CHARACTERIZING THE STABILIZATION SIZE FOR SEMI-IMPLICIT FOURIER-SPECTRAL METHOD TO PHASE FIELD EQUATIONS

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Abstract. Recent results in the literature provide computational evidence that stabilized semi-implicit time-stepping method can efficiently simulate phase field problems involving fourth-order nonlinear diffusion, with typical examples like the Cahn-Hilliard equation and the thin film type equation. The up-to-date theoretical explanation of the numerical stability relies on the assumption that the derivative of the nonlinear potential function satisfies a Lipschitz type condition, which in a rigorous sense, implies the boundedness of the numerical solution. In this work we remove the Lipschitz assumption on the nonlinearity and prove unconditional energy stability for the stabilized semi-implicit time-stepping methods. It is shown that the size of stabilization term depends on the initial energy and the perturbation parameter but is independent of the time step. The corresponding error analysis is also established under minimal nonlinearity and regularity assumptions.

1. Introduction

In this work we consider two phase field models: the Cahn-Hilliard equation and the molecular beam epitaxy equation (MBE) with slope selection. The Cahn-Hilliard (CH) equation was originally developed in [5] to describe phase separation in a two-component system (such as metal alloy). It typically takes the form

$$\begin{cases} \partial_t u = \Delta(-\nu\Delta u + f(u)), \quad (x,t) \in \Omega \times (0,\infty), \\ u\Big|_{t=0} = u_0, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is a real-valued function which represents the difference between two concentrations. Due to this fact the equation (1.1) is invariant under the sign change $u \to -u$. Another common form for CH is

$$\begin{cases} \partial_t u = \Delta w \\ w = -\epsilon\Delta u + \epsilon^{-1} f(u). \end{cases} \tag{1.2}$$

As $\epsilon \to 0$ the chemical potential w tends to a limit which solves the two-phase Hele-Shaw (Mullins-Sekerka) problem (see [21] for a heuristic derivation, [1] for a convergence proof (under the assumption that classical solution to the limiting Hele-Shaw problem exists)). In (1.1) the spatial domain Ω is taken to be the usual 2π-periodic torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$. For simplicity we only consider the periodic case but our analysis can be generalized to other settings (such as bounded domain with Neumann boundary conditions). The free energy term $f(u)$ is given by

$$f(u) = F'(u) = u^3 - u, \qquad F(u) = \frac{1}{4}(u^2-1)^2. \tag{1.3}$$

The parameter $\nu > 0$ is often called diffusion coefficient. Usually one is interested in the physical regime $0 < \nu \ll 1$ in which the dynamics of (1.1) is close to the limiting Hele-Shaw problem after some transient time.

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For smooth solutions to (1.1), the total mass is conserved:

$$\frac{d}{dt}M(t) \equiv 0, \qquad M(t) = \int_\Omega u(x,t)dx. \tag{1.4}$$

In particular $M(t) \equiv 0$ if $M(0) = 0$. Throughout this work we will only consider initial data u_0 with mean zero. At the Fourier side this implies the zero$^{\text{th}}$ mode $\hat{u}(0) = 0$. One can then define fractional Laplacian $|\nabla|^s u$ for $s < 0$. The energy functional associated with (1.1) is

$$E(u) = \int_\Omega \left(\frac{1}{2}\nu |\nabla u|^2 + F(u) \right) dx. \tag{1.5}$$

As is well known, Eq. (1.1) can be regarded as a gradient flow of $E(u)$ in H^{-1}. The basic energy identity takes the form

$$\frac{d}{dt}E(u(t)) + \||\nabla|^{-1}\partial_t u\|_2^2 = 0. \tag{1.6}$$

Note that $\partial_t u$ has mean zero and $|\nabla|^{-1}\partial_t u$ is well-defined. Alternatively to avoid using $|\nabla|^{-1}$, one can write (1.6) as

$$\frac{d}{dt}E(u(t)) + \int_\Omega |\nabla(-\nu\Delta u + f(u))|^2 dx = 0. \tag{1.7}$$

It follows from the energy identity that

$$E(u(t)) \le E(u(s)), \quad \forall\ t \ge s. \tag{1.8}$$

This gives a priori control of H^1-norm of the solution. The global wellposedness of (1.1) is not an issue thanks to this fact.

There is by now an extensive literature on the numerical simulation of the CH equation and related phase field models, see, e.g., [4, 7, 9, 10, 15, 16, 26, 30] and the references therein. On the analysis side, it is noted that Feng and Prohl [12] gave the error analysis of a semi-discrete (in time) and fully discrete finite element method for CH. Under a certain spectral assumption on the linearized CH operator (more precisely, one has to assume the existence of classical solutions to the corresponding Hele-Shaw problem), they proved an error bound which depends on $1/\nu$ polynomially.

It is known that explicit schemes usually suffer severe time step restrictions and generally do not obey energy conservation. To enforce the energy decay property and increase the time step, a good alternative is to use implicit-explicit (semi-implicit) schemes in which the linear part is treated implicitly (such as backward differentiation in time) and the nonlinear part is evaluated explicitly. For example, in [7] Chen and Shen considered the semi-implicit Fourier-spectral scheme for (1.1) (set $\nu = 1$)

$$\frac{\widehat{u^{n+1}}(k) - \widehat{u^n}(k)}{\Delta t} = -|k|^4 \widehat{u^{n+1}}(k) - |k|^2 \widehat{f(u^n)}(k), \tag{1.9}$$

where $\widehat{u^n}$ denotes the Fourier coefficient of u at time step t_n. On the other hand, the semi-implicit schemes can generate large truncation errors. As a result smaller time steps are usually required to guarantee accuracy and (energy) stability. To resolve this issue, a class of large time-stepping methods were proposed and analyzed in [13, 16, 26, 29, 30]. The basic idea is to add an $O(\Delta t)$ stabilizing term to the numerical scheme to alleviate the time step constraint whilst keeping energy stability. The choice of the $O(\Delta t)$ term is quite flexible. For example, in [30] the authors considered the Fourier spectral approximation of the modified Cahn-Hilliard-Cook equation

$$\partial_t C = \nabla \cdot \big((1 - aC^2)\nabla(C^3 - C - \kappa\nabla^2 C) \big). \tag{1.10}$$

The explicit Fourier spectral scheme is (see equation (16) therein)

$$\frac{\widehat{C^{n+1}}(k,t) - \widehat{C^n}(k,t)}{\Delta t} = ik \cdot \big\{ (1 - aC^2)[ik'(\{-C + C^3\}_{k'}^n + \kappa|k'|^2 \widehat{C^n}(k',t))]_r \big\}_k. \tag{1.11}$$

The time step for the above scheme has a severe constraint

$$\Delta t \cdot \kappa \cdot K^4 \le 1, \tag{1.12}$$

where K is the number of Fourier modes in each coordinate direction. To increase the allowed time step, the authors of [30] added a term $-Ak^4(\widehat{C^{n+1}} - \widehat{C^n})$ to the RHS of (1.11). Note that on the real side, this term corresponds to the fourth order dissipation, i.e.

$$-A\Delta^2(C^{n+1} - C^n)$$

which roughly is of order $O(\Delta t)$.

In [16], a stabilized semi-implicit scheme was considered for the CH model, with the use of an order $O(\Delta t)$ stabilization term

$$A\Delta(u^{n+1} - u^n).$$

Under a condition on A of the form:

$$A \ge \max_{x\in\Omega}\Big\{\frac{1}{2}|u^n(x)|^2 + \frac{1}{4}|u^{n+1}(x) + u^n(x)|^2\Big\} - \frac{1}{2}, \qquad \forall n \ge 0, \tag{1.13}$$

one can obtain energy stability (1.8). Note that the condition (1.13) depends nonlinearity on the numerical solution. In other words, it implicitly uses the L^∞-bound assumption on u^n in order to make A a controllable constant.

In [26], Shen and Yang proved energy stability of semi-implicit schemes for the Allen-Cahn and the CH equations with truncated nonlinear term. More precisely it is assumed that

$$\max_{u\in\mathbb{R}} |f'(u)| \le L \tag{1.14}$$

which is what we referred to as the Lipschitz assumption on the nonlinearity in the abstract. The same assumption was adopted recently in [13] to analyze stabilized Crank-Nicolson or Adams-Bashforth scheme for both the CH equations.

In a recent work of [4], Bertozzi et al. considered a nonlinear diffusion model of the form

$$\partial_t u = -\nabla\cdot(f(u)\nabla\Delta u) + \nabla\cdot(g(u)\nabla u),$$

where $g(u) = f(u)\phi'(u)$, and f, ϕ are given smooth functions. In addition f is assumed to be non-nonnegative. The numerical scheme considered in [4] takes the form

$$\frac{u^{n+1} - u^n}{\Delta t} = -A\Delta^2(u^{n+1} - u^n) - \nabla\cdot(f(u^n)\nabla\Delta u^n) + \nabla\cdot(g(u^n)\nabla u^n), \tag{1.15}$$

where $A > 0$ is a parameter to be taken large. One should note the striking similarity between this scheme and the one introduced in [30]. In particular in both papers the biharmonic stabilization of the form $-A\Delta^2(u^{n+1} - u^n)$ was used. The analysis in [4] is carried out under the additional assumption that

$$\sup_n \|f(u^n)\|_\infty \le A < \infty. \tag{1.16}$$

This is reminiscent of the L^∞ bound on u^n.

Roughly speaking, all prior analytical developments are conditional in the sense that either one makes a Lipschitz assumption on the nonlinearity, or one assumes certain a priori L^∞ bounds on the numerical solution. It is very desirable to *remove these technical restrictions* and establish a more reasonable stability theory. Thus

Problem: *prove unconditional energy stability of large time-stepping semi-implicit numerical schemes for general phase field models.*

Here unconditional means that no restrictive assumptions should be imposed on the time step. Of course one should also develop the corresponding error analysis under minimal regularity conditions.

The purpose of this work is to settle this problem for the spectral Galerkin case. In a forthcoming work [20], we shall analyze the finite difference schemes for the CH model by using a completely different approach.

We now state our main results. We first consider a stabilized semi-implicit scheme introduced in [16] following the earlier work [29]. It takes the form

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} = -\nu\Delta^2 u^{n+1} + A\Delta(u^{n+1}-u^n) + \Delta\Pi_N(f(u^n)), \quad n \geq 0, \\ u^0 = \Pi_N u_0. \end{cases} \tag{1.17}$$

where $\tau > 0$ is the time step, and $A > 0$ is the coefficient for the $O(\tau)$ regularization term. For each integer $N \geq 2$, define

$$X_N = \mathrm{span}\Big\{\cos(k\cdot x), \sin(k\cdot x) : \quad |k| \leq N,\, k \in \mathbb{Z}^2\Big\}.$$

Note that the space X_N includes the constant function (by taking $k = 0$). The L^2 projection operator $\Pi_N : L^2(\Omega) \to X_N$ is defined by

$$(\Pi_N u - u, \phi) = 0, \qquad \forall\, \phi \in X_N, \tag{1.18}$$

where $(\cdot,\cdot)$ denotes the usual L^2 inner product on Ω. In yet other words, the operator Π_N is simply the truncation of Fourier modes of L^2 functions to $|k| \leq N$. Since $\pi_N u_0 \in X_N$, by induction it is easy to check that $u^n \in X_N$ for all $n \geq 0$. Note that one can recast (1.17) into the usual weak formulation, for example:

$$(d_t u^{n+1}, v) + A(\nabla(u^{n+1}-u^n), \nabla v) + (\nabla(f(u^n)), \nabla v) + \nu(\Delta u^{n+1}, \Delta v) = 0, \quad \forall\, v \in X_N,$$

where $d_t u^{n+1} = (u^{n+1}-u^n)/\tau$. However in our analysis it is more convenient to work with (1.17). Note that u^n has mean zero for all $n \geq 0$ (since we assume u_0 has mean zero).

Theorem 1.1 (Unconditional energy stability for CH)**.** *Consider* (1.17) *with* $\nu > 0$ *and assume* $u_0 \in H^1(\Omega) \cap L^\infty(\Omega)$ *with mean zero. Denote* $E_0 = E(u_0)$ *the initial energy. There exists a constant* $\beta_c > 0$ *depending only on* E_0 *such that if*

$$A \geq \beta \cdot (\|u_0\|_\infty^2 + \nu^{-1}|\log\nu|^2 + 1), \quad \beta \geq \beta_c, \tag{1.19}$$

then

$$E(u^{n+1}) \leq E(u^n), \qquad \forall\, n \geq 0,$$

where E *is defined by (1.5).*

Remark 1.1. We stress that the above stability result works for any time step $\tau > 0$. In particular the condition on the parameter A is independent of τ. In order to keep the argument simple, we do not try to optimize the dependence of A on the diffusion coefficient ν. This can certainly be pushed further.

Remark 1.2. One should note that in (1.19), the lower bound $\nu^{-1}|\log\nu|^2$ is formally consistent with the predicted bound (1.13). In terms of the PDE solution $u(t,x)$, the bound (1.13) roughly asserts that

$$A \geq O(\|u(t)\|_\infty^2).$$

For the PDE solution, there is no L^∞ conservation and one has to trade it with the $\dot{H}^1(\mathbb{T}^2)$ bound with some logarithmic correction. The energy conservation gives $\|u(t)\|_{H^1} \lesssim \nu^{-\frac{1}{2}}$, and the log-correction gives $|\log(\nu)|$. Thus we need $A \gtrsim \nu^{-1}|\log\nu|^2$ from this heuristic.

There is an analogue of Theorem 1.1 for the MBE equation. The MBE equation has the form

$$\begin{cases} \partial_t h = -\nu\Delta^2 h + \nabla\cdot(g(\nabla h)), \qquad (x,t) \in \Omega \times (0,\infty), \\ h\Big|_{t=0} = h_0, \end{cases} \tag{1.20}$$

where $h = h(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}$ represents the scaled height function of a thin film equation, and $g(z) = (|z|^2 - 1)z$ for $z \in \mathbb{R}^2$. The domain Ω is again assumed to be the periodic torus $\mathbb{T}^2$. Eq. (1.20) can be regarded as an L^2 gradient flow of the energy functional

$$E(h) = \frac{\nu}{2}\|\Delta h\|_2^2 + \int_\Omega G(\nabla h)dx, \tag{1.21}$$

where $G(z) = \frac{1}{4}(|z|^2 - 1)^2$ for $z \in \mathbb{R}^2$. Note the striking similarity between the MBE energy (1.21) and the CH energy (1.5). Roughly speaking, ∇h is the correct scaling analogue of u in (1.1). In fact it is well known that in 1D the MBE equation can be transformed into the CH equation through the change of variable $u = \partial_x h$. In recent [19] we obtained new upper and lower gradient bounds for the MBE equation in dimensions $d \le 3$. A refined wellposedness theory is also worked out there. Some of these results will be used in the H^1 error analysis in this work. We refer to the introduction of [19] and also [11, 2, 3, 17, 18, 27, 31] for some background material and related wellposedness/ill-posedness results.

Consider the following semi-implicit scheme for MBE:

$$\begin{cases} \dfrac{h^{n+1} - h^n}{\tau} = -\nu\Delta^2 h^{n+1} + A\Delta(h^{n+1} - h^n) + \Pi_N \nabla \cdot (g(\nabla h^n)), \quad n \ge 0, \\ h^0 = \Pi_N h_0. \end{cases} \tag{1.22}$$

This scheme was introduced and analyzed in [29] (see also [22]). The authors of [29] first introduced the stabilized $O(\Delta t)$ term of the form $A\Delta(h^{n+1} - h^n)$ as given in (1.22). They also proved that the energy stability (1.8) under the condition

$$A \ge \frac{1}{2}\|\nabla h^n\|_\infty^2 + \frac{1}{4}\|\nabla(h^{n+1} + h^n)\|_\infty^2 - \frac{1}{2}, \qquad \forall\, n \ge 0. \tag{1.23}$$

Again, it is seen that A depends implicitly on the L^∞ bound on the numerical solution h^n.

The result below will provide a clean description on the size of the constant A, in the sense A is independent of the L^∞ bound on the numerical solution.

Theorem 1.2 (Unconditional energy stability for MBE)**.** *Consider* (1.22) *with* $\nu > 0$. *Assume* $h_0 \in H^2(\Omega)$ *with mean zero. Assume also* $\nabla h_0 \in L^\infty(\Omega)$. *There exists a constant* $\beta_c > 0$ *depending only on* E_0 *such that if*

$$A \ge \beta \cdot (\|\nabla h_0\|_\infty^2 + \nu^{-1}|\log \nu|^2 + 1), \quad \beta \ge \beta_c,$$

then

$$E(u^{n+1}) \le E(u^n), \qquad \forall\, n \ge 0,$$

where E *is defined by (1.21).*

We now state the results for error estimates. We start with the CH equation.

Theorem 1.3 (L^2 error estimate for CH)**.** *Let* $\nu > 0$. *Let* $u_0 \in H^s$, $s \ge 4$ *with mean zero. Let* $u(t)$ *be the solution to* (1.1) *with initial data* u_0. *Let* u^n *be defined according to* (1.17) *with initial data* $\Pi_N u_0$. *Assume* A *satisfies the same condition in Theorem 1.1. Define* $t_m = m\tau$, $m \ge 1$. *Then*

$$\|u(t_m) - u^m\|_2 \le A \cdot e^{C_1 t_m} \cdot C_2 \cdot (N^{-s} + \tau).$$

Here $C_1 > 0$ *depends only on* (u_0, ν), $C_2 > 0$ *depends on* (u_0, ν, s).

For the MBE equation, we have the following H^1 error estimate. Note that due to the use of H^1 space the error bound below involves $N^{-(s-1)}$ instead of N^{-s}.

Theorem 1.4 (H^1 error estimate for MBE)**.** *Let* $\nu > 0$ *and* $h_0 \in H^s$, $s \ge 5$ *with mean zero. Let* $h(t)$ *be the solution to the MBE equation with initial data* h_0. *Let* h^n *be defined according to* (1.22) *with initial data* $\Pi_N h_0$. *Assume* A *satisfies the same condition as in Theorem 1.2. Define* $t_m = m\tau$, $m \ge 1$. *Then*

$$\|\nabla(h(t_m) - h^m)\|_2 \le A \cdot e^{C_1 t_m} \cdot C_2 \cdot (N^{-(s-1)} + \tau),$$

where $C_1 > 0$ *depends on* (h_0, ν)*,* $C_2 > 0$ *depends on* (ν, h_0, s)*.*

We close this section by introducing some notation and preliminaries used in this paper.

We shall use $X+$ to denote $X + \epsilon$ for arbitrarily small $\epsilon > 0$. Similarly we can define $X-$. We denote by $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$ the 2π-periodic torus.

Let $\Omega = \mathbb{T}^d$. For any function $f : \Omega \to \mathbb{R}$, we use $\|f\|_{L^p} = \|f\|_{L^p(\Omega)}$ or sometimes $\|f\|_p$ to denote the usual Lebesgue L^p norm for $1 \le p \le \infty$. If $f = f(x,y) : \Omega_1 \times \Omega_2 \to \mathbb{R}$, we shall denote by $\|f\|_{L_x^{p_1} L_y^{p_2}}$ to denote the mixed-norm:

$$\|f\|_{L_x^{p_1} L_y^{p_2}} = \Big\| \|f(x,y)\|_{L_y^{p_2}(\Omega_2)} \Big\|_{L_x^{p_1}(\Omega_1)}.$$

In a similar way one can define other mixed-norms such as $\|f\|_{C_t^0 H_x^m}$ etc.

For any two quantities X and Y, we denote $X \lesssim Y$ if $X \le CY$ for some constant $C > 0$. Similarly $X \gtrsim Y$ if $X \ge CY$ for some $C > 0$. We denote $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. The dependence of the constant C on other parameters or constants are usually clear from the context and we will often suppress this dependence. We denote $X \lesssim_{Z_1,\cdots,Z_m} Y$ if $X \le CY$ where the constant C depends on the parameters $Z_1, \cdots, Z_m$.

We use the following convention for Fourier expansion on $\Omega = \mathbb{T}^d$:

$$(\mathcal{F}f)(k) = \hat{f}(k) = \int_\Omega f(x) e^{-ix\cdot k} dx, \quad f(x) = \frac{1}{(2\pi)^d} \sum_{k\in\mathbb{Z}^d} \hat{f}(k) e^{ik\cdot x}.$$

For $f : \mathbb{T}^d \to \mathbb{R}$ and $s \ge 0$, we define the H^s-norm and $\dot{H}^s$-norm of f as

$$\|f\|_{H^s} = \Big(\sum_{k\in\mathbb{Z}^d} (1 + |k|^{2s}) |\hat{f}(k)|^2 \Big)^{\frac12}, \quad \|f\|_{\dot{H}^s} = \Big(\sum_{k\in\mathbb{Z}^d} |k|^{2s} |\hat{f}(k)|^2 \Big)^{\frac12}.$$

provided of course the above sums are finite. Note that for $s = 1$

$$\|f\|_{\dot{H}^1} = \|\nabla f\|_2.$$

If f has mean zero, then $\hat{f}(0) = 0$ and in this case

$$\|f\|_{H^s} \sim \Big(\sum_{k\in\mathbb{Z}^d} |k|^{2s} |\hat{f}(k)|^2 \Big)^{\frac12}.$$

For mean zero functions, we can define the fractional Laplacian $|\nabla|^s$, $s \in \mathbb{R}$ via the relation

$$\widehat{|\nabla|^s f}(k) = |k|^s \hat{f}(k), \qquad 0 \ne k \in \mathbb{Z}^d.$$

The mean zero condition is only needed for $s < 0$.

Occasionally we will need to use the Littlewood–Paley (LP) frequency projection operators. To fix the notation, let $\phi_0 \in C_c^\infty(\mathbb{R}^d)$ and satisfy

$$0 \le \phi_0 \le 1, \quad \phi_0(\xi) = 1 \ \text{ for } |\xi| \le 1, \quad \phi_0(\xi) = 0 \ \text{ for } |\xi| \ge 2.$$

Let $\phi(\xi) := \phi_0(\xi) - \phi_0(2\xi)$ which is supported in $1/2 \le |\xi| \le 2$. For any $f \in \mathcal{S}'(\mathbb{R}^d)$, $j \in \mathbb{Z}$, define

$$\widehat{\Delta_j f}(\xi) = \phi(2^{-j}\xi)\hat{f}(\xi), \quad \widehat{S_j f}(\xi) = \phi_0(2^{-j}\xi)\hat{f}(\xi), \qquad \xi \in \mathbb{R}^d.$$

Let $f : \mathbb{T}^d \to \mathbb{R}$ be a smooth function. Note that f can be regarded as a tempered distribution on $\mathbb{R}^d$ for which $\Delta_j f$ can be defined as above. For any $1 \le p \le q \le \infty$, we recall the following Bernstein inequalities (see [19] for a standard proof)

$$\| |\nabla|^s \Delta_j f\|_{L^p(\mathbb{T}^d)} \sim 2^{js} \|\Delta_j f\|_{L^p(\mathbb{T}^d)}, \qquad s \in \mathbb{R}; \tag{1.24}$$

$$\|\Delta_j f\|_{L^q(\mathbb{T}^d)} \lesssim 2^{jd(\frac1p - \frac1q)} \|f\|_{L^p(\mathbb{T}^d)}, \qquad j \in \mathbb{Z}; \tag{1.25}$$

$$\|S_j f\|_{L^q(\mathbb{T}^d)} \lesssim 2^{jd(\frac1p - \frac1q)} \|f\|_{L^p(\mathbb{T}^d)}, \qquad j \ge -2. \tag{1.26}$$

In later sections, we will use (sometimes without explicit mentioning) the following interpolation inequality on $\mathbb{T}^2$: for $s>1$ and any $f\in H^s(\mathbb{T}^2)$ with mean zero, we have

$$\|f\|_{L^\infty(\mathbb{T}^2)} \lesssim_s \|f\|_{\dot{H}^1(\mathbb{T}^2)} \log(3+\|f\|_{H^s(\mathbb{T}^2)}). \tag{1.27}$$

Remark. The mean-zero condition is certainly needed in view of the $\|f\|_{\dot{H}^1}$ term on the RHS. If it is replaced by $\|f\|_{H^1}$ then the inequality holds for any f not necessarily with mean zero.

We include a proof of (1.27) for the sake of completeness. Since f has mean zero we have $\Delta_j f=0$ for $j<-2$. Let $j_0\in\mathbb{Z}$ whose value will be chosen later. By using the Bernstein inequality, we have

$$\begin{aligned}\|f\|_{L^\infty(\mathbb{T}^2)} &\lesssim \sum_{-2\le j\le j_0} 2^j\|\Delta_j f\|_{L^2(\mathbb{T}^2)} + \sum_{j>j_0} 2^j 2^{-js}\|f\|_{H^s(\mathbb{T}^2)},\\ &\lesssim_s (j_0+3)\|f\|_{\dot{H}^1} + 2^{-j_0(s-1)}\|f\|_{H^s}.\end{aligned}$$

Choosing $j_0=\mathrm{const}\cdot\log(3+\|f\|_{H^s})$ then yields (1.27). □

2. Proof of Stability results

In this section, we will provide rigorous proofs for the stability results, i.e., Theorems 1.1 and 1.2.

2.1. Proof of Theorem 1.1.

Rewrite (1.17) as

$$u^{n+1} = \frac{1-A\tau\Delta}{1+\nu\tau\Delta^2-A\tau\Delta}u^n + \frac{\tau\Delta\Pi_N}{1+\nu\tau\Delta^2-A\tau\Delta}f(u^n). \tag{2.1}$$

Lemma 2.1. *There is an absolute constants $c_1>0$ such that for any $n\ge 0$,*

$$\|u^{n+1}\|_{H^{\frac32}(\mathbb{T}^2)} \le c_1\cdot\left(\frac{A+1}{\nu}+\frac{1}{A\tau}\right)\cdot(E_n+1), \tag{2.2}$$

$$\|u^{n+1}\|_{\dot{H}^1(\mathbb{T}^2)} \le \left(1+\frac{1}{A}+\frac{3}{A}\|u^n\|_\infty^2\right)\cdot\|u^n\|_{\dot{H}^1(\mathbb{T}^2)}, \tag{2.3}$$

where $E_n=E(u^n)$.

Proof. In this proof for any two quantities X and Y, we shall use the notation $X\lesssim Y$ to denote $X\le CY$ where $C>0$ is an absolute constant. Also for any $f:\mathbb{T}^2\to\mathbb{R}$, we denote $\bar{f}$ as its average on $\mathbb{T}^2$. Below one should note that $f(u^n)=(u^n)^3-u^n$ generally does not have mean zero even though u^n has mean zero.

$$\begin{aligned}\|u^{n+1}\|_{H^{\frac32}} &\lesssim \Big(\frac{A}{\nu}+\frac{1}{A\tau}\Big)\|u^n\|_2 + \frac{1}{\nu}\||\nabla|^{-\frac12}\big(f(u^n)-\overline{f(u^n)}\big)\|_2\\ &\lesssim \Big(\frac{A}{\nu}+\frac{1}{A\tau}\Big)\|u^n\|_2 + \frac{1}{\nu}\|(u^n)^3-u^n-\overline{(u^n)^3}\|_{\frac43}\\ &\lesssim \Big(\frac{A+1}{\nu}+\frac{1}{A\tau}\Big)(E_n+1).\end{aligned}$$

For $\|u^{n+1}\|_{\dot{H}^1}$, we have

$$\begin{aligned}\|u^{n+1}\|_{\dot{H}^1} &\le \|u^n\|_{\dot{H}^1} + \frac{1}{A}\|(u^n)^3-u^n\|_{\dot{H}^1}\\ &\le \Big(1+\frac{1}{A}+\frac{3}{A}\|u^n\|_\infty^2\Big)\cdot\|u^n\|_{\dot{H}^1}.\end{aligned}$$

This completes the proof of Lemma 2.1. □

Lemma 2.2. *For any $n\ge 0$,*

$$\begin{aligned}&E_{n+1}-E_n+\left(A+\frac12+\sqrt{\frac{2\nu}{\tau}}\right)\|u^{n+1}-u^n\|_2^2\\ &\le \|u^{n+1}-u^n\|_2^2\cdot\Big(\|u^n\|_\infty^2+\frac12\|u^{n+1}\|_\infty^2\Big).\end{aligned} \tag{2.4}$$

Proof. In this proof we denote by $(\cdot,\cdot)$ the usual L^2 inner product. Recall

$$\frac{u^{n+1}-u^n}{\tau} = -\nu\Delta^2 u^{n+1} + A\Delta(u^{n+1}-u^n) + \Delta\Pi_N f(u^n).$$

Taking the inner product with $(-\Delta)^{-1}(u^{n+1}-u^n)$ on both sides and using the identity

$$b\cdot(b-a) = \frac{1}{2}(|b|^2 - |a|^2 + |b-a|^2), \qquad \forall\, a,b\in\mathbb{R}^d, \tag{2.5}$$

we get

$$\begin{aligned}&\frac{1}{\tau}\||\nabla|^{-1}(u^{n+1}-u^n)\|_2^2 + \frac{\nu}{2}(\|\nabla u^{n+1}\|_2^2 - \|\nabla u^n\|_2^2 + \|\nabla(u^{n+1}-u^n)\|_2^2)\\ &\quad + A\|u^{n+1}-u^n\|_2^2 = (\Delta\Pi_N f(u^n), (-\Delta)^{-1}(u^{n+1}-u^n)).\end{aligned} \tag{2.6}$$

Since all u^n have Fourier modes supported in $|k|\le N$, we have

$$(\Delta\Pi_N f(u^n), (-\Delta)^{-1}(u^{n+1}-u^n)) = -(f(u^n), u^{n+1}-u^n). \tag{2.7}$$

By the Fundamental Theorem of Calculus, we have (recall $f=F'$)

$$\begin{aligned}&F(u^{n+1}) - F(u^n)\\ &= f(u^n)(u^{n+1}-u^n) + \int_{u^n}^{u^{n+1}} f'(s)(u^{n+1}-s)ds\\ &= f(u^n)(u^{n+1}-u^n) + \int_{u^n}^{u^{n+1}} (3s^2-1)(u^{n+1}-s)ds\\ &= f(u^n)(u^{n+1}-u^n) + \frac{(u^{n+1}-u^n)^2}{4}\Big(3(u^n)^2 + (u^{n+1})^2 + 2u^n u^{n+1} - 2\Big).\end{aligned}$$

Thus

$$\begin{aligned}&\frac{1}{\tau}\||\nabla|^{-1}(u^{n+1}-u^n)\|_2^2 + E_{n+1} - E_n + \frac{\nu}{2}\|\nabla(u^{n+1}-u^n)\|_2^2 + (A+\frac{1}{2})\|u^{n+1}-u^n\|_2^2\\ &= \frac{1}{4}((u^{n+1}-u^n)^2, 3(u^n)^2 + (u^{n+1})^2 + 2u^n u^{n+1})\\ &\le \|u^{n+1}-u^n\|_2^2\cdot\frac{1}{4}\Big(3\|u^n\|_\infty^2 + \|u^{n+1}\|_\infty^2 + 2\|u^n\|_\infty\|u^{n+1}\|_\infty\Big)\\ &\le \|u^{n+1}-u^n\|_2\cdot\Big(\|u^n\|_\infty^2 + \frac{1}{2}\|u^{n+1}\|_\infty^2\Big).\end{aligned} \tag{2.8}$$

Finally observe

$$\begin{aligned}&\frac{1}{\tau}\||\nabla|^{-1}(u^{n+1}-u^n)\|_2^2 + \frac{\nu}{2}\|\nabla(u^{n+1}-u^n)\|_2^2\\ &\ge \sqrt{\frac{2\nu}{\tau}}\||\nabla^{-1}(u^{n+1}-u^n)\|_2\|\nabla(u^{n+1}-u^n)\|_2 \ge \sqrt{\frac{2\nu}{\tau}}\|u^{n+1}-u^n\|_2^2.\end{aligned}$$

The desired inequality then follows easily.

□

Remark. By using the auxiliary function $g(s) = F(u^n + s(u^{n+1}-u^n))$ and the Taylor expansion

$$g(1) = g(0) + g'(0) + \int_0^1 g''(s)(1-s)ds,$$

we get

$$\begin{aligned}F(u^{n+1}) &= F(u^n) + f(u^n)(u^{n+1}-u^n) - \frac{1}{2}(u^{n+1}-u^n)^2\\ &\quad + (u^{n+1}-u^n)^2\int_0^1 \tilde{f}'(u^n + s(u^{n+1}-u^n))(1-s)ds,\end{aligned}$$

where $\tilde{f}(z) = z^3$ and $\tilde{f}'(z) = 3z^2$ (for $z \in \mathbb{R}$). From this it is easy to see that

$$\text{LHS of (2.8)} \le \|u^{n+1} - u^n\|_2^2 \cdot \frac{3}{2}\max\{\|u^n\|_\infty^2, \|u^{n+1}\|_\infty^2\}.$$

This bound will also suffice.

Proof of Theorem 1.1. We inductively prove for all $n \ge 1$,

$$E_n \le E_0, \tag{2.9}$$

$$\|u^n\|_{H^{\frac{3}{2}}} \le c_1 \cdot \Big(\frac{A+1}{\nu} + \frac{1}{A\tau}\Big) \cdot (E_0 + 1), \tag{2.10}$$

where $c_1 > 0$ is the same absolute constant in Lemma 2.1.

We proceed in two steps. In Step 1 below, we first verify that if the statement holds for some $n \ge 1$, then it holds for $n+1$. In Step 2, we check the "base" case, namely for $n = 1$ the statement holds. We organize our whole argument in this reverse order (rather than checking the base case $n = 1$ first and then perform induction) because the verification for the base case $n = 1$ can be viewed as more or less a special case of the proof in Step 1.

`Step 1`: the induction step $n \Rightarrow n+1$. Assume the induction holds for some $n \ge 1$. We now verify the statement for $n+1$.

By Lemma 2.1, we have

$$\|u^{n+1}\|_{H^{\frac{3}{2}}} \le c_1 \cdot \Big(\frac{A+1}{\nu} + \frac{1}{A\tau}\Big) \cdot (E_n + 1) \le c_1 \cdot \Big(\frac{A+1}{\nu} + \frac{1}{A\tau}\Big) \cdot (E_0 + 1).$$

Thus we only need to check $E_{n+1} \le E_0$. In fact we shall show $E_{n+1} \le E_n$.

By Lemma 2.2, we only need to show the inequality

$$A + \frac{1}{2} + \sqrt{\frac{2\nu}{\tau}} \ge \|u^n\|_\infty^2 + \frac{1}{2}\|u^{n+1}\|_\infty^2. \tag{2.11}$$

We shall use the log-interpolation inequality (see (1.27) and choose $s = \frac{3}{2}$) for any f with mean zero:

$$\|f\|_{L^\infty(\mathbb{T}^2)} \le d_1 \cdot \|f\|_{\dot{H}^1(\mathbb{T}^2)} \cdot \log\Big(\|f\|_{H^{\frac{3}{2}}(\mathbb{T}^2)} + 3\Big), \tag{2.12}$$

where $d_1 > 0$ is an absolute constant.

In the rest of this proof, to ease the notation we shall use $X \lesssim_{E_0} Y$ to denote $X \le C_{E_0} Y$ where C_{E_0} is a constant depending only on E_0. Clearly

$$\begin{aligned}
\|u^n\|_\infty &\le d_1 \|u^n\|_{\dot{H}^1} \log\Big(\|u^n\|_{H^{\frac{3}{2}}} + 3\Big) \\
&\le d_1 \cdot \sqrt{\frac{2E_0}{\nu}} \cdot \log\Big(3 + c_1 \cdot \Big(\frac{A+1}{\nu} + \frac{1}{A\tau}\Big) \cdot (E_0+1)\Big) \\
&\lesssim_{E_0} \underbrace{\nu^{-\frac{1}{2}}(1 + \log A + |\log \nu|)}_{=:m_0} + \nu^{-\frac{1}{2}}|\log \tau|.
\end{aligned} \tag{2.13}$$

Here in the above inequality, if $\tau \gtrsim 1$ then it is not difficult to check that the $\log \tau$ term is not present. In the rest of this proof we shall just assume $0 < \tau \ll 1$ without loss of generality. The case $\tau \gtrsim 1$ is similar and even easier.

Now

$$\|u^n\|_\infty^2 \lesssim_{E_0} m_0^2 + \nu^{-1}|\log \tau|^2.$$

By (2.12) and Lemma 2.1, we have (below in the third inequality we drop $1/A$ since $A \geq 1$)

$$
\begin{aligned}
\|u^{n+1}\|_\infty &\lesssim \|u^{n+1}\|_{\dot{H}^1} \log\left(\|u^{n+1}\|_{H^{\frac{3}{2}}} + 3\right) \\
&\lesssim (1 + \frac{1}{A} + \frac{\|u^n\|_\infty^2}{A})\|u^n\|_{\dot{H}^1} \log\left(\|u^{n+1}\|_{H^{\frac{3}{2}}} + 3\right) \\
&\lesssim (1 + \frac{\|u^n\|_\infty^2}{A})\|u^n\|_{\dot{H}^1} \log\left(\|u^{n+1}\|_{H^{\frac{3}{2}}} + 3\right) \\
&\lesssim_{E_0} (1 + \frac{m_0^2 + \nu^{-1}|\log\tau|^2}{A}) \cdot \left(m_0 + \nu^{-\frac{1}{2}}|\log\tau|\right) \\
&\lesssim_{E_0} m_0 + \nu^{-\frac{1}{2}}|\log\tau| + \frac{m_0^3 + \nu^{-\frac{3}{2}}|\log\tau|^3}{A} \\
&\lesssim_{E_0} m_0 + \frac{m_0^3}{A} + 1 + \nu^{-\frac{3}{2}}|\log\tau|^3.
\end{aligned} \tag{2.14}
$$

Therefore

$$\|u^n\|_\infty^2 + \|u^{n+1}\|_\infty^2 \lesssim_{E_0} \left(m_0 + \frac{m_0^3}{A}\right)^2 + 1 + \nu^{-3}|\log\tau|^6.$$

Therefore to show the inequality (2.11), it suffices to prove

$$A + \sqrt{\frac{\nu}{\tau}} \geq C_{E_0} \cdot \left(\left(m_0 + \frac{m_0^3}{A}\right)^2 + 1 + \nu^{-3}|\log\tau|^6\right), \tag{2.15}$$

where

$$m_0 = \nu^{-\frac{1}{2}}(1 + \log A + |\log\nu|).$$

Now we discuss two cases.

`Case 1`: $\sqrt{\frac{\nu}{\tau}} \geq C_{E_0}\nu^{-3}|\log\tau|^6$. In this case we choose A such that

$$A \gg_{E_0} m_0^2 = \nu^{-1}(1 + \log A + |\log\nu|)^2.$$

Clearly for $\nu \gtrsim 1$, we just need to choose $A \gg_{E_0} 1$. On the other hand, for $0 < \nu \ll 1$, it suffices to take

$$A = \beta \cdot \nu^{-1}|\log\nu|^2,$$

with β sufficiently large depending only on E_0. Thus in both cases if we take

$$A = \beta \cdot \max\{\nu^{-1}|\log\nu|^2, 1\},$$

with $\beta \gg_{E_0} 1$, then (2.15) holds.

`Case 2`: $\sqrt{\frac{\nu}{\tau}} \leq C_{E_0}\nu^{-3}|\log\tau|^6$. In this case we have

$$|\log\tau| \lesssim_{E_0} 1 + |\log\nu|.$$

In this case we will not prove (2.15) but prove (2.11) directly. We first go back to the bound on $\|u^n\|_\infty$. Easy to check that

$$
\begin{aligned}
&\|u^n\|_\infty \lesssim_{E_0} m_0, \\
&\|u^{n+1}\|_\infty \lesssim_{E_0} \left(1 + \frac{m_0^2}{A}\right) m_0.
\end{aligned}
$$

The needed inequality on A then takes the form

$$A \geq C_{E_0} \cdot \left(1 + m_0 + \frac{m_0^3}{A}\right)^2.$$

Again we only need to choose A such that $A \gg_{E_0} m_0^2$. The same choice of A as in Case 1 (with β larger if necessary) works.

Concluding from both cases, we have proved the inequality (2.11) holds. This completes the induction step for $n \Rightarrow n + 1$.

`Step 2`: verification of the base step $n=1$. By Lemma 2.1 we have

$$\|u^1\|_{H^{\frac{3}{2}}} \le c_1 \cdot \Big(\frac{A+1}{\nu} + \frac{1}{A\tau}\Big) \cdot (E_0+1).$$

Therefore we only need to check $E_1 \le E_0$. This amounts to checking the inequality

$$A + \frac{1}{2} + \sqrt{\frac{2\nu}{\tau}} \ge \|\Pi_N u_0\|_\infty^2 + \frac{1}{2}\|u^1\|_\infty^2.$$

Now clearly the estimate of $\|u^1\|_\infty$ proceeds in the same way as in Step 1. On the other hand since

$$\|\Pi_N u_0\|_\infty \lesssim \|u_0\|_\infty,$$

we just need to choose A such that

$$A \gg_{E_0} \|u_0\|_\infty^2 + \nu^{-1}|\log \nu|^2 + 1.$$

This completes the proof of Theorem 1.1.

2.2. **Proof of Theorem 1.2.** This is similar to the proof of Theorem 1.1. Therefore we only sketch the needed modifications. In terms of scaling it is useful to think of ∇h^n as u^n in Theorem 1.1. Write (1.22) as

$$h^{n+1} = \frac{1-A\tau\Delta}{1+\nu\tau\Delta^2 - A\tau\Delta} h^n + \frac{\tau \Pi_N}{1+\nu\tau\Delta^2 - A\tau\Delta} \nabla \cdot \big(g(\nabla h^n)\big). \tag{2.16}$$

In place of Lemma 2.1 we have the following lemma. We omit the proof since it is quite similar.

Lemma 2.3. *There is an absolute constants $c_1 > 0$ such that*

$$\|h^{n+1}\|_{H^{\frac{5}{2}}(\mathbb{T}^2)} \le c_1 \cdot (\frac{A+1}{\nu} + \frac{1}{A\tau}) \cdot (E_n+1),$$
$$\|h^{n+1}\|_{\dot H^2(\mathbb{T}^2)} \le (1 + \frac{1}{A} + \frac{3}{A}\|\nabla h^n\|_\infty^2) \cdot \|h^n\|_{\dot H^2(\mathbb{T}^2)}.$$

Here $E_n = E(h^n)$.

Lemma 2.4. *For any $n \ge 0$,*

$$\begin{aligned} &E_{n+1} - E_n + (A + \frac{1}{2} + \sqrt{\frac{2\nu}{\tau}})\|\nabla(h^{n+1}-h^n)\|_2^2 \\ &\le \|\nabla(h^{n+1}-h^n)\|_2^2 \cdot \frac{3}{2}\max\{\|\nabla h^n\|_\infty^2, \|\nabla h^{n+1}\|_\infty^2\}. \end{aligned} \tag{2.17}$$

Proof. Taking inner product with $(h^{n+1}-h^n)$ on both sides of (1.22), we get

$$\begin{aligned} &\frac{1}{\tau}\|h^{n+1}-h^n\|_2^2 + \frac{\nu}{2}(\|\Delta h^{n+1}\|_2^2 - \|\Delta h^n\|_2^2 + \|\Delta(h^{n+1}-h^n)\|_2^2) + A\|\nabla(h^{n+1}-h^n)\|_2^2 \\ = &-(g(\nabla h^n), \nabla(h^{n+1}-h^n)). \end{aligned}$$

Recall $g(z) = (|z|^2-1)z = \nabla G$ and $G(z) = \frac{1}{4}(|z|^2-1)^2$. Introduce

$$H(s) = G(\nabla h^n + s(\nabla h^{n+1} - \nabla h^n)).$$

By using the expansion

$$H(1) = H(0) + H'(0) + \int_0^1 H''(s)(1-s)ds,$$

we get

$$\begin{aligned} G(\nabla h^{n+1}) - G(\nabla h^n) &= g(\nabla h^n)\cdot(\nabla h^{n+1} - \nabla h^n) \\ &+ \sum_{i,j=1}^2 \partial_i(h^{n+1}-h^n)\partial_j(h^{n+1}-h^n)\int_0^1 (\partial_{ij}G)(\nabla h^n + s(\nabla h^{n+1}-\nabla h^n))(1-s)ds, \end{aligned}$$

Now denote $\tilde{G}(z)=\frac{1}{4}|z|^4$. Then

$$\begin{aligned}&E_{n+1}-E_n+\frac{1}{\tau}\|h^{n+1}-h^n\|_2^2+\frac{\nu}{2}\|\Delta(h^{n+1}-h^n)\|_2^2+(A+\frac{1}{2})\|\nabla(h^{n+1}-h^n)\|_2^2\\ =&\sum_{i,j=1}^{2}\Big(\partial_i(h^{n+1}-h^n)\partial_j(h^{n+1}-h^n)\int_0^1(\partial_{ij}\tilde{G})(\nabla h^n+s(\nabla h^{n+1}-h^n))(1-s)ds,\ 1\Big),\end{aligned}$$

where 1 represents the constant function with value 1.

Now since $\partial_{ij}\tilde{G}(z)=|z|^2\delta_{ij}+2z_jz_i$, we have the point-wise bound $|(\partial_{ij}\tilde{G})(z)|\le 3|z|^2$. Thus

$$\|(\partial_{ij}\tilde{G})(\nabla h^n+s(\nabla h^{n+1}-h^n))\|_\infty\le 3\max\{\|\nabla h^n\|_\infty^2,\|\nabla h^{n+1}\|_\infty^2\}.$$

The desired inequality now follows from this and the interpolation inequality

$$\|\nabla h\|_2\le\|h\|_2^{\frac12}\|\Delta h\|_2^{\frac12}. \tag{2.18}$$

This completes the proof of the lemma. □

Proof of Theorem 1.2. We only need to check the induction hypothesis

$$\begin{aligned}&E_n\le E_0,\\ &\|h^n\|_{H^{\frac52}}\le c_1\cdot\Big(\frac{A+1}{\nu}+\frac{1}{A\tau}\Big)\cdot(E_0+1),\end{aligned}$$

for $n+1$. Here $c_1>0$ is the same absolute constant in Lemma 2.3.

By Lemma 2.3, we have

$$\|h^{n+1}\|_{H^{\frac52}}\le c_1\cdot\Big(\frac{A+1}{\nu}+\frac{1}{A\tau}\Big)\cdot(E_n+1)\le c_1\cdot\Big(\frac{A+1}{\nu}+\frac{1}{A\tau}\Big)\cdot(E_0+1).$$

Thus we only need to check $E_{n+1}\le E_n$. By Lemma 2.4, this amounts to proving the inequality

$$A+\frac12\ge\frac32\max\{\|\nabla h^n\|_\infty^2,\|\nabla h^{n+1}\|_\infty^2\}. \tag{2.19}$$

We shall again use the inequality

$$\|f\|_{L^\infty(\mathbb{T}^2)}\le d_1\cdot\|f\|_{\dot{H}^1(\mathbb{T}^2)}\cdot\log\Big(\|f\|_{H^{\frac32}(\mathbb{T}^2)}+3\Big), \tag{2.20}$$

where $d_1>0$ is an absolute constant, and f has mean zero. Clearly

$$\begin{aligned}\|\nabla h^n\|_\infty&\le d_1\|h^n\|_{\dot{H}^2}\log(\|h^n\|_{H^{\frac52}}+3)\\ &\le d_1\cdot\sqrt{\frac{2E_0}{\nu}}\cdot\log\Big(3+c_1\cdot\Big(\frac{A+1}{\nu}+\frac{1}{A\tau}\Big)\cdot(E_0+1)\Big).\end{aligned} \tag{2.21}$$

The rest of the argument now is similar to that in the Proof of Theorem 1.1. We omit further repetitive details.

3. Bounds on the PDE solution of CH

Consider

$$\begin{cases}\partial_t w=-\nu\Delta^2 w+\Delta(f(w)),\\ w\Big|_{t=0}=w_0.\end{cases} \tag{3.1}$$

Recall that the corresponding energy $E(\cdot)$ is defined by (1.5).

Proposition 3.1. *Let $0 < \nu \lesssim 1$. Assume the initial data $w_0 \in H^2(\mathbb{T}^2)$ with mean zero. Assume $\|w_0\|_\infty \lesssim 1$. Then*

$$\sup_{0\le t<\infty} \|w(t)\|_\infty \lesssim \sqrt{\frac{E_0}{\nu}} \cdot \Bigl(|\log \nu| + |\log E_0| + 1 \Bigr), \tag{3.2}$$

where $E_0 = E(w_0)$.

Proof. First consider the regime $0 < t \ll \nu$. Write

$$w(t) = e^{-\nu t \Delta^2} w_0 + \int_0^t \Delta e^{-\nu(t-s)\Delta^2} f(w(s)) ds.$$

Then

$$\begin{aligned} \|w(t)\|_\infty &\lesssim \|w_0\|_\infty + \int_0^t \nu^{-\frac12}(t-s)^{-\frac12} \|f(w(s))\|_\infty ds \\ &\lesssim \|w_0\|_\infty + \nu^{-\frac12} t^{\frac12} \cdot \Bigl(\|w\|^3_{L^\infty_{s,x}([0,t])} + \|w\|_{L^\infty_{s,x}([0,t])} \Bigr). \end{aligned} \tag{3.3}$$

By using a continuity argument (on the quantity $\|w\|_{L^\infty_{s,x}([0,t])}$), we get

$$\sup_{0\le t\le \epsilon_0 \nu} \|w(t)\|_\infty \lesssim 1, \tag{3.4}$$

where $\epsilon_0 > 0$ is a sufficiently small absolute constant. (Strictly speaking the value of ϵ_0 depends on the implied constants hidden in the inequalities $\|w_0\|_\infty \lesssim 1$ and $0 < \nu \lesssim 1$.)

Next we consider the L^∞ bound in the time regime $t \ge \epsilon_0 \nu$. First observe that by using energy conservation, we have

$$\|\nabla w(t)\|_2 \lesssim \sqrt{\frac{E_0}{\nu}}.$$

Set $t_1 = t - \frac12 \epsilon_0 \nu$. Then

$$w(t) = e^{-\nu(t-t_1)\Delta^2} w(t_1) + \int_{t_1}^t \Delta e^{-\nu(t-s)\Delta^2} f(w(s)) ds.$$

We bound the $\dot H^{1+}$-norm of w as

$$\begin{aligned} \|w(t)\|_{\dot H^{1+}} &\lesssim \||\nabla|^{1+} e^{-\nu(t-t_1)\Delta^2} w(t_1)\|_2 + \int_{t_1}^t \||\nabla|^{3+} e^{-\nu(t-s)\Delta^2} (w(s)^3 - w(s))\|_2 ds \\ &\lesssim (\nu(t-t_1))^{0-} \|w(t_1)\|_{\dot H^1} + \int_{t_1}^t (\nu(t-s))^{-\frac34 -} (\|w(s)\|^3_{\dot H^1} + \|w(s)\|_{\dot H^1}) ds \\ &\lesssim \nu^{-\frac12 -} \sqrt{E_0} + \nu^{-\frac34 -} \cdot \nu^{\frac14 -} \cdot \biggl(\Bigl(\frac{E_0}{\nu}\Bigr)^{\frac32} + \Bigl(\frac{E_0}{\nu}\Bigr)^{\frac12} \biggr) \\ &\lesssim \nu^{-1} \cdot \biggl(\Bigl(\frac{E_0}{\nu}\Bigr)^{\frac32} + \Bigl(\frac{E_0}{\nu}\Bigr)^{\frac12} \biggr). \end{aligned} \tag{3.5}$$

Then (recall that w has mean zero)

$$\begin{aligned} \|w(t)\|_\infty &\lesssim \|\nabla w(t)\|_2 \cdot \log(10 + \|w(t)\|_{H^{1+}}) \\ &\lesssim \sqrt{\frac{E_0}{\nu}} \cdot \Bigl(1 + |\log \nu| + |\log E_0| \Bigr). \end{aligned}$$

This completes the proof of Proposition 3.1. □

Remark. By using the method in [19], one can prove a wellposedness result for $w_0 \in L^2(\mathbb{T}^2)$. However we shall not need this refinement here.

Proposition 3.2. *Assume the initial data w_0 has mean zero and $w_0 \in H^s(\mathbb{T}^2)$, $s \geq 4$. Then*

$$\int_0^T \|\partial_t \Delta w\|_2^2 dt \lesssim_{\nu, w_0} 1 + T. \tag{3.6}$$

Proof. To ease the notation we shall write $\lesssim_{\nu,w_0}$ as $\lesssim$ throughout this proof. By using the smoothing effect, it is easy to show that

$$\sup_{1 \lesssim t < \infty} \|\partial_t w\|_{H^{100}} \lesssim 1. \tag{3.7}$$

From energy conservation, we have

$$\int_0^\infty \||\nabla|^{-1} \partial_t w\|_2^2 dt \lesssim 1. \tag{3.8}$$

Thus by interpolation,

$$\int_1^\infty \|\Delta \partial_t w\|_2^4 dt \lesssim 1. \tag{3.9}$$

This implies

$$\int_1^T \|\Delta \partial_t w\|_2^2 dt \lesssim 1 + \sqrt{T}. \tag{3.10}$$

Now we only need to show

$$\int_0^1 \|\Delta \partial_t w\|_2^2 dt \lesssim 1. \tag{3.11}$$

Observe

$$\partial_t \Delta w = -\nu \Delta^3 w + \Delta^2 f(w).$$

Multiplying both sides by $\partial_t \Delta w$ and integrating by parts, we get

$$\|\partial_t \Delta w\|_2^2 = -\frac{\nu}{2} \frac{d}{dt} (\|\Delta^2 w\|_2^2) + \int_\Omega \Delta^2 (f(w)) \partial_t \Delta w dx.$$

Thus

$$\begin{aligned} \frac{\nu}{2} \frac{d}{dt} (\|\Delta^2 w\|_2^2) &\leq -\|\partial_t \Delta w\|_2^2 + \|\Delta^2 f(w)\|_2 \cdot \|\partial_t \Delta w\|_2 \\ &\leq -\frac{1}{2} \|\partial_t \Delta w\|_2^2 + \text{const} \cdot \Big(\|w\|_{H^4}^3 + \|w\|_{H^4} \Big). \end{aligned} \tag{3.12}$$

From this (and standard H^4 global wellposedness theory), we get

$$\int_0^1 \|\partial_t \Delta w\|_2^2 dt \lesssim 1. \tag{3.13}$$

The desired inequality then follows. □

4. Error estimate for CH

In this section we give the estimate for CH in L^2.

4.1. **Auxiliary L^2 error estimate for near solutions.** Consider

$$\begin{cases} \frac{v^{n+1}-v^n}{\tau} = -\nu\Delta^2 v^{n+1} + A\Delta(v^{n+1}-v^n) + \Delta\Pi_N f(v^n) + \Delta\tilde{G}_n^1, \quad n\ge 0, \\ \frac{\tilde{v}^{n+1}-\tilde{v}^n}{\tau} = -\nu\Delta^2 \tilde{v}^{n+1} + A\Delta(\tilde{v}^{n+1}-\tilde{v}^n) + \Delta\Pi_N f(\tilde{v}^n) + \Delta\tilde{G}_2^n, \qquad n\ge 0, \\ v^0 = v_0, \quad \tilde{v}^0 = \tilde{v}_0, \end{cases} \tag{4.1}$$

where v_0 and $\tilde{v}_0$ has mean zero. Denote $\tilde{G}^n = \tilde{G}_1^n - \tilde{G}_2^n$.

We first state and prove a simple lemma.

Lemma 4.1 (Discrete Gronwall inequality)**.** *Let $\tau>0$ and $y_n\ge 0$, $\tilde{\alpha}_n\ge 0$, $\tilde{\beta}_n \ge 0$ for $n=0,1,2,\cdots$. Suppose*

$$\frac{y_{n+1}-y_n}{\tau} \le \tilde{\alpha}_n y_n + \tilde{\beta}_n, \quad \forall\, n\ge 0.$$

Then for any $m\ge 1$, we have

$$y_m \le \exp\Big(\tau\sum_{n=0}^{m-1}\tilde{\alpha}_n\Big) y_0 + \tau \sum_{k=0}^{m-1} \exp\Big(\tau \sum_{j=k+1}^{m-1}\tilde{\alpha}_j\Big)\tilde{\beta}_k. \tag{4.2}$$

In particular

$$y_m \le \exp\Big(\tau\sum_{n=0}^{m-1}\tilde{\alpha}_n\Big)(y_0 + \tau\sum_{k=0}^{m-1}\tilde{\beta}_k). \tag{4.3}$$

Proof. Clearly

$$y_{n+1} \le (1+\tilde{\alpha}_n\tau)y_n + \tau\tilde{\beta}_n \le e^{\tau\tilde{\alpha}_n} y_n + \tau\tilde{\beta}_n, \quad \forall\, n\ge 0.$$

Thus

$$\exp\Big(-\tau\sum_{j=0}^{n}\tilde{\alpha}_j\Big)y_{n+1} \le \exp\Big(-\tau\sum_{j=0}^{n-1}\tilde{\alpha}_j\Big)y_n + \tau\exp\Big(-\tau\sum_{j=0}^{n}\tilde{\alpha}_j\Big)\tilde{\beta}_n.$$

Summing n from 0 to $m-1$, we get

$$\exp\Big(-\tau\sum_{j=0}^{m-1}\tilde{\alpha}_j\Big)y_m \le y_0 + \tau\sum_{n=0}^{m-1}\exp\Big(-\tau\sum_{j=0}^{n}\tilde{\alpha}_j\Big)\tilde{\beta}_n.$$

Thus (4.2) is obtained.

□

Proposition 4.1. *For solutions of* (4.1)*, assume for some $N_1>0$, $N_2>0$,*

$$\sup_{n\ge0}\|\tilde{v}^n\|_\infty \le N_1, \quad \sup_{n\ge 0}\|\nabla v^n\|_2 \le N_2, \quad \sup_{n\ge0}\|\nabla\tilde{v}^n\|_2\le N_2. \tag{4.4}$$

Then for any $m\ge1$,

$$\begin{aligned} &\|v^m - \tilde{v}^m\|_2^2 \\ &\le \exp\Big(m\tau\cdot\frac{C_1\cdot(1+N_1^4+N_2^4)}{\nu}\Big)\cdot\Big(\|v_0-\tilde{v}_0\|_2^2 + A\tau\|\nabla(v_0-\tilde{v}_0)\|_2^2 + \frac{4\tau}{\nu}\sum_{n=0}^{m-1}\|\tilde{G}^n\|_2^2\Big), \end{aligned} \tag{4.5}$$

where $C_1>0$ is an absolute constant.

Remark. The same proposition holds if Π_N is replaced by the identity operator.

Proof of Proposition 4.1. Denote $e^n = v^n - \tilde{v}^n$. Then

$$\frac{e^{n+1}-e^n}{\tau} = -\nu\Delta^2 e^{n+1} + A\Delta(e^{n+1}-e^n) + \Delta\Pi_N(f(v^n)-f(\tilde{v}^n)) + \Delta\tilde{G}^n. \tag{4.6}$$

Taking L^2-inner product with e^{n+1} on both sides, we get

$$\begin{aligned}&\frac{1}{2\tau}(\|e^{n+1}\|_2^2 - \|e^n\|_2^2 + \|e^{n+1}-e^n\|_2^2)\\ &\qquad + \nu\|\Delta e^{n+1}\|_2^2 + \frac{A}{2}(\|\nabla e^{n+1}\|_2^2 - \|\nabla e^n\|_2^2 + \|\nabla(e^{n+1}-e^n)\|_2^2)\\ &= (\tilde{G}^n, \Delta e^{n+1}) + (f(v^n)-f(\tilde{v}^n), \Delta\Pi_N e^{n+1}).\end{aligned} \tag{4.7}$$

Obviously

$$|(\tilde{G}^n, \Delta e^{n+1})| \le \frac{2\|\tilde{G}^n\|_2^2}{\nu} + \frac{\nu}{8}\|\Delta e^{n+1}\|_2^2.$$

On the other hand, recalling $f'(z) = 3z^2 - 1$, we get

$$\begin{aligned}f(v^n) - f(\tilde{v}^n) &= \int_0^1 f'(\tilde{v}^n + se^n)ds e^n\\ &= (a_1 + a_2(\tilde{v}^n)^2)e^n + a_3\tilde{v}^n(e^n)^2 + a_4(e^n)^3,\end{aligned}$$

where a_i, $i = 1, \cdots, 4$ are constants which can be computed explicitly.

We now estimate the contribution of each term. In the rest of this proof, to ease the notation, we shall denote by C an absolute constant whose value may change from line to line. Clearly

$$\begin{aligned}&|((a_1 + a_2(\tilde{v}^n)^2)e^n, \Delta e^{n+1})|\\ &\le C\cdot(1+\|\tilde{v}^n\|_\infty^2)\|e^n\|_2\cdot\|\Delta e^{n+1}\|_2 \frac{C(1+N_1^4)}{\nu}\|e^n\|_2^2 + \frac{\nu}{8}\|\Delta e^{n+1}\|_2^2.\end{aligned} \tag{4.8}$$

By using the interpolation inequality $\|f\|_4 \lesssim \|f\|_2^{\frac12}\|\nabla f\|_2^{\frac12}$, we get

$$\begin{aligned}&|(a_3\tilde{v}^n(e^n)^2, \Delta e^{n+1})|\\ &\le C\|\tilde{v}^n\|_\infty\cdot\|e^n\|_4^2\cdot\|\Delta e^{n+1}\|_2 \le C\cdot\frac{N_1^2\cdot\|e^n\|_4^4}{\nu} + \frac{\nu}{8}\|\Delta e^{n+1}\|_2^2\\ &\le C\cdot\frac{N_1^2}{\nu}\|\nabla e^n\|_2^2\|e^n\|_2^2 + \frac{\nu}{8}\|\Delta e^{n+1}\|_2^2 \le C\frac{N_1^2N_2^2}{\nu}\|e^n\|_2^2 + \frac{\nu}{8}\|\Delta e^{n+1}\|_2^2.\end{aligned} \tag{4.9}$$

Similarly

$$\begin{aligned}&|(a_4(e^n)^3, \Delta e^{n+1})|\\ &\le \frac{C}{\nu}\|e^n\|_6^6 + \frac{\nu}{8}\|\Delta e^{n+1}\|_2^2 \le \frac{C}{\nu}\|e^n\|_2^2\|\nabla e^n\|_2^4 + \frac{\nu}{8}\|\Delta e^{n+1}\|_2^2 \le C\frac{N_2^4}{\nu}\|e^n\|_2^2 + \frac{\nu}{8}\|\Delta e^{n+1}\|_2^2.\end{aligned} \tag{4.10}$$

Collecting the estimates, we get

$$\begin{aligned}&\frac{\|e^{n+1}\|_2^2 - \|e^n\|_2^2}{\tau} + A(\|\nabla e^{n+1}\|_2^2 - \|\nabla e^n\|_2^2)\\ &\le \frac{4}{\nu}\|\tilde{G}^n\|_2^2 + C\frac{1+N_1^4+N_2^4}{\nu}\|e^n\|_2^2.\end{aligned} \tag{4.11}$$

Define

$$y_n = \|e^n\|_2^2 + A\tau\|\nabla e^n\|_2^2, \quad \tilde{\alpha} = C\frac{1+N_1^4+N_2^4}{\nu}, \quad \tilde{\beta}_n = \frac{4}{\nu}\|\tilde{G}^n\|_2^2.$$

Then obviously

$$\frac{y_{n+1}-y_n}{\tau} \le \tilde{\alpha}y_n + \tilde{\beta}_n.$$

The desired result then follows from Lemma 4.1. □

4.2. L^2 **error estimate for CH (Proof of Theorem 1.3).** In this proof to ease the notation, we shall denote by C a constant depending only on (ν, u_0). The value of C may vary from line to line. For any two quantities X and Y, we shall write $X \lesssim Y$ if $X \le CY$. Note that we shall still keep track of the dependence on the parameter A and also the regularity index s.

We need to compare

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} = -\nu\Delta^2 u^{n+1} + A\Delta(u^{n+1}-u^n) + \Delta\Pi_N f(u^n), \\ \partial_t u = -\nu\Delta^2 u + \Delta f(u), \\ \tilde{u}^0 = \Pi_N u_0, \quad u(0) = u_0. \end{cases} \tag{4.12}$$

We first rewrite the PDE solution u in the discretized form. Note that for one-variable function $h = h(t)$, we have the formulae

$$\frac{1}{\tau}\int_{t_n}^{t_{n+1}} h(t)dt = h(t_n) + \frac{1}{\tau}\int_{t_n}^{t_{n+1}} h'(t)\cdot(t_{n+1}-t)dt, \tag{4.13}$$

$$\frac{1}{\tau}\int_{t_n}^{t_{n+1}} h(t)dt = h(t_{n+1}) + \frac{1}{\tau}\int_{t_n}^{t_{n+1}} h'(t)\cdot(t_n-t)dt. \tag{4.14}$$

By using the above formulae and integrating the PDE for u on the time interval $[t_n, t_{n+1}]$, we get

$$\begin{aligned} &\frac{u(t_{n+1})-u(t_n)}{\tau} \\ = &-\nu\Delta^2 u(t_{n+1}) + A\Delta(u(t_{n+1})-u(t_n)) + \Delta\Pi_N f(u(t_n)) + \Delta\Pi_{>N} f(u(t_n)) + \Delta\tilde{G}^n, \end{aligned} \tag{4.15}$$

where $\Pi_{>N} = \mathrm{Id} - \Pi_N$ (Id is the identity operator) and

$$\tilde{G}^n = -\frac{\nu}{\tau}\int_{t_n}^{t_{n+1}} \partial_t\Delta u\cdot(t_n-t)dt + \frac{1}{\tau}\int_{t_n}^{t_{n+1}} \partial_t(f(u))\cdot(t_{n+1}-t)dt - A\int_{t_n}^{t_{n+1}} \partial_t u dt. \tag{4.16}$$

Now

$$\|\tilde{G}^n\|_2 \le \nu\int_{t_n}^{t_{n+1}} \|\partial_t\Delta u\|_2 dt + \int_{t_n}^{t_{n+1}} \|\partial_t u\|_2 dt(\|f'(u)\|_\infty + A). \tag{4.17}$$

By Proposition 3.1, we have $\|u\|_\infty \lesssim 1$. Since $\|\partial_t u\|_2 \lesssim \|\Delta\partial_t u\|_2$, we get

$$\begin{aligned} \|\tilde{G}^n\|_2 &\lesssim (1+A)\int_{t_n}^{t_{n+1}} \|\partial_t\Delta u\|_2 dt \\ &\lesssim (1+A)\cdot\left(\int_{t_n}^{t_{n+1}} \|\partial_t\Delta u\|_2^2 dt\right)^{\frac{1}{2}}\cdot\sqrt{\tau}. \end{aligned}$$

Therefore by Proposition 3.2,

$$\sum_{n=0}^{m-1} \|\tilde{G}^n\|_2^2 \lesssim (1+A)^2\tau\int_0^{t_m} \|\partial_t\Delta u\|_2^2 dt \lesssim (1+A)^2\tau\cdot(1+t_m).$$

It is easy to check that

$$\sup_{t\ge 0} \|u(t)\|_{H^s} \lesssim_s 1$$

which implies

$$\sup_{t\ge 0} \|f(u(t))\|_{H^s} \lesssim_s 1. \tag{4.18}$$

This gives

$$\sum_{n=0}^{m-1} \|\Pi_{>N} f(u(t_n))\|_2^2 \lesssim_s \; N^{-2s} t_m/\tau. \tag{4.19}$$

Therefore

$$\tau \sum_{n=0}^{m-1} (\|\tilde{G}^n\|_2^2 + \|\Pi_{>N} f(u(t_n))\|_2^2) \lesssim_s (1+t_m)(\tau^2 + N^{-2s})(1+A)^2. \tag{4.20}$$

Note that

$$\|u_0 - \Pi_N u_0\|_2 \lesssim_s N^{-s}, \quad \|\nabla u_0 - \nabla \Pi_N u_0\|_2 \lesssim_s N^{-(s-1)}.$$

By Proposition 4.1, we then get

$$\|u^m - u(t_m)\|_2^2 \lesssim_s (1+A)^2 e^{Ct_m} \Big(N^{-2s} + \tau \cdot N^{-2(s-1)} + (1+t_m)(\tau^2 + N^{-2s}) \Big).$$

Since by assumption we have $s \geq 4$, clearly by Cauchy-Schwartz

$$\tau \cdot N^{-2(s-1)} \lesssim \tau^2 + N^{-4(s-1)} \lesssim \tau^2 + N^{-2s}.$$

This implies

$$\|u^m - u(t_m)\|_2 \lesssim_s (1+A) e^{Ct_m} (N^{-s} + \tau).$$

Remark 4.1. From the above analysis, it is clear that our regularity assumption H^s, $s \geq 4$ on the initial data comes from bounding the term

$$\int \|\partial_t \Delta u\|_2^2 dt$$

which in turn arose from rewriting the diffusion term $-\nu \Delta^2 u$ into the time-discretized form. Recall $\partial_t u = -\nu\Delta^2 u + \Delta(f(u))$. For $0 < t \ll 1$, the linear effect is dominant and one can roughly regard $\partial_t u \sim \Delta^2 P_{<t^{-\frac14}} u$, where $P_{<t^{-\frac14}}$ is the Littlewood-Paley projection to the frequency regime $|\xi| \lesssim t^{-\frac14}$. Heuristically speaking

$$\|\partial_t \Delta u\|_2^2 \sim (t^{-\frac12} \|P_{<t^{-\frac14}} \Delta^2 u\|_2)^2 \sim t^{-1} \|P_{<t^{-\frac14}} \Delta^2 u\|_2^2$$

which is barely non-integrable in t, provided we assume H^4 regularity on u. Of course a well-known technique in these situations is to use the maximal regularity estimates of the linear semi-group to get integrability in t. In the L^2 case the usual energy estimate suffices and this is why we need H^4 regularity on the initial data.

5. Error estimate for MBE

5.1. **Auxiliary H^1 estimate for MBE.** For MBE we need to consider

$$\begin{cases} \dfrac{q^{n+1} - q^n}{\tau} = -\nu \Delta^2 q^{n+1} + A\Delta(q^{n+1} - q^n) + \nabla \cdot \Pi_N(g(\nabla q^n)) + \Delta \tilde{G}_1^n, \\ \dfrac{\tilde{q}^{n+1} - \tilde{q}^n}{\tau} = -\nu \Delta^2 \tilde{q}^{n+1} + A\Delta(\tilde{q}^{n+1} - \tilde{q}^n) + \nabla \cdot \Pi_N(g(\nabla \tilde{q}^n)) + \Delta \tilde{G}_2^n, \\ q^0 = q_0, \quad \tilde{q}^0 = \tilde{q}_0, \end{cases} \tag{5.1}$$

where we recall $g(z) = (|z|^2 - 1)z$ for $z \in \mathbb{R}^2$. As before q_0 and $\tilde{q}_0$ are assumed to have mean zero. Denote $\tilde{G}^n = \tilde{G}_1^n - \tilde{G}_2^n$.

Proposition 5.1. *Assume for some* $N_1 > 0$

$$\sup_{n\geq 0} \Big(\|\nabla \tilde{q}^n\|_\infty + \|\Delta \tilde{q}^n\|_2 + \|\Delta q^n\|_2 \Big) \leq N_1. \tag{5.2}$$

Then for any $m \geq 1$,

$$\|\nabla(q^n - \tilde{q}^n)\|_2^2 \leq e^{m\tau \cdot \frac{C_1 \cdot (1+N_1^4)}{\nu}} \Big(\|\nabla(q^0 - \tilde{q}^0)\|_2^2 + A\tau \|\Delta(q^0 - \tilde{q}^0)\|_2^2 + \frac{2\tau}{\nu} \sum_{n=0}^{m-1} \|\nabla \tilde{G}^n\|_2^2 \Big), \tag{5.3}$$

where $C_1 > 0$ *is an absolute constant.*

Proof. Denote $e^n = q^n - \tilde{q}^n$. Then

$$\frac{e^{n+1}-e^n}{\tau} = -\nu\Delta^2 e^{n+1} + A\Delta(e^{n+1}-e^n) + \nabla\cdot\Pi_N(g(\nabla q^n) - g(\nabla\tilde{q}^n)) - \Delta\tilde{G}^n.$$

Taking L^2-inner product with $(-\Delta)e^{n+1}$ on both sides, we get

$$\begin{aligned}
&\frac{1}{2\tau}(\|\nabla e^{n+1}\|_2^2 - \|\nabla e^n\|_2^2 + \|\nabla(e^{n+1}-e^n)\|_2^2) + \nu\|\Delta\nabla e^{n+1}\|_2^2 \\
&\quad + \frac{A}{2}(\|\Delta e^{n+1}\|_2^2 - \|\Delta e^n\|_2^2 + \|\Delta(e^{n+1}-e^n)\|_2^2) \\
=\;& \underbrace{-(\tilde{\nabla} G^n, \Delta\nabla e^{n+1})}_{I_1} + \underbrace{(g(\nabla q^n) - g(\nabla\tilde{q}^n),\, \nabla\Delta\Pi_N e^{n+1})}_{I_2}.
\end{aligned} \tag{5.4}$$

For the first term on the RHS of (5.4), we simply bound it as

$$|I_1| \le \frac{1}{\nu}\|\nabla\tilde{G}^n\|_2^2 + \frac{\nu}{4}\|\Delta\nabla e^{n+1}\|_2^2. \tag{5.5}$$

For the second term I_2, recalling $g(z) = (|z|^2-1)z$, we have

$$g(\nabla q^n) - g(\nabla\tilde{q}^n) = O(\partial e^n) + O((\partial\tilde{q}^n)^2\cdot\partial e^n) + O((\partial\tilde{q}^n)\cdot(\partial e^n)^2) + O((\partial e^n)^3).$$

Then

$$\begin{aligned}
\|g(\nabla q^n) - g(\nabla\tilde{q}^n)\|_2 &\lesssim (1+N_1^2)\|\nabla e^n\|_2 + N_1\|\nabla e^n\|_4^2 + \|\nabla e^n\|_6^3 \\
&\lesssim (1+N_1^2)\|\nabla e^n\|_2 + N_1\|\nabla e^n\|_2\|\Delta e^n\|_2 + \|\nabla e^n\|_2\|\Delta e^n\|_2^2 \\
&\lesssim (1+N_1^2)\|\nabla e^n\|_2.
\end{aligned} \tag{5.6}$$

Thus

$$|I_2| \le C\cdot\frac{1+N_1^4}{\nu}\|\nabla e^n\|_2^2 + \frac{\nu}{2}\|\Delta\nabla e^{n+1}\|_2^2. \tag{5.7}$$

We then obtain

$$\begin{aligned}
&\frac{\|\nabla e^{n+1}\|_2^2 - \|\nabla e^n\|_2^2}{\tau} + A\Big(\|\Delta e^{n+1}\|_2^2 - \|\Delta e^n\|_2^2\Big) \\
\le\;& C\cdot\frac{1+N_1^4}{\nu}\|\nabla e^n\|_2^2 + \frac{2}{\nu}\|\nabla\tilde{G}^n\|_2^2.
\end{aligned} \tag{5.8}$$

The desired result then follows from Lemma 4.1. □

5.2. **Proof of Theorem 1.4.** Similar to the proof of Theorem 1.3, we need to compare

$$\begin{cases}
\dfrac{h^{n+1}-h^n}{\tau} = -\nu\Delta^2 h^{n+1} + A\Delta(h^{n+1}-h^n) + \nabla\cdot\Pi_N(g(\nabla h^n)), \\
\partial_t h = -\nu\Delta^2 h + \nabla\cdot(g(\nabla h)), \\
h^0 = \Pi_N h_0, \quad h(0) = h_0.
\end{cases}$$

On the time interval $[t_n, t_{n+1}]$, we have

$$\begin{aligned}
\frac{h(t_{n+1}) - h(t_n)}{\tau} &= -\nu\Delta^2 h(t_{n+1}) + A\Delta(h(t_{n+1}) - h(t_n)) + \nabla\cdot\Pi_N(g(\nabla h(t_n))) \\
&\quad + \nabla\cdot\Pi_{>N}(g(\nabla h(t_n))) + \Delta\tilde{G}^n,
\end{aligned} \tag{5.9}$$

where

$$\tilde{G}^n = -\frac{\nu}{\tau}\int_{t_n}^{t_{n+1}}\partial_t\Delta h\cdot(t_n - t)dt + \frac{1}{\tau}\int_{t_n}^{t_{n+1}}\Delta^{-1}\partial_t\nabla\cdot(g(\nabla h(t)))\cdot(t_{n+1}-t)dt - A\int_{t_n}^{t_{n+1}}\partial_t h dt.$$

Now we only need to verify the estimates:

$$\int_0^T \|\partial_t \nabla \Delta h\|_2^2 dt \lesssim_{\nu,h_0} 1+T, \tag{5.10}$$

$$\int_0^T \|\Delta^{-1}\nabla\partial_t\nabla\cdot(g(\nabla h))\|_2^2 dt \lesssim_{\nu,h_0} 1+T. \tag{5.11}$$

Recall

$$\partial_t h = -\nu\Delta^2 h + \nabla\cdot(g(\nabla h)).$$

Multiplying both sides by $-\Delta^3\partial_t h$ and integrating by parts, we get

$$\|\Delta\nabla\partial_t h\|_2^2 = -\frac{\nu}{2}\frac{d}{dt}(\|\Delta^2\nabla h\|_2^2) + \int \Delta\nabla\nabla\cdot(g(\nabla h))\cdot\Delta\nabla\partial_t h dx,$$

and

$$\begin{aligned}\frac{\nu}{2}\frac{d}{dt}\|\Delta^2\nabla h\|_2^2 &\le -\|\partial_t\Delta\nabla h\|_2^2 + \|\Delta\nabla\nabla\cdot(g(\nabla h))\|_2\cdot\|\partial_t\Delta\nabla h\|_2\\ &\le -\frac{1}{2}\|\partial_t\Delta\nabla h\|_2^2 + \mathrm{const}\cdot(\|h\|_{H^5}^3 + \|h\|_{H^5}).\end{aligned} \tag{5.12}$$

This (together with standard local wellposedness theory, cf. [19] for more refined results) yields

$$\int_0^1 \|\partial_t\Delta\nabla h\|_2^2 dt \lesssim_{\nu,h_0} 1.$$

The smoothing effect gives control for $t\ge 1$. Thus

$$\int_0^T \|\partial_t\Delta\nabla h\|_2^2 dt \lesssim_{\nu,h_0} 1+T.$$

For the term $\|\Delta^{-1}\nabla\partial_t\nabla\cdot(g(\nabla h))\|_2$, we note that

$$\begin{aligned}&\|\Delta^{-1}\nabla\nabla\cdot(\partial_t(g(\nabla h)))\|_2\\ &\lesssim \|\partial_t(|\nabla h|^2\nabla h - \nabla h)\|_2 \lesssim (\|\nabla h\|_\infty^2 + 1)\|\nabla\partial_t h\|_2 \lesssim_{\nu,h_0} \|\nabla\partial_t h\|_2.\end{aligned} \tag{5.13}$$

Thus

$$\int_0^T \|\Delta^{-1}\nabla\partial_t\nabla\cdot(g(\nabla h))\|_2^2 dt \lesssim_{\nu,h_0} 1+T. \tag{5.14}$$

Finally we get

$$\|\nabla(h(t_m)-\tilde{h}^m)\|_2 \lesssim (1+A)e^{Ct_m}\cdot(N^{-(s-1)}+\tau).$$

The theorem is proved.

6. Concluding remarks

In this work we considered a class of large time-stepping methods for the phase field models such as the Cahn-Hilliard equation and the thin film equation with fourth order dissipation. We analyzed the representative case (see (1.17) and (1.22)) which is first order in time and Fourier-spectral in space, with a stabilization $O(\Delta t)$ term of the form

$$A\Delta(u^{n+1}-u^n). \tag{6.1}$$

For A sufficiently large ($A \ge O(\nu^{-1}|\log\nu|^2)$), we proved unconditional energy stability independent of the time step. The corresponding error analysis is also carried out in full detail (L^2 for CH and H^1 for MBE). It is worth emphasizing that our analysis does not require any additional Lipschitz assumption on the nonlinearity, or any a priori bounds on the numerical solution. It is expected our theoretical framework can be extended in several directions. We discuss a few such possibilities below the fold.

- General stabilization techniques. There are a myriad of ways of introducing the stabilization term. Taking the first order in time methods as an example, instead of (6.1), one can consider a more general form
$$AB(u^{n+1} - u^n), \tag{6.2}$$
where B is a general operator. One example is $B = -\Delta^2$ which is already used in the aforementioned works [30, 4]. Similarly one can consider $B = -(-\Delta)^s$ ($s > 0$ is real) or even a general pseudo-differential operator. It will be interesting to carry out a comparative study of these different stabilization techniques and identify the corresponding stability regions. Another issue is to investigate the lower bound on the parameter A. In typical numerical simulations the stability is observed to hold for relatively small values of A (the threshold value exhibits a weak dependence on the time step τ and the diffusion coefficient ν, cf. the numerical simulation results in [16]). This certainly merits further study and probably one has to fine-tune our analysis with some numerically verifiable bounds.
- Higher order time stepping methods. In [29], Xu and Tang considered a second order scheme for MBE:
$$\begin{aligned}&\frac{3h^{n+1} - 4h^n + h^{n-1}}{2\tau} + \nu\Delta^2 h^{n+1}\\ &= A\Delta(h^{n+1} - 2h^n + h^{n-1}) + \nabla \cdot \Pi_N g(\nabla(2h^n - h^{n-1})), \quad n \geq 1,\end{aligned} \tag{6.3}$$
where h^0 is the initial condition and h^1 is computed by the first order scheme (1.22). Here to keep some consistency with our setup we have added the projection operator Π_N in front of the nonlinear term. This scheme is called BD2/EP2 since it is obtained by combining a second-order backward differentiation (BD2) for the time derivative term and a second-order extrapolation (EP2) for the explicit treatment of the nonlinear term. A similar higher order BD3/EP3 scheme is also presented in [29]. The stability analysis in [29] is conditional in the sense that the choice of A depends on the a priori gradient bound on the numerical solution. Moreover, quite different from the first order (in time) methods, the energy stability for higher order methods typically takes the form
$$E(h^n) \leq E(h^0) + O(\tau), \qquad n\tau \leq T, \tag{6.4}$$
where the implied constant in the $O(\tau)$ term usually depends on the time interval $[0, T]$. In yet other words one cannot achieve strict monotonic decay of energy as in the first order case. A very natural problem is to extend our analysis to cover these cases. By using our analysis it is also possible to refine the stability results in [26] and remove the Lipschitz assumption on the nonlinearity in the case of second order implicit scheme. For second order semi-implicit schemes it is expected that our method can be extended to prove an unconditional stability result at least for time steps which are moderately small. We plan to address these issues in a future publication.
- General phase field models (possibly) with higher order dissipations. In [9], the authors considered the sixth order scalar model
$$\partial_t u = \Delta(\epsilon^2\Delta - W''(u) + \epsilon^2\eta)(\epsilon^2\Delta u - W'(u)), \tag{6.5}$$
where $W(u) = \frac{1}{4}(u^2-1)^2$ and $\eta > 0$ is a given constant. This equation arises in the modeling of pore formation in functionalized polymers [14]. The numerical experiments in [9] used implicit time stepping together with Newton's method at each time step. From our point of view it will be interesting to use the numerical schemes similar to (1.17) and establish the corresponding stability and error convergence results. In a similar vein one can also consider the volume-preserving vector CH model in the same paper (see equation (7) in [9]) and also the nonlinear diffusion model in [4]. Yet another possibility is to study the model with general *fractional* dissipation which is already mentioned in the introduction of [19]. Also one can extend our analysis to the phase-fields models

of two-phase complex fluids (see [28] for a pioneering study in this direction). In any case a first step in the analysis is to establish similar results as in [19].

The above list is certainly not exhaustive. For example we did not include the analysis of the Allen-Cahn model which will be quite similar to the CH case from our point of view. To keep the presentation simple we leave out the case of dimensions $d = 1$ and $d = 3$ which can be similarly handled. One can also consider generalizing the analysis herein to finite difference schemes and even some hybrid schemes. In [20] we will introduce a completely new approach to tackle some of these problems. Another direction is to consider the phase field models with stochastic noises. One can introduce similar numerical stabilization techniques as in the deterministic case and prove stability and convergence in these settings. We plan to investigate these problems in the future.

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