Numerical simulations for space-time fractional diffusion equations

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Abstract

We consider the solutions of a space-time fractional diffusion equation on the standard interval $[-1, 1]$. The equation is obtained from the standard diffusion equation by replacing the second order space derivative by a Riemann-Liouville fractional derivative of order between one and two, and the first order time derivative by a Caputo fractional derivative of order between zero and one. As the fundamental solution of this fractional equation is unknown (if exist), an eigenfunction approach is applied to obtain approximate fundamental solutions which are then used to solve the space-time fractional diffusion equation with initial and boundary values. Numerical results are presented to demonstrate the effectiveness of the proposed method in long time simulations.

1 Introduction

In the waste management guidance, it is stated that any free-flowing solvent removed from the industrial wipers (i.e. shop towels, rags and disposable wipes used in commercial and industrial settings such as auto repair and printing shops) must be collected and managed as hazardous wastes. In reality, potentially hazardous solvents are widely used in many industrial processes that, if spilt or improperly treated, will seriously contaminate the environment. A variety of industrial users

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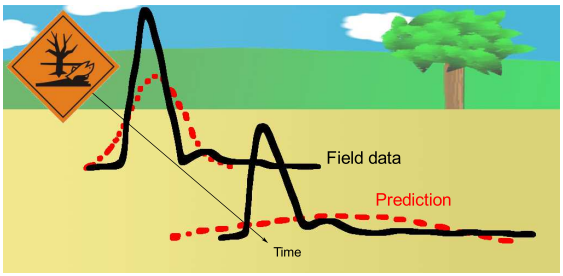


Figure 1: *The reality (solid) and prediction from conventional model (dotted) of pollutant spread.*

have sought guidance for environment safety level. Making such guidance would require the understanding of the spread of pollutants underground. When the wastes are disposed, underground pollution is then caused by the diffusion and the possible penetration by natural groundwater flow. Many highly industrialized sites and their nearby residential areas are therefore at risk of the leaking of solvents from factories, waste disposal, treatment areas, and dumping grounds. One of the most typical environmental problems to be considered is the estimation of large-scale and long term environmental pollution; see [1] for more detailed review.

Due to the heterogeneousity of soils (micro-scale $\sim 100\mu$m), field data from [2] first showed that water flow in aquifer is better modeled by anomalous diffusion instead of conventional diffusion. The analysis based on a 20-month natural gradient tracer study in the saturated zone of a highly heterogeneous aquifer reveals dramatically non-Gaussian behavior and asystematic mass loss in concentration distributions versus time and spatial moments, see Figure 1 for a schematic demonstration. A long-term goal is to take a step towards a better fundamental understanding and quantification of pollutants transported by underground water in non-saturated soils and develop suitable numerical methods for practical applications. Site assessment and monitoring of soils affected are often difficult and expensive. Understanding the transport processes of underground pollutants and the ability to correctly predict the concentration downstream (with limited noisy data from laboratory testing) are essential tools in environmental protection—the quantitative targets of soil purification in real site (marco-scale $\sim$100m–10km) can be set and the performance of pollutant-barrier can be determined.

It is nowadays well-known that the anomalous phenomena can be better simulated using the model of continuous-time random walk that leads to diffusion equations of fractional order as macroscopic model. In particular, we are interested in diffusion processes of contaminants in porous media; see [3] and the references therein. Fractional differential equations can also be derived for other anomalous processes in many applied fields. As for works on numerical methods for fractional diffusion equations, we can refer to the recent articles [4, 5, 6, 7, 8, 9, 10, 11, 12, 13]

and the references within. Most of these papers treat the case where the spatial dimension is one; see also [10] for a nonlinear fractional diffusion equation. Despite many references for the more-or-less completed ordinary equations of fractional order, theory of the partial differential equations of fractional order still has many gaps; this situation holds for both the direct and inverse problems. To our best knowledge, the first uniqueness of an inverse problem for 1D fractional diffusion equation is proven recently in [14].

In this paper, we are interested in developing novel numerical algorithm for solving the *Space-Time Fractional Diffusion Equations* (STFDE): for $0 < \alpha < 1 < \beta < 2$,

$$\begin{array}{rcll} {}_0^c\partial_t^\alpha u(x,t) & = & {}_{-1}\mathbf{D}_x^\beta u(x,t), & -1 \le x \le 1, \quad 0 < t, \\ u(x,0) & = & u_0(x), & -1 \le x \le 1, \\ u(\pm 1,t) & = & 0, & 0 < t, \end{array} \tag{1}$$

where ${}_0^c\partial_t^\alpha$ is the *Caputo fractional derivative* of order–α:

$${}_0^c\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\eta)}{\partial \eta} \frac{d\eta}{(t-\eta)^\alpha},$$

and ${}_{-1}\mathbf{D}_x^\beta$ is the *Riemann-Liouville fractional derivative* of order–β:

$${}_{-1}\mathbf{D}_x^\beta u(x,t) = \frac{1}{\Gamma(2-\beta)} \frac{\partial^2}{\partial x^2} \int_{-1}^x \frac{u(\xi,t)d\xi}{(x-\xi)^{(\beta-1)}}. \tag{2}$$

Existence and uniqueness results of a symmetric version of (1) can be found in [15]. When $\alpha \to 1$ and $\beta \to 2$, the STFDE (1) becomes to the well-known diffusion equation (Markovian process) in the limit, $\partial_t u(x,t) = \partial_{xx} u(x,t)$. In the cases $0 < \alpha < 1$, the current solution depends on its history (non-Markovian process); this is so–called the *memory-effect* of fractional derivatives. Whereas, setting $\alpha = 1$ and $\beta = 1$ yields an overspecified wave equation that has no solution in general.

The method of fundamental solutions (MFS) and its related methods have been developed for the numerical solutions of elliptic [16] and parabolic type differential equations and their corresponding inverse problems [17]. Detailed reviews of the method can be found in [18] and [19]. The MFS belongs to a general class of boundary collocation methods (see [20]) in which the unknown solution is represented as a linear combination of fundamental solutions with their source points located outside the computational domain [21]. The boundary conditions are satisfied by collocation or least squares fitting. Unlike traditional methods, the MFS does not involve integral evaluations and hence provides an efficient computational alternative for problems in higher dimension with irregular domains. If MFS can be used in solving the STFDE (1), the extra dimension in time will be canceled out.

The aim of this paper is to develop an approximate fundamental solution for (1). We focus our study in 2D which will already be useful for long-time simulations; generalization to higher dimensions is straightforward. After a brief introduction of STFPE, we will detail the procedure of constructing approximate fundamental solutions in Section 2 with the help of the Jacobi spectral-collocation method. In Section 3, an MFS-like expansion is derived and the STFDE can be solved by collocation methods as seen in the standard MFS approach. As exact solution to (1) is difficult to obtain analytically, we demonstrate some simulated results in Section 4. Consecutive differences are shown (instead of the exact error) to give evidences of convergence. Note that if an existing method can solve a STFDE from time $t = 0$ to $t = 1$, either the linear system or the memory requirement has to be enlarged by 100 times to obtain solution at $t = 100$. Long time simulations are shown to justify the ability of the proposed approach as above claimed.

2 Approximate fundamental solution

We begin by considering the following eigenfunction problem

$$\begin{aligned} {}_{-1}\mathbf{D}_x^{\beta}\varphi_k(x) &= \lambda_k\varphi_k(x), \qquad -1 < x < 1,\ 1 < \beta < 2, \\ \varphi_k(\pm 1) &= 0. \end{aligned} \tag{3}$$

The general solution to (3) is given by [22]

$$Ax^{\beta-1}E_{\beta,\beta}(\lambda_k(x+1)^{\beta}) + Bx^{\beta-2}E_{\beta,\beta-1}(\lambda_k(x+1)^{\beta}),$$

where the generalized two-parameter Mittag-Leffler function is given as

$$E_{\mu,\nu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k + \nu)}.$$

Because the Mittag-Leffler function is not oscillating (for real z), the properties of (3) will be completely different from its integer order counterparts with sin or cos. There is no theoretical support as of today to ensure the existence of $A \neq 0 \neq B$ satisfying (3). In this section, the eigenvalue problem (3) will be solved by the Jacobi spectral-collocation method which gives some numerical evidences to the existence of nontrivial solution.

To numerically solve (3), We will soon see that we need to approximate integrals of the form

$$\int_{-1}^{1} G(x)(1-x)^p(1+x)^q. \tag{4}$$

The Gauss-Jacobi quadrature is a well-known method for numerical quadrature based on Gaussian quadrature over the interval $[-1, 1]$ with the weight function

$\omega(x) = (1-x)^p(1+x)^q$, where f is a smooth function on $[-1,1]$ and $p, q > -1$. The orthogonal polynomials associated to the weight function $\omega(x)$ consist of Jacobi polynomials. Thus, the Gauss-Jacobi quadrature rule on n points has the form

$$\int_{-1}^{1} G(x)(1-x)^p(1+x)^q dx \approx \sum_{i=1}^{n} \omega_i G(x_i), \tag{5}$$

where $\{x_j\}_{j=1}^n$, called Jacobi-collocation points, are the roots of the Jacobi polynomial of degree n and the weights ω_i are giving by

$$\omega_i = -\frac{2n+p+q+2}{n+p+q+1} \frac{\Gamma(n+p+1)\Gamma(n+q+1)}{\Gamma(n+p+q+1)(n+1)!} \frac{2^{p+q}}{P_n^{(p,q)'}(x_i)P_{n+1}^{(p,q)}(x_i)}, \tag{6}$$

where $\Gamma(\cdot)$ denotes the Gamma function. The Jacobi polynomials $P_n^{(p,q)}$ are solution of

$$(1-x^2)y'' + \big(q-p-(p+q+2)x\big)y' + n(n+p+q+1)y = 0, \tag{7}$$

having the following explicit expression

$$P_n^{(p,q)}(x_i) = \frac{\Gamma(p+n+1)}{n!\Gamma(p+q+n+1)} \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(p+q+n+k+1)}{\Gamma(p+k+1)} \left(\frac{x_i-1}{2}\right)^k, \tag{8}$$

and satisfy the orthogonality condition

$$\begin{aligned} &\int_{-1}^{1} (1-x)^p(1-x)^q P_m^{(p,q)}(x) P_n^{(p,q)}(x) dx \\ &\quad = \frac{2^{p+q+1}}{2n+p+q+1} \frac{\Gamma(n+p+1)\Gamma(n+q+1)}{\Gamma(n+p+q+1)n!} \delta_{nm}. \end{aligned}$$

From [23], if the eigenfunction $\varphi(x)$ (index-k is dropped for simplicity) is two-time continuous differentiable in $[-1,1]$, then the Riemann-Liouville fractional derivatives (2) coincides with the Griinwald-Letnikov derivatives; that is

$${}_{-1}\mathbf{D}_x^{\beta}\varphi(x) = \sum_{j=0}^{1} \frac{\varphi^{(j)}(-1)(x+1)^{j-\beta}}{\Gamma(j+1-\beta)} + \frac{1}{\Gamma(2-\beta)} \int_{-1}^{x} (x-\xi)^{1-\beta} \varphi''(\xi)\, d\xi. \tag{9}$$

Assuming sufficient smoothness in the STFDE allows us to use reformulate the eigenfunction problem using the Grünwald-Letnikov definitions. Because of the zero Dirichlet boundary conditions, $\varphi(\pm 1) = 0$, we can further simplify (9) to the form

$${}_{-1}\mathbf{D}_x^{\beta}\varphi(x) = \frac{\varphi'(-1)(x+1)^{1-\beta}}{\Gamma(2-\beta)} + \frac{1}{\Gamma(2-\beta)} \int_{-1}^{x} (x-\xi)^{1-\beta} \varphi''(\xi)\, d\xi. \tag{10}$$

To approximate the integral term in (10), we apply the following transformation

$$\int_{-1}^{x}(x-\xi)^{1-\beta}\varphi''(\xi)\,d\xi = \left(\frac{x+1}{2}\right)^{2-\beta}\int_{-1}^{1}(1-\tau)^{1-\beta}\varphi''\left(\frac{x+1}{2}\tau+\frac{x-1}{2}\right)d\tau, \quad (11)$$

by using the transformation (in which the variable x can be considered as fixed),

$$\xi = \frac{x+1}{2}\tau+\frac{x-1}{2},\ \tau\in[-1,1]. \quad (12)$$

Using the n-point Gauss-Jacobi quadrature formula relative to the Jacobi weight $\{\omega_i\}_{i=1}^n$, the integration term can be approximated by

$$\int_{-1}^{1}(1-\tau)^{1-\beta}\varphi''\left(\frac{x+1}{2}\tau+\frac{x-1}{2}\right)d\tau \sim \sum_{k=1}^{n}\varphi''\left(\frac{x+1}{2}x_k+\frac{x-1}{2}\right)\omega_k \quad (13)$$

where τ is discretized at the collocation or Jacobi points $\{x\}_{i=1}^n$.

Imposing the collocation conditions, also, at the Jacobi points $\{x\}_{i=1}^n$ yields,

$$\lambda\varphi(x_i) = \frac{\varphi'(-1)(x_i+1)^{1-\beta}}{\Gamma(2-\beta)}+\frac{1}{\Gamma(2-\beta)}\int_{-1}^{x_i}(x_i-\xi)^{1-\beta}\varphi''(\xi)\,d\xi,\ \xi\in[-1,x_i] \quad (14)$$

or after discretization,

$$\lambda\varphi(x_i) = \frac{\varphi'(-1)(x_i+1)^{1-\beta}}{\Gamma(2-\beta)}+\frac{1}{\Gamma(2-\beta)}\sum_{k=1}^{n}\varphi''\left(\frac{x_i+1}{2}x_k+\frac{x_i-1}{2}\right)\omega_k. \quad (15)$$

Here, the Jacobi weight $\{\omega_i\}_{i=1}^n$ with $p=1-\beta$ and $q=0$ is used to compute the integral term in (10). At this stage, we approximate the eigenfunction in the form

$$\varphi(x) = \sum_{i=1}^{n}\varphi(x_i)L_i(x), \quad (16)$$

where $L_i(x)$ is a cardinal basis function associated with the Jacobi collocation points $\{x_i\}_{i=1}^n$; namely, $L_i(x_j)=\delta_{ij}\in\{0,1\}$. The JSCM was first apply to solve the Volterra integral equations of second kind with a weakly singular kernel, and it is proven in the same paper that the method enjoys exponential convergence. Following [24], we adopt the Lagrange interpolation basis as L_i.

As in the Trefftz method, the zero boundary conditions can be satisfied *exactly* by the construction of basis functions. This is done by adding the points ±1 into the construction of the Lagrange interpolation basis. Now, equation (14) becomes

$$\begin{aligned}\lambda\varphi(x_i) &= \frac{(x_i+1)^{1-\beta}}{\Gamma(2-\beta)}\sum_{j=1}^{n}\varphi(x_j)\mathrm{L}_j'(-1)\\ &\quad+\frac{1}{\Gamma(2-\beta)}\left(\frac{x_i+1}{2}\right)^{2-\beta}\sum_{j=1}^{n}\varphi(x_j)\sum_{k=1}^{n}L_j''\left(\frac{x_i+1}{2}x_k+\frac{x_i-1}{2}\right)\omega_k\end{aligned}$$

With a total of n unknown $\varphi(x_i)$, the collocation system can be rewritten in the following $n \times n$ matrix form:

$$\lambda \begin{bmatrix} \varphi(x_1) \\ \varphi(x_2) \\ \vdots \\ \vdots \\ \varphi(x_n) \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & \cdots & \cdots & D_{1n} \\ D_{21} & D_{22} & \cdots & \cdots & D_{2n} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ D_{n1} & D_{n2} & \cdots & \cdots & D_{nn} \end{bmatrix} \begin{bmatrix} \varphi(x_1) \\ \varphi(x_2) \\ \vdots \\ \vdots \\ \varphi(x_n) \end{bmatrix},$$

where the matrix entries are given as

$$D_{ij} = \frac{(x_i+1)^{1-\beta}}{\Gamma(2-\beta)} Ł'_j(-1) + \frac{1}{\Gamma(2-\beta)} \left(\frac{x_i+1}{2}\right)^{2-\beta} \sum_{k=1}^{n} L''_j \left(\frac{x_i+1}{2} x_k + \frac{x_i-1}{2}\right) \omega_k.$$

The eigenvectors φ_k corresponding to the eigenvalues λ_k $(k = 1, \ldots, n)$ of matrix D are the approximation of function $\varphi(x)$ evaluated at the Jacobi collocation points $\{x_i\}_{i=1}^n$.

From the fact that the Mittag-Leffler functions are the eigenfunctions of the Caputo fractional derivative [23], the solution

$$\psi_k(t) = E_{\alpha,1}\big(\lambda_k t^\alpha\big), \tag{17}$$

can be obtained for the *k-th* fractional eigenfunction problem in time

$$\begin{aligned} {}_0^c\partial_t^\alpha \psi_k(t) &= \lambda_k \psi_k(t), \qquad t > 0,\ 0 < \alpha < 1, \\ \psi_k(0) &= 1. \end{aligned} \tag{18}$$

We now have the approximate fundamental solutions in hand. Besides of the many unanswered theoretical questions, we construct the approximate solution to the STFDE (1) in the following form:

$$u(x,t) = \sum_{k=1}^{\infty} \sigma_k \, F_k(x,t) = \sum_{k=1}^{\infty} \sigma_k \varphi_k(x) \psi_k(t), \tag{19}$$

where $\{\sigma_k\}$ is the set unknown coefficients to be determined.

3 Numerical expansion and collocation system

It is easy to verify that each complex-valued approximate fundamental solution $F_k(x,t)$, that is product of eigenfunctions (3) and (18) with appropriated eigenvalues, satisfies the STDFE and the boundary conditions in (1), but not the initial

condition in general. In the STFDE, as the order β decreases, complex eigenvalues appears in the eigenvalue problem (3) as the property of the Riemann-Liouville differs more from that of the Laplacian. If β is away below 1, our numerical simulations also show that $\mathfrak{Re}(\lambda_n) < 0$ all the time. In order to avoid the complexity of calculations that involves complex numbers, we separate the real and imaginary part of those complex eigenvalues according to their conjugation property. Since complex eigenvalues always appear in the form of complex conjugates, we reorder the set of eigenvalues of (14) whenever both real and complex eigenvalues exist. Let

$$\Lambda = \{\lambda_1, \overline{\lambda_1}, \lambda_2, \overline{\lambda_2}, \ldots, \lambda_{n'}, \overline{\lambda_{n'}}, \lambda_{2n'+1}, \ldots, \lambda_n, \},$$

for some $0 \leq n' \leq \lfloor n/2 \rfloor$, where n' is the number of the pairs of complex conjugate and let φ_j be the corresponding eigenvectors. The j^{th}–eigenfunctions φ_j $(j = 1, \ldots, n)$ of the Riemann-Liouville eigenproblem corresponding to λ_j will be approximated by the Lagrange interpolation formula (16) using its values $\varphi_j(x_n)$ at the Jacobi points.

By rearrange our eigenvalue set, we can further expand the solution U_n to the exact solution u of (1) as

$$\begin{aligned} U_n(x,t) := &\sum_{j=1}^{n'} \sigma_{2j-1}\, \mathfrak{Re}\big(E_{\alpha,1}(\lambda_j t^\beta)\varphi_j(x)\big) \\ &+ \sum_{j=1}^{n'} \sigma_{2j}\, \mathfrak{Im}\big(E_{\alpha,1}(\lambda_j t^\beta)\varphi_j(x)\big) + \sum_{j=n'}^{n} \sigma_j E_{\alpha,1}(\lambda_j t^\beta)\varphi_j(x), \end{aligned} \tag{20}$$

in which all functions are real-valued and σ here differs from the ones in (19). To determine the set unknown coefficients $\{\sigma_j\}_{j=1}^n$, we follow the standard procedure in MFS and take a large set of (equispaced) collocation points $X' = \{x'_1, \ldots, x'_m\}$ $(m \gg n)$ in $[-1, 1]$. Via the standard least-squares fit to the $m \times n$ matrix system, we seek $\{\sigma_k\}_{k=1}^n$ that satisfy the initial condition

$$U_n(x'_i, 0) \approx u_0(x'_i),\ (i = 1, \ldots, m).$$

After the determination of $\{\sigma_k\}_{k=1}^n$, the numerical solution (20) can be evaluated anywhere in $(x, t) \in [-1, 1] \times [0, \infty)$.

4 Numerical examples

To demonstrate the effectiveness of the proposed method for solving STFDE, we consider the following equations with boundary conditions and initial condition:

N	$\max_x \left(U_n(x,t) - U_{n-2}(x,t)\right)$
13	1.74×10^{-4}
15	8.38×10^{-5}
17	4.02×10^{-5}
19	2.06×10^{-5}
21	1.12×10^{-5}
23	6.44×10^{-6}
25	3.74×10^{-6}
27	2.38×10^{-6}
29	1.60×10^{-6}
31	1.06×10^{-6}

Table 1: *Maximum differences of consecutive numerical approximation.*

for $0 < \alpha < 1 < \beta < 2$,

$$\begin{aligned} {}_0^c\partial_t^\alpha u(x,t) &= {}_{-1}\mathbf{D}_x^\beta u(x,t), \quad -1 \le x \le 1, \quad 0 < t, \\ u(x,0) &= \cos(\pi x/2), \qquad -1 \le x \le 1, \\ u(\pm 1,t) &= 0, \qquad\qquad 0 < t. \end{aligned} \tag{21}$$

Using a downloadable script [25], all Mittag-Leffler functions are evaluated to an accuracy of 10 decimal places. First, we test the obvious limiting case when $\alpha = 1$ and $\beta = 2 - 10^{-6}$; note that the proposed algorithm does not allowed $\beta = 2$ exactly but $\alpha = 1$ is possible. As the fractional derivative approaches to the integer order derivative, and the STFDE solution $u(x,t)$ approaches that of the standard heat diffusion $\exp(-\pi^2 t/4)\cos(\pi x/2)$. With $m = 100$ collocation points, the maximum *differences* between the numerical approximations and the *nearby* standard heat solution are respectively 1.42×10^{-5} and 1.36×10^{-5} for $n = 21$ and 31. To verify convergence, we set $\alpha = 0.8$, $\beta = 1.8$ and Table 1 reports the maximum differences of consecutive approximation $U_n(x,t) - U_{n-2}(x,t)$, $n = 13, 15, \ldots, 31$, over the computation domain $-1 \le x \le 1$, $0 \le t \le 1$. The observed monotonic behavior provides some evidences of convergence despite the lack of exact solution for comparison.

As our proposed method only requires collocations for the initial condition, the memory effect does no effect to the complexity of the solution process. Once the unknown coefficients are obtained, the numerical solution can be evaluated at any time t as wish. With $n = 21$, the solution profiles can be plotted easily up to time $t = 100$ (or larger) as shown in Figure 2. A slight asymmetry in the solution can be observed from the 2D snapshots, also see Figure 3(a).

To further parameter analysis, Figure 3 demonstrate the approximation solutions to (21) for four pairs of α and β values. All figures show the solutions for

$0 \leq t \leq 2$ and employ the same color scheme for easy comparison. On the first row, in Figure 3(a) and 3(b), the numerical solutions for fixed $\alpha = 0.8$ and, respectively $\beta = 1.8$ and $\beta = 1.4$ are shown. As β decreases, the solution of the STFDE skews to the left towards the $x = -1$ boundary due to the asymmetry in the Riemann-Liouville derivative; the same asymmetric behavior is also observed in the field data [2]. When we compare the figures columnwise, the effect of decreasing α can be observed. Smaller α yields "slower diffusion" with a rapid temperature drop at the initial time. In the second row with $\alpha = 0.4$, it is difficult to observe any region with solution value greater than 0.8.

In [26], the STFDE (1) with a piecewise linear initial condition

$$u(x,0) = \min\left(2(1+x),\ \frac{2}{3}(1-x)\right)$$

with singularity at $x = -1/2$ was studied with a finite difference scheme. For standard heat equation, it is common to start the simulation with some low order difference schemes and switch back to high order spectral after the solution is smoothed by diffusion. For the STFDE we consider, due to the memory effect, the initial singularity will in theory never leave the picture of this non-Markov process. When the number of collocation is being set large (i.e. $m = 100$ as in the above), the corner will be rounded in the numerical approximation. For suitable small m so that the a collocation point coincides with the corner, the singularity could be well preserved; Figure 4 demonstrates two numerical results obtained by $m = n = 21$.

5 Conclusions

In this paper, we propose a numerical procedure for solving a space-time fractional diffusion equation in a boundary domain. We use the Jacobi Spectral-Collocation Method to solve the eigenvalue problems of the Riemann-Liouville fractional derivative. Using the resulting eigen-information and the generalized two-parameter Mittag-Leffler functions, we construct approximate fundamental solutions for the space-time fractional diffusion equation. Due to the lack of exact solution, convergence and accuracy of the proposed scheme are partially verified by numerical results. A numerical solution is shown for $1 \leq t \leq 100$ as a demonstration of time algorithm's capability on long time simulation. To illustrate the diffusion pattern in STFDE, simulations for various fractional orders in space and time are included. Our last example shows that the proposed spectral based approach can still well handle initial conditions with singularity.

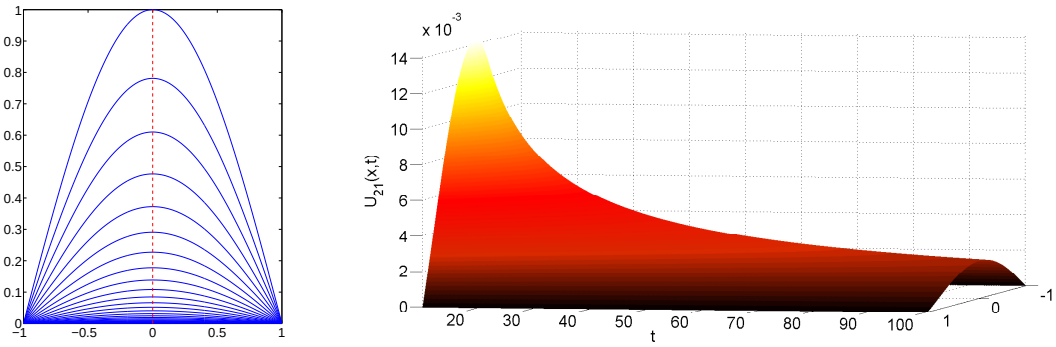
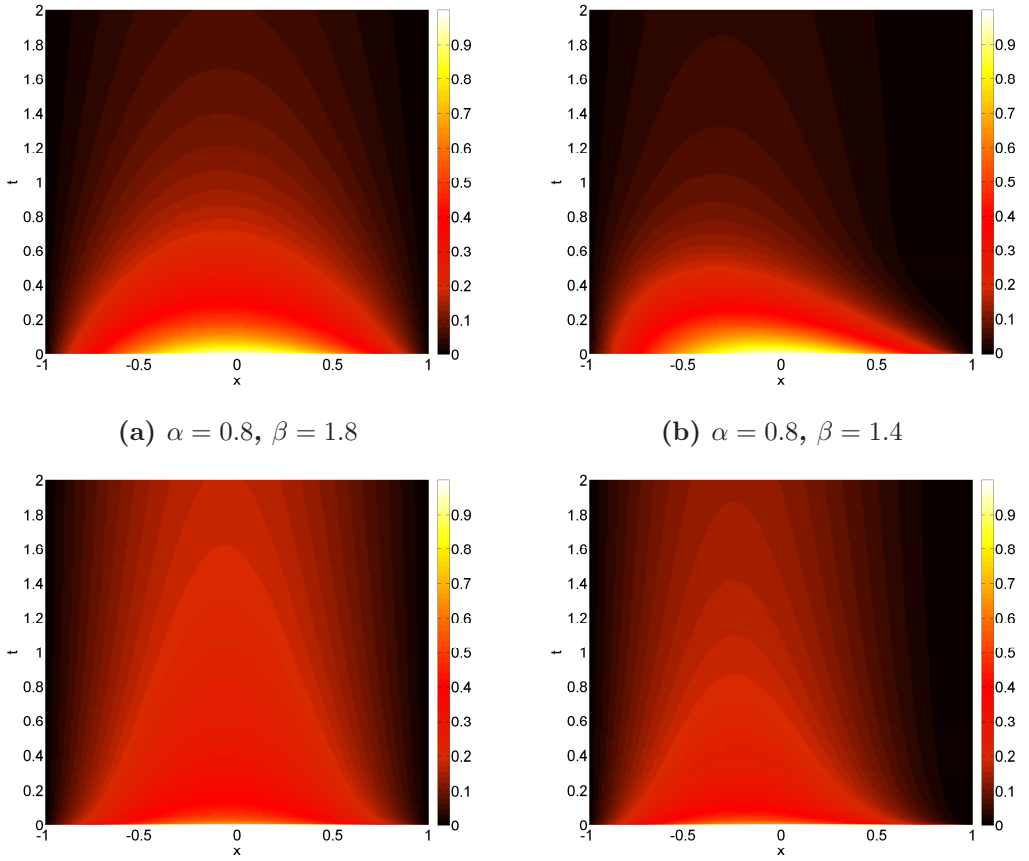


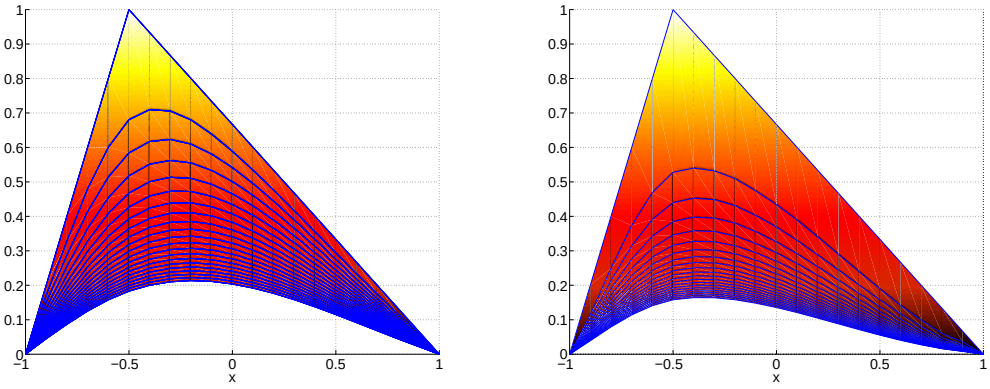
Figure 2: *Numerical solutions for the STFDE with* $\alpha = 0.8$ *and* $\beta = 1.8$. (Left) *Snapshots at various time.* (Right) *Solution profiles.*



(a) $\alpha = 0.8$, $\beta = 1.8$ **(b)** $\alpha = 0.8$, $\beta = 1.4$

(c) $\alpha = 0.4$, $\beta = 1.8$ **(d)** $\alpha = 0.4$, $\beta = 1.4$

Figure 3: *Dispersion patterns for various orders of fractional derivative* α *and* β.



(a) $\alpha = 0.8$, $\beta = 1.8$ **(b)** $\alpha = 0.6$, $\beta = 1.6$

Figure 4: *Decaying profiles for STFDE from a singular initial condition to time* $t = 0.4$.

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