed his net worth, could be catastrophic. In this case, the decision maker might well be willing to pay more than expected loss of 10,000 in order to obtain insurance.

Utility Function $u(w)$ the function specified to attach to wealth of amount w, measured in monetary units.

A linear transformation

$$u^*(w) = au(w) + b, \quad a > 0,$$

yields a function $u^*(w)$, which is essentially equivaluent to $u(w)$.

Example Fix $u(0) = -1$ and $u(20,000) = 0$. Suppose you face a loss of $20,000$ with probability 0.5, and will remain at your level of wealth with probability 0.5. What is the maximum amount G you would be willing to pay for complete insurance protection against this random loss.

Express this question in the following way: For what value of G does

$$u(20,000 - G) = 0.5u(20,000) + 0.5u(0) = 0.5(0) + 0.5(-1) = -0.5?$$

Consistency requirements There is a utility function $u(w)$ such that if the distribution of X is preferred to the distribution of Y, $E[u(X)] > E[u(Y)]$, and if the decision maker is indifferent between the two distribution, $E[u(X)] = E[u(Y)]$.

Observation about utility

- Utility theory is built on the assumed existed and consistency of preferences for probability distribution of outcomes. It is a numerical description of existing preferences.

- A utility function need not,in fact can not be determined uniquely . For example if $u^*(w) = au(w) + b, a > 0$, then $E[u(X)] > E[u(Y)]$ is equivalent to $E[u^*(X)] > E[u^*(Y)]$

- If $u(w) = aw + b, a > 0$, then if $E(X) = \mu_X$ and $E(Y) = \mu_Y$, we have

$$E[u(X)] = a\mu_X + b > E[u_Y(Y)] = au_Y + b$$

if and only if $u_X > u_Y$.

1.2. Insurance and Utility

- **Insurer** An insurance organization was established to help reduce the financial consequences of the damage or destruction of property.

- **Policies** The insurer would issue contracts that would promise to pay the owner of a property a defined amount to or less than the financial loss if the property were damaged or destroyed during the period of the policy.

- **Claim payment** The contingent payment linked to the amount of the loss.

- **Insured** the owner of the property

- **Premium** In return for the promise contained in the policy, the owner of the property pays a consideration.

- ***Pure or net premium*** μ ***for the 1-period insurance policy*** The basic price for full insurance coverage as the expected loss $\mathrm{E}[X] = \mu$.

- ***loaded premium*** To provide for expense , taxes, and profit and for some security against adverse loss experience, the premium is set by loading, adding to the pure premium.

$$H = (1 + a)\mu + c, \quad a > 0, \quad c > 0.$$

$a\mu$ can be viewed as being associated with expenses that vary with expected losses and with the risk that claim experience will deviate from expected.

The constant c provides for expected expenses that do not vary with losses.

- **Increasing linear utility function** $u(w) = bw + d$

$$u(w - G) = b(w - G) + d \geq \mathrm{E}[u(w - X)] = \mathrm{E}[b(w - X) + d],$$

$$b(w - G) + d \geq b(w - \mu) + d \Rightarrow G \leq \mu.$$

  There seems to be little opportunity for a mutually advantageous insurance contract.

- **Consistency requirement**
  To ensure the existence of a utility function. (linear,quadratic, exponential, logarithmic, or any other particular form)
  Reflect decision maker's preferences.

- **Decreasing marginal utility in Economics**
  Each additional equal increment of wealth results in a smaller increment of associated utility.

Two properties suggested by observations for $u(w)$

$$u'(w) > 0 \quad \text{and} \quad u''(w) < 0.$$

Jensen's inequations

For a random variable X and function $u(w)$,

$$\text{if } u''(w) < 0, \quad \text{then} \quad \mathrm{E}[u(X)] \leq u(\mathrm{E}[X]),$$

$$\text{if } u''(w) > 0, \quad \text{then} \quad \mathrm{E}[u(X)] \geq u(\mathrm{E}[X]).$$

Applications to Insurance

- Assuming the **decision maker's** preferences are that $u'(w) > 0$ and $u''(w) < 0$.

$$u(w - G) = \mathrm{E}[u(w - X)] \le u(w - \mu) \Rightarrow w - G \le w - \mu$$

The decision maker will pay an amount greater than the expected loss for insurance. Hence there is an opportunity for mutually advantageous insurance policy.

- ***Risk averse*** if and only if $u''(w) < 0$.

- General utility function for the **insurer**,

$$u_I(w_I) = \mathrm{E}[u_I(w_I + H - X)] \le u_I(w_I + H - \mu) \Rightarrow H \ge \mu.$$

H, collecting premium. X, paying random loss.

- A utility function is based on the decision maker's preferences for various distribution outcomes. An insurer need not be an individual. It may be a partnership, corporation or government agency.

Exponential utility function $u(w) = -e^{-\alpha w}$ for all w and for a fixed $\alpha > 0$.

- Risk-averse

$$u'(w) = \alpha e^{-\alpha w} > 0 \quad \text{and} \quad u''(w) = -\alpha^2 e^{-\alpha w} < 0.$$

-
$$\mathrm{E}[-e^{-\alpha X}] = -\mathrm{E}[e^{-\alpha X}] = -M_X(-\alpha).$$

The same as finding the moment generating function(m.g.f.) of X, $M_X(t) = \mathrm{E}[e^{tX}]$.

- Insurance premiums do not depend on the wealth of the decision maker. For the insured

$$-e^{-\alpha(w-G)} = \mathrm{E}[-e^{-\alpha(w-X)}] \Rightarrow G = \frac{\log M_X(\alpha)}{\alpha}.$$

For the insurer

$$-e^{-\alpha_I w_I} = \mathrm{E}[-e^{-\alpha_I(w_I+H-X)}] \Rightarrow H = \frac{\log M_X(\alpha_I)}{\alpha_I}.$$

Example 1. A decision maker's utility function is given by $u(w) = -e^{-5w}$. The decision maker has two random economic prospects (gains) available. The outcome of the first, denoted by X, has a normal distribution with mean 5 and variance 2. Henceforth, a statement about a normal distribution with mean μ and variance σ^2 will be abbreviated as $\mathcal{N}(\mu, \sigma^2)$. The second prospect denoted by Y, is distributed as $\mathcal{N}(6, 2.5)$. What prospect will be preferred ?

Solution:

$$\mathrm{E}[u(X)] = \mathrm{E}[-e^{-5X}] = -M_X(-5) = -e^{[-5(5)+(5^2)(2)/2]} = -1,$$

$$\mathrm{E}[u(Y)] = \mathrm{E}[-e^{-5Y}] = -M_Y(-5) = -e^{[-5(6)+(5^2)(2.5)/2]} = -e^{-1.25}.$$

Therefore,

$$\mathrm{E}[\mu(X)] = -1 > \mathrm{E}[\mu(Y)] = -e^{-1.25},$$

and the distribution of X is preferred to the distribution of Y.

Fractional power utility function

$$u(w) = w^{\gamma}, \quad w > 0, 0 < \gamma < 1.$$

Risk-averse preference

$$u'(w) = \gamma w^{\gamma-1} > 0, \quad \text{and} \quad u''(w) = \gamma(\gamma - 1)w^{\gamma-2} < 0.$$

Example 2 A decision maker's utility function is given by $u(w) = \sqrt{w}$. The decision maker has wealth of $w = 10$ and faces a random loss X with a uniform distribution on $(0, 10)$. what is the maximum amount this decision maker will pay for complete insurance against the random loss ?

Solution:
$$\sqrt{10 - G} = \mathrm{E}[\sqrt{10 - G}] = \int_0^{10} \sqrt{10 - x}\, 10^{-1} dx = \frac{2}{3}\sqrt{10},$$

$$\Rightarrow G = 5.5556 > \mathrm{E}[X] = 5.$$

Quadratic utility function

$$u(w) = w - \alpha w^2, \quad w < (2\alpha)^{-1}, \quad \alpha > 0.$$

Risk-averse preference

$$u'(w) = 1 - 2\alpha w > 0 \quad \text{and} \quad u''(w) = -2\alpha.$$

Example 3 A decision maker's utility of wealth function is given by

$$u(w) = w - 0.01w^2, \; w < 50.$$

The decision maker will retain wealth of amount w with probability p and suffer a financial loss of amount c with probability $1 - p$. For the values of w, c and p exhibited in the table below, find the maximum insurance premium that the decision maker will pay for complete insurance. Assume $c \leq w < 50$.

Solution: For the facts stated, we have

$$u(W - G) = pu(w) + (1 - p)u(w - c),$$

$$(w - G) - 0.01(w - G)^2 = p(w - 0.01w^2) + (1 - p)[(w - c) - 0.01(w - c)^2].$$

For given value of w, p and c this expression becomes a quadratic equation. Two solution are shown.

Wealth	Loss	Probability	Insurance Premium
w	c	p	G
10	10	0.5	5.28
20	10	0.5	5.37

- A maximum insurance premium increasing with wealth is a property of quadratic utility functions.

- The premium G for an exponential utility function does not depend on w, the amount of wealth. of

Example 4 The probability that property will not be damaged in the next period is 0.75. The probability density function (p.d.f) of a positive loss is given by $f(x) = 0.25(0.01e^{-0.01x}), x > 0$. The owner of the property has a utility function given by $u(w) = -e^{-0.005w}$. Calculate the expected loss and the maximum insurance premium the property owner will pay for complete insurance.

Solution: The expected loss is given by

$$\mathrm{E}[X] = 0.75(0) + 0.25 \int_0^\infty x(0.01e^{-0.01x})dx = 25.$$

Then we could determine the maximum premium that the owner will pay for complete insurance.

$$u(w - G) = 0.75u(w) + 0.25 \int_0^\infty u(w - x)f(x)dx,$$

$$-e^{0.05(w-G)} = -0.75e^{-0.05w} - 0.25 \int_0^\infty e^{-0.005(w-x)}(0.01e^{-0.01x})dx,$$

$$e^{0.005G} = 0.75 + (0.25)(2) = 1.25 \Rightarrow G = 200\log 1.25 = 44.63.$$

Therefore, in accord with the property owner's preferences, he will pay up to $44.63 - 25 = 19.63$ in excess of the expected loss to purchase insurance covering all losses in the next period.

Example 5 The property owner in Example 4 is offered an insurance policy that will pay $1/2$ of any loss during the next period. The expected value of the partial loss payment is $\mathrm{E}[X/2] = 12.5$. Calculate the maximum premium that the property owner will pay for this insurance.

Solution: Consistent with his attitude toward risk, as summarized in his utility function, the premium is determined from

$$0.75u(w-G)+\int_0^\infty u\left(w-G-\frac{x}{2}\right)f(x)dx = 0.75u(w)+\int_0^\infty u(w-x)f(x)dx.$$

For the exponential utility function and p.d.f of losses specified in Example 4. It can be shown $G = 28.62$. The property owner is willing to pay up to $G - \mu = 28.62 - 12.50 = 16.12$ more than the expected partial loss for the partial insurance coverage.

1.3. Elements of Insurance

- An insurance system can be organized only after the identification of a class of situations where random losses may occur.

- Because most insurance systems operate under dynamic conditions, it is important that a plan exist for collecting and analyzing insurance operating data so that the insurance system can adapt.

- Deviations should exhibit no pattern that might be exploited by the insured or insurer to produce consistent gains.
  The classification of risk into homogeneous groups is an important function within a market-based insurance system. Experience deviations that are random indicate efficiency or equity in classification.

- For insurance systems organized to serve groups rather than individuals, the issue is no longer whether deviations insurance experience are random for each individual. Group insurance plans are based on a collective decision on whether the system increases the total welfare of the group.

1.4. Individual Risk Models for A Short Term

Individual Risk Model

$$S = X_1 + X_2 + \cdots + X_n$$

where S is the random loss of a segment of an insuring organization's risk, and X_i is the loss on insured unit i and n is the number of risk unit insured.

- X_i's are postulated to be independent random variables.

- This model does not recognize the time value of money.

- Here we just consider only closed models, that is the number of insured unit n is known and fixed at the beginning of the period.

Basic concept with a life insurance product

In a one-year term life insurance

- Insurer pay amount b if the insured dies with in a year of policy issue, and nothing to pay if the insured survives the year.

- q, the probability of a claim during the year.

- X, the claim random variable. Its p.f is

$$f_X(x) = \Pr(X = x) = \begin{cases} 1 - q, & x = 0 \\ q, & x = b \\ 0, & \text{elsewhere,} \end{cases}$$

and the d.f. is

$$F_X(x) = \Pr(X \le x) = \begin{cases} 0, & x < 0 \\ 1 - q, & 0 \le x < b \\ 1, & x \ge b \end{cases}$$

$$\mathrm{E}[X] = bq, \quad \mathrm{E}[X^2] = b^2q, \quad \mathrm{Var}(X) = b^2q(1-q).$$

Writing

$$X = Ib$$

where $\Pr(I = 0) = 1 - q$ and $\Pr(I = 1) = q$.

$$\mathrm{E}(I) = q, \quad \mathrm{Var}(I) = q(1-q) \Longrightarrow$$

$$\mathrm{E}(X) = b\mathrm{E}(I) = bq, \quad \text{and} \quad \mathrm{Var}(X) = q^2\mathrm{Var}(I) = b^2q(1-q)$$

$$X = IB$$

where X is the claim random variable for the period, B gives the total claim amount incurred during the period, and I is the indicator for the event that at least one claim has occurred.

Example Consider 1-year term life insurance paying an extra benefit in case of accidental death. If the death is accidental, the benefit amount is 50,000. For other causes of death, the benefit amount is 25,000. Assume that for the age, health, and occupation of a specific individual, the probability of an accidental death within the year is 0.0005, while the probability of nonaccidental death is 0.0020. More specifically

$$\Pr(I = 1, B = 50,000) = 0.0005 \quad \text{and} \quad \Pr(I = 1, B = 25,000) = 0.0020.$$

Example Consider an automobile insurance providing collision coverage (this indemnifies the owner for collision damage to his car) above a 250 deductible up to maximum claim of 2,000. Assuming that for a particular individual the probability of one claim in a period is 0.15 and the chance of more than one claim is 0:

$$\Pr(I = 0) = 0.85, \quad \text{and} \quad \Pr(I = 1) = 0.15$$

Furthermore, assume that the claim amount is between 0 and 2,000 can be modeled by a continuous distribution with a p.d.f proportional to $1 - x/2,000$ for $0 < x < 2000$ and the probability mass at the maximum claim size of 2,000 is 0.1.

$$\Pr(B \le x | I = 1) = \begin{cases} 0, & x \le 0 \\ 0.9 \left[1 - \left(1 - \frac{x}{2,000} \right)^2 \right], & 0 < x < 2000 \\ 1, x \ge 2,000 \end{cases}$$

Some formulas relating to the moments of random variables to certain conditional expectations

$$\mathrm{E}[W] = \mathrm{E}[\mathrm{E}[W|V]]$$

$$\mathrm{Var}(W) = \mathrm{Var}(\mathrm{E}[W|V]) + \mathrm{E}[\mathrm{Var}(W|V)]$$

Assume $X = IB$

$$\mathrm{E}[X] = \mathrm{E}[\mathrm{E}[X|I]]$$

$$\mathrm{Var}(X) = \mathrm{Var}(\mathrm{E}[X|I]) + \mathrm{E}[\mathrm{Var}(X|I)]$$

Assume $\mathrm{E}[X|I=1] = \mu$, $\mathrm{E}[X|I=0] = 0$ $\mathrm{Var}(B|I=1) = \sigma^2$, $\mathrm{Pr}(I = 1) = q$, **and** $\mathrm{Pr}(I = 0) = 1 - q$.

$$\mathrm{E}[X] = \mathrm{E}[\mathrm{E}[X|I]] = \mu\mathrm{E}[I] = \mu q$$

$$\mathrm{Var}(X) = \mathrm{Var}(\mathrm{E}[X|I]) + \mathrm{E}[\mathrm{Var}(X|I)] = u^2 q(1-q) + \sigma^2 q.$$

Sums of Independent Random Variables

First, consider the sum of two random variables

$$S = X + Y$$

For two discrete, non-negative random variables,

$$F_s(s) = \sum_{\text{all } y \leq s} \mathsf{Pr}(X + Y \leq s | Y = y)\mathsf{Pr}(Y = y)$$

$$= \sum_{\text{all } y \leq s} \mathsf{Pr}(X \leq s - y | Y = y)\mathsf{Pr}(Y = y)$$

When X and Y and independent

$$F_s(s) = \sum_{\text{all } y \leq s} F_X(s - y) f_Y(y)$$

and the p.f. corresponding to this d.f.

$$f_s(s) = \sum_{\text{all } y \leq s} f_X(s - y) f_Y(y).$$

For continuous, non-negative random variables, the formulas corresponding to the formulas above are

$$F_s(s) = \int_0^s \Pr(X \le s - y | Y = y) f_Y(y) dy.$$

$$F_s(s) = \int_0^s F_X(s - Y) f_Y(y) dy.$$

$$f_s(s) = \int_0^s f_X(s - y) f_Y(y) dy.$$

- **Convolution** $F_X * F_Y$

- **Sum of** $S = X_1 + X_2 + \cdots + X_n$,
  Let $F^{(k)}$ is the d.f. of $X_1 + X_2 + \cdots + X_k$

$$F^{(2)} = F_2 * F^{(1)} = F_2 * F_1$$
$$F^{(3)} = F_3 * F^{(3)} = F_3 * (F_2 * F_1)$$
$$\vdots$$
$$F^{(3)} = F_n * F^{(n-1)}$$

Example 4.1 Let X have a uniform distribution on $(0,2)$ and let Y be independent of X with a uniform distribution over $(0,3)$ Determine the d.f. of $S = X + Y$.

Example 4.2 Consider three independent random variables X_1, X_2, X_3. For $i = 1, 2, 3$, X_i has an exponential distribution and $\mathrm{E}[X_i] = 1/i$. Derive the p.d.f of $S = X_1 + X_2 + X_3$ by the convolution process.

Moment generating function (m.g.f.) $M_X(t) = \mathrm{E}[e^{tX}]$.
$S = X_1 + X_2 + \cdots + X_n$,

$$M_S(t) = \mathrm{E}[e^{tS}] = \mathrm{E}[\exp(tX_1 + tX_2 + \cdots + tX_n)] = \mathrm{E}[e^{tX_1}e^{tX_2}\cdots e^{tX_n}].$$

If $X_1, X_2, \ldots, X_n$ are independent, then

$$M_S(t) = \mathrm{E}[e^{tX_1}]\mathrm{E}[e^{tX_2}]\cdots \mathrm{E}[e^{tX_n}] = M_{X_1}(t)M_{X_2}(t)\cdots M_{X_n}(t).$$

Example 4.3 Consider the random variables of Example 4.2. Derive the p.d.f of $S = X_1 + X_2 + X_3$ by recognition of the m.g.f of S.

Approximations for the distribution of the sum

Central Limit Theorem

For a sequence of independent and identically distribution variables $X_1, X_2, \ldots$, with $\mathrm{E}[X_i] = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$. For each n, the distribution of $\sqrt{n}(\bar{X}_n - \mu)/\sigma$, where $\bar{X}_n = (X_1 + X_2 + \cdots + X_n)/n$, has mean 0 and variance 1.

When $X_i, i = 1, \ldots, n$ are **not identically distribution**, a normal approximation to the distribution of

$$S = X_1 + X_2 + \ldots + X_n$$

where $X_i, i = 1, \ldots, n$ are independent. Compute

$$\mathrm{E}[S] = \sum_{k=1}^{n} \mathrm{E}[X_k], \mathrm{Var}(S) = \sum_{k=1}^{n} \mathrm{Var}(X_k)$$

Then apply the normal approximation.

Applications to Insurance

Example 4.4 A life insurance company issue 1-year term life contract for benefit amount of 1 and 2 units to individuals with probabilities of death 0.02 or 0.10. The following table gives the number of individuals n_k in each of the four classes created by a benefit amount b_k and a probability of claim q_k.

k	q_k	b_k	n_k
1	0.02	1	500
2	0.02	2	500
3	0.10	1	300
4	0.10	2	500

The company want to collect, from this population of 1,800 individuals, an amount equal to the 95th percentile of the distribution of total claims. Moreover, it wants each individual's share of this amount to be proportional to that individual's expected claim. The share for individual j with mean $\mathrm{E}[X_j]$ wound be $(1 + \theta)\mathrm{E}[X_i]$. The 95th percentile requirement suggest that $\theta > 0$. This extra amount, $\theta\mathrm{E}[X_j]$, is the *security loading* and θ is the *relative security loading*. Calculate θ.

Example 4.5 The policyholders of an automobile insurance company fall into two classes

Class	Number in Class	Claim Probability	Distribution of Claim amount, B_k, Parameters of Truncated Exponential	
k	n_k	q_k	λ	L
1	500	0.10	1	2.5
2	2000	0.05	2	5.0

A truncated exponential distribution is defined by the d.f.

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & 0 \leq x < L \\ 1, & x \geq L. \end{cases}$$

This is a mixed distribution with p.d.f $f(x) = \lambda e^{-\lambda x}, 0 < x < L$, and a probability mass $e^{-\lambda L}$ at L.

Again, the probability that total claim exceeds the amount collected from policyholder is 0.05. We assume that the relative security load, θ, is the same for the two classes. Calculate θ.