SOLVING PARTIAL DIFFERENTIAL EQUATIONS ON SURFACES WITH FUNDAMENTAL SOLUTIONS

MENG CHEN*, KA CHUN CHEUNG†, AND LEEVAN LING‡

Abstract. The aim of this paper is to present partial differential equations (PDEs) on surface to the community of methods of fundamental solutions (MFS). First, we present an embedding formulation to *embed* surface PDEs into a domain so that MFS can be applied after the PDEs is homogenized with a particular solution. Next, we discuss how the domain-MFS method can be used to directly collocate surface PDEs. Some numerical demonstrations were included to study the effect of basis functions and source point locations.

1. Partial differential equations on surfaces. In this paper, we focus on second-order elliptic partial differential equations (PDEs) posed on some sufficiently smooth, connected, and compact surface $\mathcal{S} \subset \mathbb{R}^d$ with bounded geometry. Without loss of generality, we assume $\dim(\mathcal{S}) = d-1$, a.k.a., $\mathcal{S}$ has co-dimension 1. We denote the unit outward normal vector at $\boldsymbol{x} \in \mathcal{S}$ as $\mathbf{n} = \mathbf{n}(\boldsymbol{x})$ and the corresponding projection matrix to the tangent space of $\mathcal{S}$ at $\boldsymbol{x}$ as

$$\mathcal{P}(\boldsymbol{x}) = [\vec{\mathcal{P}}_1, \ldots, \vec{\mathcal{P}}_d](\boldsymbol{x}) := I_d - \mathbf{n}\mathbf{n}^T \in \mathbb{R}^{d\times d}, \tag{1.1}$$

where I_d is the $d \times d$ identity matrix.

Example 1. *Let $\mathcal{S}$ be the unit circle in $\mathbb{R}^2$. Then, we have $\mathbf{n} = (x, y)^T = \boldsymbol{x}$ for every $\boldsymbol{x} \in \mathcal{S}$ and the projection matrix is*

$$\mathcal{P}(\boldsymbol{x}) = \begin{pmatrix} 1-x^2 & -xy \\ -xy & 1-y^2 \end{pmatrix} = \begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix}.$$

Unlike standard PDEs posed in some bounded domains with flat geometry, curvatures of our computational domain $\mathcal{S}$ plays key roles in solution behaviours of the PDEs. The surface gradient $\nabla_{\mathcal{S}}$ can be defined in terms of the standard Euclidean gradient ∇ for $\mathbb{R}^d$ via projection $\mathcal{P}$ as

$$\nabla_{\mathcal{S}} := \mathcal{P}\nabla \tag{1.2}$$

and the Laplace-Beltrami $\Delta_{\mathcal{S}}$ operators (a.k.a. the surface Laplacian) can then be

*ASTRI Hong Kong Applied Science and Technology Research Institute Company Limited
†NVIDIA AI Technology Center (NVAITC), NVIDIA, USA
‡Department of Mathematics, Hong Kong Baptist University, Kowloon Tong, Hong Kong.

defined using the surface gradient operator by

$$\Delta_{\mathcal{S}} := \nabla_{\mathcal{S}} \bullet \nabla_{\mathcal{S}}. \tag{1.3}$$

Surface gradient is straightforward to compute, whereas the analytic expression of surface Laplacian involves derivatives of normal vector $\mathbf{n}$. In [4], one can find the following formula: for any sufficient smooth function $u_{\mathcal{S}} : \mathcal{S} \to \mathbb{R}$, we have

$$\Delta_{\mathcal{S}} u_{\mathcal{S}} = \sum_{i=1}^{d} \Big(\operatorname{trace}(\mathcal{P} \cdot J(\vec{\mathcal{P}}_i)^T) \frac{\partial u_{\mathcal{S}}}{\partial x_i} + \mathcal{P}_{ii} \frac{\partial^2 u_{\mathcal{S}}}{\partial x_i^2} \Big) + \sum_{\substack{i,j=1 \\ i \neq j}}^{d} \mathcal{P}_{ij} \frac{\partial^2 u_{\mathcal{S}}}{\partial x_i \partial x_j}, \tag{1.4}$$

where J denotes the Jacobian operator, $\vec{\mathcal{P}}_i$ and $\mathcal{P}_{ij}$ denotes the i-th column and ij-th entries of the projection matrix $\mathcal{P}$ in (1.1) respectively.

Example 2. *Let $\mathcal{S}$ be the unit circle in $\mathbb{R}^2$. The surface gradient operator takes the form*

$$\nabla_{\mathcal{S}} = \begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} y^2 \partial_x - xy \partial_y \\ -xy \partial_x + x^2 \partial_y \end{pmatrix}.$$

We can also compute the coefficients of surface Laplacian with simplifications based on the parametric equation $x^2 + y^2 = 1$ as follows

$$\begin{aligned} \operatorname{trace}(\mathcal{P} \cdot J(\vec{\mathcal{P}}_1)^T) &= -xy^2 - x^3 = -x, \\ \operatorname{trace}(\mathcal{P} \cdot J(\vec{\mathcal{P}}_2)^T) &= -x^2 y - y^3 = -y. \end{aligned}$$

Together, we have $\Delta_{\mathcal{S}} = y^2 \dfrac{\partial^2}{\partial x^2} - 2xy \dfrac{\partial^2}{\partial x \partial y} + x^2 \dfrac{\partial^2}{\partial y^2} - x \dfrac{\partial}{\partial x} - y \dfrac{\partial}{\partial y}$ on the unit circle.

Any linear second-order elliptic differential operators on surfaces $\mathcal{S} \subset \mathbb{R}^d$ can be expressed in the form of

$$\mathcal{L}_{\mathcal{S}} := a_{\mathcal{S}} \Delta_{\mathcal{S}} + \mathbf{b}_{\mathcal{S}} \bullet \nabla_{\mathcal{S}} + c_{\mathcal{S}}, \tag{1.5}$$

where $a_{\mathcal{S}}, c_{\mathcal{S}} : \mathcal{S} \to \mathbb{R}$, and $\mathbf{b}_{\mathcal{S}} : \mathcal{S} \to \mathbb{R}^d$ are bounded coefficients that can be functions of $\boldsymbol{x} \in \mathcal{S}$. The elliptic surface PDEs is then defined as

$$\mathcal{L}_{\mathcal{S}} u_{\mathcal{S}} = f_{\mathcal{S}}, \tag{1.6}$$

for some sufficiently smooth right hand function $f_{\mathcal{S}}$, and, because $\mathcal{S}$ is closed, without any other conditions. Under some standard smoothness assumptions [14], we know (1.6) has classical solutions $u_{\mathcal{S}}^* : \mathcal{S} \to \mathbb{R}$.

2. Traditional MFS approach. Since the surface PDE (1.6) is inhomogeneous, the traditional approach is to find a particular solution in order to obtain a homogeneous PDE. However, finding *the* particular solution to (1.6) is equivalent to solving the uniquely solvable PDE; see [4] for some intrinsic algorithms. This leaves no room for applying MFS.

We now present an embedding framework to solve (1.6) by dual reciprocity method (DRM) [2,9] and MFS. Our approach is based on the closest point method [13]. Let the *closest point map* be defined as

$$\mathrm{cp}(\mathbf{x}) = \arg\inf_{\boldsymbol{x}\in\mathcal{S}} \|\boldsymbol{x} - \mathbf{x}\|_{\ell^2(\mathbb{R}^d)}.$$

The differentiability of the closest point map is directly related to the smoothness of $\mathcal{S}$, see [5]. For smooth $\mathcal{S}$, there exits $\varepsilon > 0$ depending on the curvature of $\mathcal{S}$ so that the closest point map is well-defined in the *narrow band domain*

$$\Omega := \left\{\mathbf{x} \in \mathbb{R}^d : \inf_{\boldsymbol{s}\in\mathcal{S}} \|\boldsymbol{s} - \mathbf{x}\|_{\ell^2(\mathbb{R}^d)} < \varepsilon\right\} \subset \mathbb{R}^d.$$

The first aim is to embed (1.6) to another PDE in Ω. To do so, we define the constant-along-normal extension operator $\mathcal{E}_{\mathbf{n}} := \mathcal{E}_{\mathbf{n},\mathcal{S}\to\Omega}$ so that, for every function $w_{\mathcal{S}} : \mathcal{S} \to \mathbb{R}$, its extension has the property that

$$w_\Omega(\mathbf{x}) := (\mathcal{E}_{\mathbf{n}} w_{\mathcal{S}})(\mathbf{x}) = w_{\mathcal{S}}(\mathrm{cp}(\mathbf{x})) \quad \text{for all } \mathbf{x} \in \Omega. \tag{2.1}$$

Now we consider an *embedding PDE* to (1.5)-(1.6)

$$\mathcal{L}_\Omega u_\Omega = f_\Omega \quad \text{with} \quad \mathcal{L}_\Omega := a_\Omega \Delta + \mathbf{b}_\Omega \bullet \nabla + c_\Omega, \tag{2.2}$$

where $f_\Omega = \mathcal{E}_{\mathbf{n}} f_{\mathcal{S}}$, $a_\Omega = \mathcal{E}_{\mathbf{n}} a_{\mathcal{S}}$, $c_\Omega = \mathcal{E}_{\mathbf{n}} c_{\mathcal{S}}$, and $\mathbf{b}_\Omega = \mathcal{E}_{\mathbf{n}} \mathbf{b}_{\mathcal{S}}$ with componentwise extension. The governing equation (2.2) alone is not yet well-posed and we need the following *embedding conditions*

$$\partial_{\mathbf{n}} u_\Omega = 0 \quad \text{and} \quad \partial^{(2)}_{\mathbf{n}} u_\Omega := \mathbf{n}^T J(\nabla u_\Omega)\mathbf{n} = 0 \quad \text{on } \mathcal{S}, \tag{2.3}$$

to ensure unique solution. The connection between surface PDE (1.5)-(1.6) and embedding PDE (2.2)-(2.3) is that the restriction of the embedding PDE solution u_Ω to $\mathcal{S}$ coincides with the surface PDE solution $u_{\mathcal{S}}$. This is an immediate consequence of the following equalities [7]:

$$\nabla_{\mathcal{S}} u_\Omega := \nabla u_\Omega - \mathbf{n}\partial_{\mathbf{n}} u_\Omega \quad \text{and} \quad \Delta_{\mathcal{S}} u_\Omega := \Delta u_\Omega - H_{\mathcal{S}} \partial_{\mathbf{n}} u_\Omega - \partial^{(2)}_{\mathbf{n}} u_\Omega \quad \text{on } \mathcal{S},$$

where $H_{\mathcal{S}}(\boldsymbol{x}) = \mathrm{trace}\left(J(\mathbf{n})(I - \mathbf{n}\mathbf{n}^T)\right)$. In the same article, readers can find Kansa-

type algorithms for solving (2.2)-(2.3) directly.

At this point, we have (2.2)-(2.3) posed in the narrow band domain (2.1), which is analogue to the standard elliptic PDEs and is DRM-MFS ready. We can decompose the embedding PDE into

$$(I)\ \mathcal{L}_\Omega u_p = f_\Omega, \quad \text{and} \qquad (II)\ \begin{cases} \mathcal{L}_\Omega u_h = 0 & \text{in } \Omega, \\ \partial_{\mathbf{n}} u_h = -\partial_{\mathbf{n}} u_p & \text{on } \mathcal{S}, \\ \partial^{(2)}_{\mathbf{n}} u_h = -\partial^{(2)}_{\mathbf{n}} u_p & \text{on } \mathcal{S}. \end{cases}$$

Solving the above by some appropriate means, e.g., (I) by DRM and (II) by MFS, yields the embedding PDE solution $u_\Omega = u_h + u_p$, whose restriction is the surface PDE solution $u_\mathcal{S} = u_{\Omega|\mathcal{S}}$. An Dirichlet-type alternative to problem (II) is

$$(II')\ \begin{cases} \mathcal{L}_\Omega u_h = 0 & \text{in } \Omega, \\ u_h - u_h \circ \mathrm{cp} = u_p \circ \mathrm{cp} - u_p & \text{on } \partial\Omega, \end{cases}$$

in which we require $u_\Omega = u_\Omega \circ \mathrm{cp}$ on the whole boundaries $\partial\Omega$. When the width ε of the narrow band domain is small, we can easily see that (II') is a finite difference approximations to the embedding conditions (2.3). This idea is essentially the orthogonal gradient method [12] and help avoiding differentiation to u_p.

2.1. DRM for solving (I). Although problem (I) is posed in Ω, our interest is only on $\mathcal{S}$ and, thus, it is sufficient to have interpolation conditions solely on $\mathcal{S}$ in DRM. Suppose we use n_I interpolation points $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_{n_I}\}$ on $\mathcal{S}$ to solve (I) by DRM . The DRM requires two couple sets of basis functions $(\{\psi_k\}_{k=1}^{n_I}, \{\varphi_k\}_{k=1}^{n_I})$ that satisfy

$$\mathcal{L}_\Omega \psi_k = \varphi_k, \quad 1 \le k \le n_I.$$

One can find some closed form formulas of analytic particular solution ψ in [3] of commonly used RBFs φ for some standard elliptic operators $\mathcal{L}_\Omega$. In cases of RBF symmetric interpolation, we want to center RBFs at the same set of points, i.e., $\varphi_k = \varphi(\|\cdot - \mathbf{x}_k\|)$, and solve

$$\sum_{k=1}^{n_I} a_k\, \varphi(\|\mathbf{x}_i - \mathbf{x}_k\|) = f(\mathbf{x}_i), \quad 1 \le i \le n_I.$$

Once the coefficients $\vec{a} = \{a_k\}_{k=1}^{n_I}$ were obtained, the approximate particular solution is given as

$$U_p = \sum_{k=1}^{n_I} a_k\, \psi_k(\cdot) = \sum_{k=1}^{n_I} a_k\, \psi(\|\cdot - \mathbf{x}_k\|).$$

The computation cost here is dominated by the cost of solving an $n_I \times n_I$ symmetric matrix system.

2.2. MFS for solving (II). When apply MFS to (II), all collocation points for the two embedding conditions are on $\mathcal{S}$ and there is real role for Ω. Let $Y = \{\mathbf{y}_1, \ldots, \mathbf{y}_{n_{II}/2}\}$ be a set of $n_{II}/2$ points on $\mathcal{S}$. Going after a square system, one can simply put n_{II} source points $Z = \{\mathbf{z}_1, \ldots, \mathbf{z}_{n_{II}}\}$ in $\partial\Omega$ in order to avoid singularity. Let G denote the fundamental solution of $\mathcal{L}_\Omega$ such that

$$\mathcal{L}_\Omega G(\|\cdot - \mathbf{z}\|) = \delta(\|\cdot - \mathbf{z}\|), \quad \mathbf{z} \in \mathbb{R}^d.$$

The standard approach is to numerical expand the approximate homogenous solution by

$$U_h = \sum_{j=1}^{n_{II}} \lambda_j \, G(\|\cdot - \mathbf{z}_j\|), \tag{2.4}$$

and solve the following collocation matrix system for $\vec{\lambda} = \{\lambda_j\}_{j=1}^{n_{II}}$:

$$\begin{pmatrix} \mathbf{n}^T \nabla G(\|\mathbf{y}_i - \mathbf{z}_j\|) \\ \mathbf{n}^T J(\nabla G(\|\mathbf{y}_i - \mathbf{z}_j\|))\mathbf{n} \end{pmatrix} \vec{\lambda} = \begin{pmatrix} -\mathbf{n}^T \nabla \psi(\|\mathbf{y}_i - \mathbf{x}_k\|) \\ -\mathbf{n}^T J(\nabla \psi(\|\mathbf{y}_i - \mathbf{x}_k\|))\mathbf{n} \end{pmatrix} \vec{a},$$

for $1 \le i \le n_{II}/2$, $1 \le j \le n_{II}$, and $1 \le k \le n_I$. Note that all differential operators here act upon the variable $\mathbf{y}$ and the right hand vector is computed based on the approximate particular solution U_p. The computation cost here is dominated by the cost of solving an $n_{II} \times n_{II}$ asymmetric matrix system.

2.3. MFS for solving (II'). When MFS is applied to solve (II'), four layers of points are required. First, we have to distribute collocations points on $\partial\Omega$. Here, we want ε as small as possible to minimize the finite difference error. Their respective closest points have to be identified so that we can evaluate $u_h \circ \mathrm{cp}$, i.e., fundamental solutions, and $u_p \circ \mathrm{cp}$ on $\mathcal{S}$.

A simpler approach is to put a set of $n_{II}/2$ points $Y' = \{\mathbf{y}'_1, \ldots, \mathbf{y}'_{n_{II}/2}\}$ on $\mathcal{S}$. Then, we extend Y' by $\pm\varepsilon\mathbf{n}$ as in the orthogonal gradient method to generate collocation points on $\partial\Omega$, i.e., the set of n_{II} collocation points is given by

$$Y = \{\mathbf{y}'_1 \pm \varepsilon\mathbf{n}, \ldots, \mathbf{y}'_{n_{II}/2} \pm \varepsilon\mathbf{n}\}.$$

By construction, if we operate on sets, we have $\mathrm{cp}(Y) = Y'$ and no closest point search is required. For convenience, let $Y'' =< Y', Y' >$ be a set of n_{II} entries that is ordered in such a way that $cp(\mathbf{y}_j) = \mathbf{y}''_j$ for all collocation points $\mathbf{y}_j \in Y$.

In terms of source points placement, there is now a requirement on the fictitious boundary imposed from the narrow band domain, i.e., our choice of ε. Away from Ω, we have to place the inner and outer source points to complete the set up of MFS. One can, of course, repeat the same data points extension technique to complete the job. When $\mathcal{S}$ is nonconvex, however, feasible choice of extension may be limited.

For now, we assume the set of source points Z is fixed by some appropriate means. The collocation system to be solved is in the form of

$$\big(G(\|\mathbf{y}_i - \mathbf{z}_j\|) - G(\|\mathbf{y}_i^{''} - \mathbf{z}_j\|)\big)\vec{\lambda} = \big(\varphi(\|\mathbf{y}_i^{''} - \mathbf{x}_k\|) - \varphi(\|\mathbf{y}_i - \mathbf{x}_k\|)\big)\vec{a}.$$

The approximate homogenous solution is given by (2.4) once we found $\vec{\lambda}$. The costs of (II) and (II') are similar, but (II') requires no numerical differentiations.

The above algorithms inherit all limitations in the traditional MFS approach that heavily relies on our knowledge of the fundamental solution to $\mathcal{L}_\Omega$. Method in the next section will circumvent the problem.

3. Using fundamental solutions for collocations. In this section, we focus on Kansa-type collocation method for solving surface PDEs in the form of (1.5)-(1.6). The idea is to analytically carry out some calculations, similar to our Example 2 above, so that we can collocate the surface operator directly. Our convergent analysis in [4] applies to kernels that satisfy

$$c_{\Phi_m}(1 + \|\omega\|_2^2)^{-m} \le \widehat{\Phi_m}(\omega) \le C_{\Phi_m}(1 + \|\omega\|_2^2)^{-m} \quad \text{for all } \omega \in \mathbb{R}^d, \tag{3.1}$$

for some constants $0 < c_{\Phi_m} \le C_{\Phi_m}$ and smoothness order $m > d/2$. Simply by restricting the global kernels Φ_m on $\mathcal{S}$, we have a surface kernel

$$\Psi_{m-1/2} := \Phi_{m|\mathcal{S}\times\mathcal{S}} : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$$

that reproduces $\mathcal{H}^{m-1/2}(\mathcal{S})$ provided the smoothness assumption on $\mathcal{S}$ stated in [8,11] are satisfied. Commonly used kernels in this class include Whittle-Matérn-Sobolev kernels $\Phi_m(\mathbf{x}) := \|\mathbf{x}\|_2^{m-d/2}\mathcal{K}_{m-d/2}(\|\mathbf{x}\|_2)$ and the family of compactly supported piecewise polynomial Wendland functions. The $m > d/2$ requirement is commonly seen in RBF theories. In both $\mathbb{R}^2$ and $\mathbb{R}^3$, we must take $m \ge 2$ if we insist on integer order smoothness. This means that $\mathcal{H}^{3/2}(\mathcal{S})$ is the largest solution space on which our theories applied. This is the motivation of using fundamental solutions as basis is a density result in [1]. In our notation, it reads as follows: let Υ_k be the fundamental solutions of the modified Helmholtz equation

$$-(\Delta - k^2)\Upsilon_k(\|\cdot - \mathbf{z}\|) = \delta(\|\cdot - \mathbf{z}\|), \quad \mathbf{z} \in \mathbb{R}^d,$$

and $\hat{\mathcal{S}} \supset \mathcal{S}$ be some sufficiently smooth artificial surface containing $\mathcal{S}$. Then,

$$\operatorname{span}\{\Upsilon_k(\|\cdot - \boldsymbol{s}\|)_{|\mathcal{S}} \ : \ \boldsymbol{s} \in \hat{\mathcal{S}}\}$$

is dense in $\mathcal{H}^{1/2}(\mathcal{S})$. This is beneficial for surface PDEs with low regularity, i.e., when there are near-singularity [6] on $\mathcal{S}$. In other words, we are considering a collocation method using basis functions

$$\Upsilon_k(r) := \begin{cases} \dfrac{1}{2\pi} K_0(kr) & \text{in 2D}, \\ \dfrac{1}{4\pi} \dfrac{\exp(-kr)}{r} & \text{in 3D}, \end{cases}$$

for some $k > 0$. As we do not require Υ_k to satisfy any governing equations, one can drop the constant for simplicity. The dimension here should match with that of the embedding space, i.e., 2D and 3D basis are for solving PDEs on curve and surface respectively. This collocation approach comes with a computational cost of solving an $n_Z \times n_Z$ asymmetric matrix system.

Example 3. *Consider modified Helmholtz $\mathcal{L}_{\mathcal{S}} u_{\mathcal{S}} := (\Delta_{\mathcal{S}} - I)u_{\mathcal{S}} = f_{\mathcal{S}}$ on the unit circle. Define sets of source points $Z \subset \Omega \setminus \mathcal{S}$ and collocation points $Y \subset \mathcal{S}$. By fixing some $k > 0$ and therefore a basis function Υ_k, we can obtain a collocation system*

$$\big(\mathcal{L}_{\mathcal{S}} \Upsilon_k(\|\mathbf{y}_i - \mathbf{z}_j\|)\big)\vec{\alpha} = f(\mathbf{y}_i), \quad \mathbf{y}_i \in Y, \, \mathbf{z}_j \in Z$$

for the unknown $\vec{\alpha} = \{\alpha_j\}_{j=1}^{n_Z}$. Based on the results in Example 2, we know that

$$\mathcal{L}_{\mathcal{S}} = y^2 \frac{\partial^2}{\partial x^2} - 2xy \frac{\partial^2}{\partial x \partial y} + x^2 \frac{\partial^2}{\partial y^2} - x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} - I, \quad (x, y) \in \mathcal{S},$$

which can be used to analytically evaluate $\mathcal{L}_{\mathcal{S}} \Upsilon_k(\|(x, y) - \cdot\|)$. The numerical solution is of the form

$$U_{\mathcal{S}} = \sum_{j=1}^{n_Z} \alpha_j \, \Upsilon_k(\|\cdot - \mathbf{z}_j\|).$$

Note that the selection of wavenumber k in Υ_k does not need to match with the wavenumber in $\mathcal{L}_{\mathcal{S}}$, which may not even be of Helmholtz in general.

Example 4. *We now consider a modified Helmholtz $\mathcal{L}_{\mathcal{S}} u_{\mathcal{S}} := (\Delta_{\mathcal{S}} - I)u_{\mathcal{S}} = f_{\mathcal{S}}$ on the unit sphere. In this example, we shall use the 3D basis function*

$$\Upsilon_k(r) = \frac{1}{4\pi} \frac{\exp(-kr)}{r}.$$

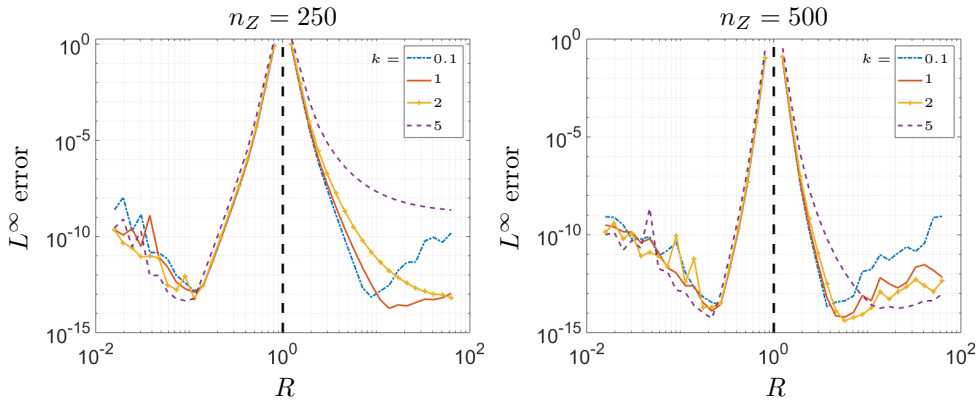

FIG. 3.1. *Maximum error against source points location R on one side of $\mathcal{S}$ of collocation methods using modified Helmholtz fundamental solution with various wavenumbers k.*

A smooth solution $u_{\mathcal{S}} = 10xyz + 5xy + z$ is considered; we aim to study the effects of wavenumber k and source points distribution. For $n_Z = \{250, 500\}$, the set of n_Z collocation points $Y \subset \mathcal{S}$ is quasi-uniformly distributed on the unit sphere. The set of source points Z is uniformly distributed on a sphere with radius $R > 0$. Taking advantage of the simple geometry, we simply generate Z by an extension of the set Y along normal direction by $Z = RY$. Figure 3.1 shows the resulting maximum error over a range of R for various values of k, i.e., basis functions. When $R = 10^0$, source points are right on the surface and, as expected, we see error blow up nearby. When $R < 10^0$, i.e., source points are placed inside the unit sphere, the selection of wavenumber k has nearly no effect on the accuracy and only the value of n_Z matters. Like the traditional MFS, n_Z also affect the location of the Goldilocks zone: $R \approx 0.1$ and $R \approx 0.2$ for $n_Z = 250$ and 500 respectively. The error profiles on the $R > 10^0$ sides is less consistent. In this example, we can see that large k has better conditioning but lower accuracy.

Example 5. *It is commonly seen that MFS source points are placed on both sides of annulus domains. We repeat Example 4 with source points distributed as*

$$Z = \{Rz'_1, (2-R)z'_1, \ldots, Rz'_{n_Z/2}, (2-R)z'_{n_Z/2}\} \quad 0 < R < 1$$

where $Z' = \{z'_1, \ldots, z'_{n_Z/2}\}$ is a set $n_Z/$ uniformly distributed data points on the unit sphere. The parameter R here still measures the source points to surface distance, but now we have source points both inside and outside of the unit sphere. The resulting effects is obvious, see Figure 3.2, in a bad way. The condition numbers increases dramatically and accuracy drop significantly by five orders. We did a quick fix by employing an adaptive basis selection algorithm [10]; it helps stabilize the numerical solutions for $R \lesssim 10^0$, i.e., when source points are close to $\mathcal{S}$. This example only means to show how the accuracy of this collocation method is sensitive to source point

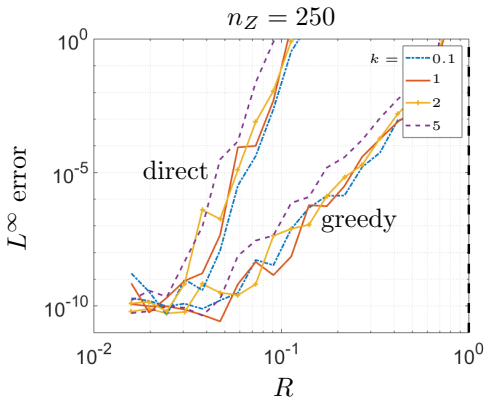

FIG. 3.2. *Maximum error against source points location R on both sides of $\mathcal{S}$ of collocation methods using modified Helmholtz fundamental solution with various wavenumbers k.*

locations. We do not attempt to search for the sweet spot that balances the inner/outer number of source points and their respectively optimal distances to surface for the best accuracy.

4. Conclusions. We present some methodologies for solving surface PDEs with fundamental solutions. Our discussion focus on elliptic types, which can be extended to time-dependent problems on surface by some time discretization schemes. This introductory paper only employs traditional MFS techniques for simplicity; we do not include any regularization, optimal data points place, etc. Yet, cutting edge developments in the MFS community are necessary in order to make MFS an attractive alternative for solving surface PDEs. In particular, singularity-free alternatives to MFS are attractive when dealing with surfaces with complex geometry. We invite our readers to explore further in this research topic.

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