Recovery Type a Posteriori Error Estimate for Distributed Convex Optimal Control Problems Governed by Integral-differential Equations

Ningning Yan

Abstract. In this paper, we present a recovery type a posteriori error estimate and the superconvergence analysis for the finite element approximation of the distributed convex optimal control problems governed by integral-differential equations. We provide the recovery type a posteriori error estimates for both the control and the state approximation, which is equivalent to the exact error generally. Under some strong conditions, it is not only equivalent, but also asymptotically exact.

1. Introduction

The problem that we are interested in is the following distributed convex optimal control problem:

$$\min_{u\in K}\left\{\frac{1}{2}\|y-y_0\|^2_{0,\Omega}+\frac{\alpha}{2}\|u\|^2_{0,\Omega_U}\right\}, \tag{1.1}$$

$$-\mathrm{div}(A\nabla y)-\int_\Omega G(s,t)y(s)ds=f+Bu \quad \text{in } \Omega\,, \quad y|_{\partial\Omega}=0,$$

where K is a closed convex set. The details will be specified later.

It is well known that finite element method is the most widely used numerical method in computing optimal control problems, and the adaptive finite element approximation is among the most important means to boost accuracy and efficiency of the finite element discretization (see, i.e., [**1**], [**14**]). Although adaptive finite element approximation is widely used in numerical simulations, it has not yet been fully utilized in optimal control problems. Recent years, there have been some papers worked on the a posteriori error estimate for optimal control problems (see, i.e., [**12**], [**9**]). But to our knowledge, there are still only a few work on the gradient recovery type a posteriori error estimate for optimal control problems, say, [**13**] provided a kind of gradient recovery type a posteriori error estimate for optimal control problem governed by the standard elliptic partial differential equations recently.

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The constrained convex optimal control problem governed by integral-differential equations is an interesting and useful problem. It is different with the problem governed by partial differential equations, which has sone new property, say, nonlocal nature, and need some extension of existing technique. This kind of problem has recently been discussed in [**4**]. A priori error estimate and the residual type a posteriori error estimate were provided in that paper. In this paper, we extend the result of [**13**] to the problem governed by integral-differential equations. We investigate the the recovery type a posteriori error estimate and the global supercovergence for the constrained convex optimal control problems governed by integral-differential equations. The recovery type a posteriori error estimates for both the control and the state approximation are provided. It is proved for rather general meshes and solutions, the recovery type a posteriori error estimator provided in this paper is equivalent to the exact error. Moreover, the global superconvergence is derived in the problem under some strong conditions, which have a half or one order higher approximation accuracy than the optimal a priori error estimate. Based on the superconvergence analysis, it is shown that the recovery type a posteriori error estimator provided in this paper is not only equivalent, but also asymptotically exact under the superconvergence conditions.

The plan of the paper is as follows. In Section 2, we shall give a brief review of the model problem and the finite element method, and then we construct the finite element approximation for the distributed convex optimal control problems governed by integral-differential equations. In Section 3, we provide the recovery type a posteriori error estimator. It is shown that for the rather general meshes and solutions, the recovery type a posteriori error estimator provides the equivalent upper and lower error bounds. In Section 4, the supercovergence analysis are derived. Based on the superconvergence analysis, the asymptotical exactness of the a posteriori error estimator is proved.

2. Finite element approximation of optimal control problems

Let Ω (or Ω_U) be a bounded open set in R^2 with a Lipschitz boundary $\partial\Omega$ ($\partial\Omega_U$). In this paper we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with the norm $\|\cdot\|_{m,q,\Omega}$ and the seminorm $|\cdot|_{m,q,\Omega}$. We set $W_0^{m,q}(\Omega) \equiv \{w \in W^{m,q}(\Omega) : w|_{\partial\Omega} = 0\}$. We denote $W^{m,2}(\Omega)$ ($W_0^{m,2}(\Omega)$) by $H^m(\Omega)$ ($H_0^m(\Omega)$) with the norm $\|\cdot\|_{m,\Omega}$ and the seminorm $|\cdot|_{m,\Omega}$. In addition, c or C denotes a general positive constant independent of h.

In this section, we study the finite element approximation of the distributed convex optimal control problems governed by integral-differential equations. In the rest of the paper, we shall take the state space $V = H_0^1(\Omega)$, the control space $U = L^2(\Omega_U)$, and $H = L^2(\Omega)$. Let the observation space $Y = L^2(\Omega)$. Let B be a linear continuous operator from U to H. Let K be a closed convex set in the control space U such that

$$K = \{v \in U : \; v \geq 0\}.$$

We are interested in the following optimal control problem:

$$\min_{u\in K\subset U} \left\{ \frac{1}{2}\|y - y_0\|_{0,\Omega}^2 + \frac{\alpha}{2}\|u\|_{0,\Omega_U}^2 \right\}, \tag{2.1}$$

$$-\mathrm{div}(A\nabla y) - \int_\Omega G(s,t)y(s)ds = f + Bu \quad \text{in } \Omega\ , \quad y|_{\partial\Omega} = 0, \tag{2.2}$$

where α is a positive constant, $f \in L^2(\Omega)$, and $A(\cdot) = (a_{i,j}(\cdot))_{n\times n} \in R^{n\times n}$ is a symmetric positive definite matrix. To consider the finite element approximation of the above optimal control problem, we have to give some weak formulas. Let

$$a(y,w) = \int_\Omega (A\nabla y)\cdot\nabla w \qquad g(y,w) = \int_\Omega\int_\Omega G(s,t)y(s)w(t)dsdt \quad \forall y,w \in V,$$

$$(f_1,f_2) = \int_\Omega f_1 f_2 \quad \forall (f_1,f_2) \in H\times H, \qquad (u,v)_U = \int_{\Omega_U} uv \quad \forall (u,v) \in U\times U.$$

It follows from the assumptions on A that there are constants c and $C > 0$ such that $\forall y,w \in V$

$$a(y,y) \geq c\|y\|_V^2, \quad |a(y,w)| \leq C\|y\|_V\|w\|_V. \tag{2.3}$$

The inequalities in (2.3) describe coercivity and continuity of the bilinear form $a(\cdot,\cdot)$. Moreover, the state equation (2.2) is an integral-differential equation, instead of the standard partial differential equation. The extra integral term has the non-local nature, and can be used to describe some useful physical phenomena. For example, In the integral-differential equation which models the behavior of certain micromachined microsensor devices, the integral term describes heat losses to the surrounding gas (see [**2**] for more details). In this paper, we assume that $G(\cdot,\cdot) \in L^\infty$ is defined such that the equation

$$-div(A\nabla y) - \int_\Omega G(s,t)y(s)ds = F \quad \text{in } \Omega \quad y|_{\partial\Omega} = 0$$

has a unique solution $y \in H^1(\Omega)$ for any $F \in L^2(\Omega)$, and $\|y\|_{1,\Omega} \leq C\|F\|_{0,\Omega}$. Then the standard weak formula for the state equation reads as follows: find $y(u) \in V$ such that

$$a(y(u),w) - g(y(u),w) = (f + Bu, w) \qquad \forall w \in H_0^1(\Omega). \tag{2.4}$$

Therefore, the control problem (2.1)-(2.2) can be restated as the following, which we shall label (OCP):

$$\min_{u\in K\subset U}\{\frac{1}{2}\|y-y_0\|_{0,\Omega}^2 + \frac{\alpha}{2}\|u\|_{0,\Omega_U}^2\}, \tag{2.5}$$

$$a(y(u),w) - g(y(u),w) = (f + Bu, w) \qquad \forall w \in V = H_0^1(\Omega). \tag{2.6}$$

It is well known (see, for example, [**8**]) that the control problem (OCP) has a solution (y,u), and that if a pair (y,u) is the solution of (OCP), then there is a co-state $p \in V$ such that the triplet (y,p,u) satisfies the following optimality condition, which we shall label (OCP-OPT):

$$a(y,w) - g(y,w) = (f + Bu, w) \qquad \forall w \in V = H_0^1(\Omega), \tag{2.7}$$

$$a(q,p) - g(q,p) = (y - y_0, q) \qquad \forall q \in V = H_0^1(\Omega), \tag{2.8}$$

$$(\alpha u + B^*p, v - u)_U \geq 0 \quad \forall v \in K \subset U = L^2(\Omega_U), \tag{2.9}$$

where B^* is the adjoint operator of B.

By employing the technique in [**3**], it can be verified that for any $v \in H_0^1(\Omega)$ there exist a constant c and a function $w \in H_0^1(\Omega)$, $w \neq 0$, such that

$$c\|v\|_{1,\Omega}\|w\|_{1,\Omega} \leq a(v,w) - g(v,w). \tag{2.10}$$

Let us consider the finite element approximation of the control problem (OCP). Here we consider only n-simplex elements, as they are among the most widely used ones. Also we consider only conforming finite elements.

Let Ω^h be a polygonal approximation to Ω with a boundary $\partial\Omega^h$. For simplicity, we assume that $\Omega^h = \Omega$ in this paper. Let T^h be a partitioning of Ω^h into disjoint regular n-simplices τ, so that $\bar{\Omega}^h = \bigcup_{\tau \in T^h} \bar{\tau}$. Associated with T^h is a finite dimensional subspace S^h of $C(\bar{\Omega}^h)$, such that $\chi|_\tau$ are polynomials of order m ($m \geq 1$) $\forall \chi \in S^h$ and $\tau \in T^h$. Let $V^h = S^h \cap H_0^1(\Omega)$. It is easy to see that $V^h \subset V$, and it also can be proved that (2.10) is still valid for the finite element space V^h.

Let T_U^h be a partitioning of Ω_U^h into disjoint regular n-simplices τ_U, so that $\bar{\Omega}_U^h = \bigcup_{\tau_U \in T_U^h} \bar{\tau}_U$. Again, assume that $\Omega_U^h = \Omega_U$. Associated with T_U^h is another finite dimensional subspace W_U^h of $L^2(\Omega_U^h)$, such that $\chi|_{\tau_U}$ are polynomials of order m ($m \geq 0$) $\forall \chi \in W_U^h$ and $\tau_U \in T_U^h$. Here there is no requirement for the continuity. Let $U^h = W_U^h$.

In this paper, we will only consider the simplest finite element spaces, i.e., $m = 1$ for V^h and $m = 0$ for U^h. We choose the piecewise constant element space for U^h because of the limited regularity of the optimal control u (in $H^1(\Omega_U)$ in general). The piecewise linear element space for V^h is chosen just for simplicity. The higher-degree FE space can be used for V^h. But the theoretical analysis should be more complicated because of the limited regularity of the auxiliary solutions $y(u_h)$ and $p(u_h)$ (see (3.4) and (3.5)).

Let h_τ (h_{τ_U}) denote the maximum diameter of the element τ (τ_U) in T^h (T_U^h), let $h = \max_{\tau \in T^h}\{h_\tau\}$, $h_U = \max_{\tau_U \in T_U^h}\{h_{\tau_U}\}$. Note that the order of the finite element space for U^h ($m = 0$) is lower than the one for V^h ($m = 1$). The size of the element in T_U^h is smaller than the one in T^h generally. Therefore, it is reasonable to assume that $(h_U/h) \leq C$ in this paper.

Then a possible finite element approximation of (OCP), which we shall label $(OCP)^h$, is as follows:

$$\min_{u_h \in K^h \subset U^h} \{\frac{1}{2}\|y_h - y_0\|_{0,\Omega}^2 + \frac{\alpha}{2}\|u_h\|_{0,\Omega_U}^2\} , \tag{2.11}$$

$$a(y_h, w_h) - g(y_h, w_h) = (f + Bu_h, w_h) \qquad \forall w_h \in V^h, \tag{2.12}$$

where $K^h = U^h \cap K$ is a closed convex set in U^h. This is a finite dimensional optimization problem and may be solved by existing mathematical programming methods.

It follows that the control problem $(OCP)^h$ has a solution (y_h, u_h) and that if a pair $(y_h, u_h) \in V^h \times U^h$ is the solution of $(OCP)^h$, then there is a co-state $p_h \in V^h$ such that the triplet (y_h, p_h, u_h) satisfies the following optimality condition, which we shall label $(OCP - OPT)^h$:

$$a(y_h, w_h) - g(y_h, w_h) = (f + Bu_h, w_h) \qquad \forall w_h \in V^h \subset V = H_0^1(\Omega), \tag{2.13}$$

$$a(q_h, p_h) - g(q_h, p_h) = (y_h - y_0, q_h) \qquad \forall q_h \in V^h \subset V = H_0^1(\Omega), \tag{2.14}$$

$$(\alpha u_h + B^* p_h, v_h - u_h)_U \geq 0 \quad \forall v_h \in K^h \subset U = L^2(\Omega_U). \tag{2.15}$$

It has been proved (see [**4**] for details) that for the problem (2.7)-(2.9) and its finite element approximation (2.13)-(2.15), the following error estimate holds:

$$\|u - u_h\|_{0,\Omega} + \|y - y_h\|_{1,\Omega} + \|p - p_h\|_{1,\Omega} \leq C(h + h_U), \tag{2.16}$$

if $y, p \in H^2(\Omega)$ and $u \in H^1(\Omega_U)$.

3. Recovery type a posteriori error estimates

In order to discuss the recovery type a posteriori error estimate, let us construct the recovery operator R_h. Let $R_h v \in S^h$ be a continuous piecewise linear function (without zero boundary constraint). The values of $R_h v$ on the nodes are defined by least-squares argument on an element patches surrounding the nodes as follows. Let z be a node, $\omega_z = \cup_{\tau_U \in T_U^h, z \in \bar{\tau}_U} \tau_U$, V_z be the linear function space on ω_z. Set $R_h v(z) = \sigma_z(z)$ where

$$E(\sigma_z) = \min_{w \in V_z} E(w), \quad E(w) = \sum_{\tau_U \subset \omega_z} (\int_{\tau_U} w - \int_{\tau_U} v)^2.$$

When $z \in \partial\Omega_U$, we should add a few extra neighbor elements to ω_z such that ω_z contains more than three elements. For the regular mesh and the suitable choice of ω_z, we can conclude that for any $v \in L^2(\Omega)$, $R_h v$ exists. Moreover, for any domain $D \subset \Omega$, $R_h v = v$ on D if v is a linear function on $\hat{D}$, where $\hat{D} = \{\cup \tau_U : \ \bar{\tau}_U \cap \bar{D} \neq \emptyset\}$. Note that the basic idea for above recovery operator is similar as Z-Z patch recovery (see, e.g., [**18**] and [**19**]).

For gradient of y and p, we construct the gradient recovery operator: $G_h v = (R_h v_x, R_h v_y)$, where R_h is the recovery operator defined above, $v_x = \frac{\partial v}{\partial x}$ and $v_y = \frac{\partial v}{\partial y}$. It should be pointed that G_h is similar as the Z-Z gradient recovery (see e.g., [**18**] and [**19**]) in our piecewise linear case.

Based on the recovery operator R_h and G_h defined above, we can define the recovery type a posteriori error estimator:

$$\eta^2 = \|G_h y_h - \nabla y_h\|_{0,\Omega}^2 + \|G_h p_h - \nabla p_h\|_{0,\Omega}^2 + \|R_h u_h - u_h\|_{0,\Omega_U}^2. \tag{3.1}$$

Let us divide the domain Ω_U into three disjoint subdomains:

$$\bar{\Omega}_U = \bar{\Omega}_U^- \cup \bar{\Omega}_U^{00} \cup \bar{\Omega}_U^{++},$$

where

$$\begin{aligned} \Omega_U^- &= \{x \in \Omega_U : \ B^* p_h(x) \leq 0\}, \\ \Omega_U^{00} &= \{x \in \Omega_U : \ B^* p_h(x) > 0, \ u_h(x) = 0\}, \\ \Omega_U^{++} &= \{x \in \Omega_U : \ B^* p_h(x) > 0, \ u_h(x) > 0\}. \end{aligned}$$

Let

$$\Omega_U^{-+} = \Omega_U^- \cup \Omega_U^{++}, \quad \Omega_h^+ = \{x \in \Omega_U : \ u_h(x) > 0\}.$$

Then, it is easy to see that

$$\Omega_U^{-+} = \Omega_h^+ \cup \Omega_U^-.$$

THEOREM 3.1. *Let (y, p, u) and (y_h, p_h, u_h) be the solutions of the systems (2.7)-(2.9) and (2.13)-(2.15), respectively. Then,*

$$|y_h - y|_{1,\Omega}^2 + |p_h - p|_{1,\Omega}^2 + \|u_h - u\|_{0,\Omega_U}^2 \leq C\eta^2 + C(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2), \tag{3.2}$$

where η is defined in (3.1), and

$$\epsilon_1^2 = \int_{\Omega_U^{-+}} |B^* p_h + \alpha R_h u_h|^2,$$

$$\epsilon_2^2 = \sum_{z \in \Lambda} \int_{\omega_z} h_z^2 |f + Bu_h + \int_\Omega G(s,t) y_h(s) ds - \overline{(f + Bu_h + \int_\Omega G(s,t) y_h(s) ds)_z}|^2,$$

$$\epsilon_3^2 = \sum_{z\in\Lambda}\int_{\omega_z} h_z^2|y_h - y_0 + \int_\Omega G(s,t)p_h(t)dt - (\overline{y_h - y_0 + \int_\Omega G(s,t)p_h(t)dt})_z|^2,$$

where

$$\bar{v}_z = \frac{\int_{\omega_z} v}{\int_{\omega_z} 1},$$

ω_z is the support of ϕ_z, ϕ_z is the basis function on the node point z, and Λ is the set of all inner nodes.

Proof. Note that

$$\begin{aligned}
\alpha\|u-u_h\|^2_{0,\Omega_U} &= (\alpha u, u-u_h)_U - (\alpha u_h, u-u_h)_U \\
&\leq -(B^*p, u-u_h)_U - (\alpha u_h, u-u_h)_U \\
&= (B^*p_h+\alpha u_h, u_h-u)_U + (B^*(p_h-p(u_h)), u-u_h)_U \\
&\quad +(B^*(p(u_h)-p), u-u_h)_U,
\end{aligned} \tag{3.3}$$

where $p(u_h)$ is the solution of the auxiliary equation:

$$a(y(u_h),w) - g(y(u_h),w) = (f+Bu_h, w) \qquad \forall w\in V = H_0^1(\Omega), \tag{3.4}$$

$$a(q,p(u_h)) - g(q,p(u_h)) = (y(u_h)-y_0, q) \qquad \forall q\in V = H_0^1(\Omega). \tag{3.5}$$

It is easy to show that

$$(B^*(p_h-p(u_h)), u-u_h)_U \leq C\|p_h-p(u_h)\|^2_{0,\Omega} + C\delta\|u-u_h\|^2_{0,\Omega_U}, \tag{3.6}$$

and

$$(B^*(p(u_h)-p), u-u_h)_U = (y(u_h)-y, y-y(u_h)) \leq 0. \tag{3.7}$$

For the first term of (3.3), we have that

$$(B^*p_h+\alpha u_h, u_h-u)_U = \int_{\Omega_U^{-+}}(B^*p_h+\alpha u_h)(u_h-u) + \int_{\Omega_U^{00}}(B^*p_h+\alpha u_h)(u_h-u), \tag{3.8}$$

$$\begin{aligned}
&\int_{\Omega_U^{-+}}(B^*p_h+\alpha u_h)(u_h-u) \leq C\int_{\Omega_U^{-+}}(B^*p_h+\alpha u_h)^2 + C\delta\|u-u_h\|^2_{0,\Omega_U} \\
\leq\;& C\int_{\Omega_U^{-+}}(B^*p_h+\alpha R_hu_h)^2 + C\int_{\Omega_U^{-+}}\alpha^2(u_h-R_hu_h)^2 + C\delta\|u-u_h\|^2_{0,\Omega_U} \\
\leq\;& C\|u_h-R_hu_h\|^2_{0,\Omega_U} + C\epsilon_1^2 + C\delta\|u-u_h\|^2_{0,\Omega_U},
\end{aligned} \tag{3.9}$$

and

$$\int_{\Omega_U^{00}}(B^*p_h+\alpha u_h)(u_h-u) = -\int_{\Omega_U^{00}} B^*p_hu \leq 0. \tag{3.10}$$

Therefore, it follows from (3.8)-(3.10) that

$$(B^*p_h+\alpha u_h, u_h-u)_U \leq C\|u_h-R_hu_h\|^2_{0,\Omega_U} + C\epsilon_1^2 + C\delta\|u-u_h\|^2_{0,\Omega_U}. \tag{3.11}$$

Then, (3.3), (3.6), (3.7) and (3.11) imply that

$$\|u-u_h\|^2_{0,\Omega_U} \leq C\|u_h-R_hu_h\|^2_{0,\Omega_U} + C\epsilon_1^2 + C\|p_h-p(u_h)\|^2_{0,\Omega}. \tag{3.12}$$

Now let us consider the last term $\|p_h - p(u_h)\|_{0,\Omega}^2$. Using the averaging interpolation (see [**5**]) and the technique used in [**4**], [**5**] and [**16**], it can be proved that

$$\|p(u_h) - p_h\|_{1,\Omega}^2 \le C \sum_{l\cap\partial\Omega=\emptyset} h_l \int_l [A^*\nabla p_h \cdot n]^2 + C\|y(u_h) - y_h\|_{0,\Omega}^2 + C\epsilon_3^2. \tag{3.13}$$

Note that $G_h p_h$ is continuous on Ω. We have that

$$\begin{aligned}\sum_{l\cap\partial\Omega=\emptyset} h_l \int_l [A^*\nabla p_h \cdot n]^2 &\le C \sum_{l\cap\partial\Omega=\emptyset} h_l \int_l [\nabla p_h]^2 = C \sum_{l\cap\partial\Omega=\emptyset} h_l \int_l [\nabla p_h - G_h p_h]^2 \\ &\le C \sum_{l\cap\partial\Omega=\emptyset} \sum_{l\cap\partial\tau\ne\emptyset} \int_\tau (\nabla p_h - G_h p_h)^2 \le C\|\nabla p_h - G_h p_h\|_{0,\Omega}^2. \end{aligned}\tag{3.14}$$

It follows from (3.13) and (3.14) that

$$\|p(u_h) - p_h\|_{1,\Omega}^2 \le C\|\nabla p_h - G_h p_h\|_{0,\Omega}^2 + C\|y(u_h) - y_h\|_{0,\Omega}^2 + C\epsilon_3^2. \tag{3.15}$$

Similarly, it can be proved that

$$\|y(u_h) - y_h\|_{1,\Omega}^2 \le C\|\nabla y_h - G_h y_h\|_{0,\Omega}^2 + C\epsilon_2^2. \tag{3.16}$$

Therefore, it follows from (3.12), (3.15) and (3.16) that

$$\|y(u_h)-y_h\|_{1,\Omega}^2 + \|p(u_h)-p_h\|_{1,\Omega}^2 + \|u-u_h\|_{0,\Omega_U}^2 \le C\eta^2 + C\epsilon_1^2 + C\epsilon_2^2 + C\epsilon_3^2. \tag{3.17}$$

Note that

$$\begin{aligned}\|y - y_h\|_{1,\Omega} &\le \|y - y(u_h)\|_{1,\Omega} + \|y(u_h) - y_h\|_{1,\Omega},\\ \|p - p_h\|_{1,\Omega} &\le \|p - p(u_h)\|_{1,\Omega} + \|p(u_h) - p_h\|_{1,\Omega},\end{aligned}$$

and

$$\begin{aligned}\|y - y(u_h)\|_{1,\Omega} &\le C\|u - u_h\|_{0,\Omega_U},\\ \|p - p(u_h)\|_{1,\Omega} &\le C\|y - y(u_h)\|_{0,\Omega} \le C\|u - u_h\|_{0,\Omega_U}.\end{aligned}$$

We have that

$$\|y - y_h\|_{1,\Omega}^2 + \|p - p_h\|_{1,\Omega}^2 + \|u - u_h\|_{0,\Omega_U}^2 \le C\eta^2 + C\epsilon_1^2 + C\epsilon_2^2 + C\epsilon_3^2.$$

This proves the theorem.

THEOREM 3.2. *Let (y,p,u) and (y_h,p_h,u_h) be the solutions of the systems (2.7)-(2.9) and (2.13)-(2.15), respectively. Then,*

$$\eta^2 \le C(|y_h - y|_{1,\Omega}^2 + |p_h - p|_{1,\Omega}^2 + \|u_h - u\|_{0,\Omega_U}^2) + C(\hat\epsilon_1^2 + \hat\epsilon_2^2 + \hat\epsilon_3^2), \tag{3.18}$$

where η is defined in (3.1), and

$$\begin{aligned}\hat\epsilon_1^2 &= \int_{\Omega_U^d\setminus\Omega_U^-} (\alpha u_h + B^* p_h)^2\\ \hat\epsilon_2^2 &= \sum_\tau \int_\tau h_\tau^2 |f + Bu_h + \int_\Omega G(s,t) y_h(s) ds - \overline{(f + Bu_h + \int_\Omega G(s,t) y_h(s) ds)}_\tau|^2,\\ \hat\epsilon_3^2 &= \sum_\tau \int_\tau h_\tau^2 |y_h - y_0 + \int_\Omega G(s,t) p_h(t) dt - \overline{(y_h - y_0 + \int_\Omega G(s,t) p_h(t) dt)}_\tau|^2,\end{aligned}$$

where

$\Omega_U^d = \{\cup\tau_U : u_h|_{\tau_U} = 0,$ *and there exists* $\tau_U' \in T_U^h$ *such that* $\bar\tau_U \cap \bar\tau_U' \ne \emptyset,\ u_h|_{\tau_U'} > 0\}$, *and*

$$\bar{v}_\tau = \frac{\int_\tau v}{\int_\tau 1}.$$

Proof. Note that for any node z,

$$G_h y_h(z) = \sum_{j=1}^{J_z} \alpha_z^j (\nabla y_h)_{\tau_z^j}, \quad \sum_{j=1}^{J_z} \alpha_z^j = 1.$$

Then, on any element τ,

$$\begin{aligned} |\nabla y_h - G_h y_h|^2 &= |(\nabla y_h)_\tau - \sum_{z\cap\bar{\tau}\neq\emptyset} \phi_z(\sum_{j=1}^{J_z} \alpha_z^j (\nabla y_h)_{\tau_z^j})|^2 \\ = \ & | \sum_{z\cap\bar{\tau}\neq\emptyset} \phi_z(\sum_{j=1}^{J_z} \alpha_z^j((\nabla y_h)_\tau - (\nabla y_h)_{\tau_z^j}))|^2 \leq C \sum_{\tau'\subset S_\tau} |(\nabla y_h)_\tau - (\nabla y_h)_{\tau'}|^2, \end{aligned}$$

where $S_\tau = \cup_{\tau'\in T_{S_\tau}} \tau'$, $T_{S_\tau} = \{\tau' \in T^h : \ \bar{\tau}' \cap \bar{\tau} \neq \emptyset\}$. Given an edge l of an element in T^h, let $[w]_l$ be the jump across l. Note that $\forall \tau, \tau' \subset S_\tau$, there exist a finite positive integer m_τ, which is independent of h, and elements $\tau_i \subset S_\tau$, $i = 0, 1, \cdots, m_\tau$, such that $\bar{\tau}_{i-1} \cap \bar{\tau}_i = l_i$, $i = 1, \cdots, m_\tau$, where $l_i \subset S_\tau$ are edges of the elements, $l_i \cap \partial S_\tau = \emptyset$, and $\tau = \tau_0$, $\tau' = \tau_{m_\tau}$. Hence,

$$|(\nabla y_h)_\tau - (\nabla y_h)_{\tau'}| = |\sum_{i=1}^{m_\tau} [\nabla y_h]_{l_i}| \leq \sum_{i=1}^{m_\tau} |[\nabla y_h]_{l_i}| \leq \sum_{l\subset S_\tau\backslash\partial S_\tau} |[\nabla y_h]_l|.$$

Since y_h is continuous on Ω, $[\frac{\partial y_h}{\partial t}]_l = 0$, if $l \cap \partial\Omega = \emptyset$, where $\frac{\partial y_h}{\partial t}$ is the tangent derivative of y_h. Therefore, $|[\nabla y_h]_l| = |[\frac{\partial y_h}{\partial n}]_l|$, if $l \cap \partial\Omega = \emptyset$. Then, we have that

$$\|Gy_h - \nabla y_h\|_{0,\tau}^2 \leq Ch_\tau^2 \sum_{l\subset(S_\tau\backslash\partial S_\tau)} |[\frac{\partial y_h}{\partial n}]_l|^2 \leq C \sum_{l\subset(S_\tau\backslash\partial S_\tau)} h_l \int_l [\nabla y_h \cdot n]_l^2,$$

where h_l is the size of the edge l. Hence, noting that A is positive definite, and $n = k[\nabla y_h]_l$, with $k = \pm(1/|[\nabla y_h]_l|)$, because that $|[\nabla y_h]_l \cdot n| = |[\nabla y_h]_l|$. we have that

$$\begin{aligned} [\nabla y_h \cdot n]_l^2 &= k^2 |[\nabla y_h]_l|^4 \leq Ck^2((A[\nabla y_h]_l) \cdot [\nabla y_h]_l)^2 = C((A[\nabla y_h]_l) \cdot k[\nabla y_h]_l)^2 \\ &= C((A[\nabla y_h]_l) \cdot n)^2 = C[(A\nabla y_h) \cdot n]_l^2. \end{aligned}$$

Then,

$$\sum_\tau \|Gy_h - \nabla y_h\|_{0,\tau}^2 \leq C \sum_{l\cap\partial\Omega=\emptyset} h_l \int_l [\nabla y_h \cdot n]_l^2 \leq C \sum_{l\cap\partial\Omega=\emptyset} h_l \int_l [A\nabla y_h \cdot n]^2. \tag{3.19}$$

Using the standard bubble technique (see, e.g. [**1**], [**4**], [**14**]), it can be shown that

$$\sum_{l\cap\partial\Omega=\emptyset} h_l \int_l [A\nabla y_h \cdot n]^2 \leq C\|y - y_h\|_{1,\Omega}^2 + C\|u - u_h\|_{0,\Omega_U}^2 + C\hat{\epsilon}_2^2. \tag{3.20}$$

Then, it follows from (3.19) and (3.20) that

$$\|Gy_h - \nabla y_h\|_{0,\Omega}^2 \leq C\|y - y_h\|_{1,\Omega}^2 + C\|u - u_h\|_{0,\Omega_U}^2 + C\hat{\epsilon}_2^2. \tag{3.21}$$

Similarly, it can be shown that

$$\|Gp_h - \nabla p_h\|_{0,\Omega}^2 \le C\|p - p_h\|_{1,\Omega}^2 + C\|y - y_h\|_{1,\Omega}^2 + C\hat{\epsilon}_3^2. \tag{3.22}$$

Repeating the steps of the above proof again, it can be shown that

$$\begin{aligned}
\|R_h u_h - u_h\|_{0,\Omega_U}^2 &\le C \sum_{l\subset(\Omega_U^d\cup\Omega_h^+),\, l\cap\partial(\Omega_U^d\cup\Omega_h^+)=\emptyset} h_l^2 [u_h]^2 \\
&= \frac{C}{\alpha^2} \sum_{l\subset(\Omega_U^d\cup\Omega_h^+),\, l\cap\partial(\Omega_U^d\cup\Omega_h^+)=\emptyset} h_l \int_l [\alpha u_h + B^* p_h]^2 \\
&\le C \int_{\Omega_U^d\cup\Omega_h^+} (\alpha u_h + B^* p_h)^2 \\
&= C \int_{\Omega_U^{-+}} (\alpha u_h + B^* p_h)^2 + C \int_{(\Omega_U^d\cup\Omega_h^+)\setminus\Omega_U^{-+}} (\alpha u_h + B^* p_h)^2 \qquad (3.23) \\
&= C \int_{\Omega_U^{-+}} (\alpha u_h + B^* p_h)^2 + C \int_{\Omega_U^d\setminus\Omega_U^-} (\alpha u_h + B^* p_h)^2 \\
&= C \int_{\Omega_U^{-+}} (\alpha u_h + B^* p_h)^2 + C\hat{\epsilon}_1^2.
\end{aligned}$$

It follows from the definition of Ω_U^{-+} that

$$\int_{\Omega_U^{-+}} (\alpha u_h + B^* p_h)^2 = \int_{\Omega_U^-} (\alpha u_h + B^* p_h)^2 + \int_{\Omega_U^{++}} (\alpha u_h + B^* p_h)^2. \tag{3.24}$$

Note that $(B^*p + \alpha u) = 0$ when $u > 0$ and $B^*p \ge 0$ when $u = 0$. Let

$$\Omega_0^0 = \{x \in \Omega_U^- :\ u(x) = 0\}.$$

We have that

$$\begin{aligned}
\int_{\Omega_U^-} (\alpha u_h + B^* p_h)^2 &= \int_{\Omega_0^0} (\alpha u_h + B^* p_h - \alpha u)^2 + \int_{\Omega_U^-\setminus\Omega_0^0} (\alpha u_h + B^* p_h - \alpha u - B^* p)^2 \\
&\le C\Big(\|u - u_h\|_{0,\Omega_U}^2 + \|p - p_h\|_{0,\Omega}^2 + \int_{\Omega_0^0} (B^* p_h)^2 \Big) \qquad (3.25) \\
&\le C\Big(\|u - u_h\|_{0,\Omega_U}^2 + \|p - p_h\|_{0,\Omega}^2 + \int_{\Omega_0^0} (B^* p_h - B^* p)^2 \Big) \\
&\le C\Big(\|u - u_h\|_{0,\Omega_U}^2 + \|p - p_h\|_{0,\Omega}^2 \Big),
\end{aligned}$$

where we used the fact that $B^* p_h \le 0 \le B^* p$ on Ω_0^0.

Now consider the second term of (3.24). If $u_h|_{\tau_U} > 0$, there exists an $\epsilon > 0$ and a function $\psi \in U^h$, such that $\psi \ge 0$, $\|\psi\|_{L^\infty(\tau_U)} = 1$ and $(u_h - \epsilon\psi)|_{\tau_U} \ge 0$. Let

$$v_h = \begin{cases} u_h - \epsilon\psi & x \in \tau_U \\ u_h & \text{otherwise} \end{cases}.$$

Then, $v_h \in K^h$. It follows from (2.15) that

$$\int_{\Omega_U} (\alpha u_h + B^* p_h)(u_h - v_h) = \int_{\tau_U} (\alpha u_h + B^* p_h)(u_h - (u_h - \epsilon\psi)) = \epsilon \int_{\tau_U} (\alpha u_h + B^* p_h)\psi \le 0.$$

Note that on Ω_U^{++}, $(\alpha u_h + B^* p_h) \geq B^* p_h > 0$. We have that

$$\begin{aligned}
&\int_{\tau_U \cap \Omega_U^{++}} |\alpha u_h + B^* p_h| \psi = \int_{\tau_U \cap \Omega_U^{++}} (\alpha u_h + B^* p_h) \psi \\
&\leq \quad -\int_{\tau_U \cap \Omega_U^{-}} (\alpha u_h + B^* p_h) \psi \leq \int_{\tau_U \cap \Omega_U^{-}} |\alpha u_h + B^* p_h|. \qquad (3.26)
\end{aligned}$$

Let $\hat{\tau}_U$ be the reference element of τ_U, $\tau_U^0 = \tau_U \cap \Omega_U^{++}$, and $\hat{\tau}_U^0 \subset \hat{\tau}_U$ be a part mapped from τ_U^0. It follows from the equivalence of the norm in finite dimensional space and (3.26) that

$$\begin{aligned}
&\int_{\tau_U^0} |\alpha u_h + B^* p_h|^2 \leq C h_{\tau_U}^2 \int_{\hat{\tau}_U^0} |\alpha u_h + B^* p_h|^2 \\
\leq \quad & C h_{\tau_U}^2 \Big(\int_{\hat{\tau}_U^0} |\alpha u_h + B^* p_h| \psi\Big)^2 \leq C h_{\tau_U}^{-2} \Big(\int_{\tau_U^0} |\alpha u_h + B^* p_h| \psi\Big)^2 \\
\leq \quad & C h_{\tau_U}^{-2} \Big(\int_{\tau_U \cap \Omega_U^{-}} |\alpha u_h + B^* p_h|\Big)^2 \leq C \int_{\tau_U \cap \Omega_U^{-}} |\alpha u_h + B^* p_h|^2.
\end{aligned}$$

Therefore,

$$\int_{\Omega_U^{++}} (\alpha u_h + B^* p_h)^2 \leq C \int_{\Omega_U^{-}} (\alpha u_h + B^* p_h)^2. \qquad (3.27)$$

It follows from (3.24), (3.25) and (3.27) that

$$\int_{\Omega_U^{-+}} (\alpha u_h + B^* p_h)^2 \leq C(\|u - u_h\|_{0,\Omega_U}^2 + \|p - p_h\|_{0,\Omega}^2). \qquad (3.28)$$

Hence, (3.23) and (3.28) imply that

$$\|R_h u_h - u_h\|_{0,\Omega_U}^2 \leq C(\|u - u_h\|_{0,\Omega_U}^2 + \|p - p_h\|_{0,\Omega}^2) + C\hat{\epsilon}_1^2. \qquad (3.29)$$

Then, (3.18) follows from (3.21), (3.22) and (3.29).

REMARK 3.1. It can be shown that ϵ_i and $\hat{\epsilon}_i$, $i = 1, 2, 3$, defined in Theorem 3.1 and 3.2 should be the higher order terms comparing the exact error under some strong conditions (say, f and y_0 are smooth enough, the discrete free boundary is not too bad). Therefore, it follows from Theorem 3.1 and 3.2 that the recovery type a posteriori error estimator provided in this section is equivalent to the exact error in many cases. The discussions about them are complicated, we omit them in this paper (see [**13**] for details).

4. Superconvergence analysis and recovery

In this section, we will provide the superconvergence results. Firstly, let us consider the supercovergence analysis for the control u. Let

$$\Omega_U^+ = \{\cup \tau_U : \ \tau_U \subset \Omega_U, \ u|_{\tau_U} > 0\},$$

$$\Omega_U^0 = \{\cup \tau_U : \ \tau_U \subset \Omega_U, \ u|_{\tau_U} = 0\},$$

$$\Omega_U^b = \Omega_U \setminus (\Omega_U^+ \cup \Omega_U^0).$$

In this section, we assume that u and T_U^h are regular such that $meas(\Omega_U^b) \leq C h_U$.

LEMMA 4.1. *Let u and u_h be the solutions of (2.9) and (2.15), respectively. Let $u_I \in K^h$ be the L^2-projection of u, such that*

$$u_I|_{\tau_U} = \frac{\int_{\tau_U} u}{\int_{\tau_U} 1}.$$

Assume that $p \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$ and $u \in W^{1,\infty}(\Omega_U)$. Then,

$$\|u_h - u_I\|_{0,\Omega_U} \le C(h_U^{\frac{3}{2}} + h^2). \tag{4.1}$$

Proof. Note that $u_h, u_I \in K^h \subset K$. It follows from (2.9) and (2.15) that

$$(\alpha u + B^*p, u - u_h) \le 0, \quad (\alpha u_h + B^*p_h, u_h - u_I) \le 0.$$

Then,

$$\begin{aligned}
\alpha\|u_h - u_I\|^2_{0,\Omega_U} &= \alpha(u_h - u_I, u_h - u_I)_U \\
&\le -(B^*p_h, u_h - u_I)_U - (\alpha u_I, u_h - u_I)_U \\
&= (B^*p, u - u_h)_U + (B^*p, u_I - u)_U \\
&\quad +(\alpha u_I, u_I - u_h)_U + (B^*(p - p_h), u_h - u_I)_U \\
&\le -\alpha(u, u - u_h)_U + (B^*p, u_I - u)_U \\
&\quad +\alpha(u_I, u_I - u_h)_U + (B^*(p - p_h), u_h - u_I)_U \\
&= \alpha(u_I - u, u_I - u_h)_U + (\alpha u + B^*p, u_I - u)_U \\
&\quad +(B^*(p - p(u_h)), u - u_I)_U + (B^*(p - p(u_h)), u_h - u)_U \\
&\quad +(B^*(p(u_h) - p_h), u_h - u_I)_U,
\end{aligned} \tag{4.2}$$

where $p(u_h)$ is defined by (3.4)-(3.5). Let π^c be the integral averaging operator such that $\pi^c u = u_I$. The definition of u_I implies that

$$(u_I - u, u_I - u_h)_U = \sum_{\tau_U}(u_I - u_h)\int_{\tau_U}(\pi^c u - u) = 0, \tag{4.3}$$

and

$$\begin{aligned}
(B^*(p - p(u_h)), u - u_I)_U &= \sum_{\tau_U}\int_{\tau_U}\Big(B^*(p - p(u_h)) - \pi^c(B^*(p - p(u_h)))\Big)(u - \pi^c u) \\
&\le C\sum_{\tau_U} h^2_{\tau_U}|B^*(p - p(u_h))|_{1,\tau_U}|u|_{1,\tau_U} \le Ch_U^2\|p - p(u_h)\|_{1,\Omega}\|u\|_{1,\Omega}.
\end{aligned} \tag{4.4}$$

Using (2.7)-(2.8) and (3.4)-(3.5), we have that

$$\begin{aligned}
\|p - p(u_h)\|_{1,\Omega} &\le C\|y - y(u_h)\|_{0,\Omega} \le C\|B(u - u_h)\|_{0,\Omega} \le C\|u - u_h\|_{0,\Omega_U} \\
&\le C\|u - u_I\|_{0,\Omega_U} + C\|u_I - u_h\|_{0,\Omega_U} \\
&\le Ch_U|u|_{1,\Omega_U} + C\|u_I - u_h\|_{0,\Omega_U}
\end{aligned} \tag{4.5}$$

and

$$(B^*(p-p(u_h)), u_h-u)_U = (p-p(u_h), B(u_h-u)) = (y-y(u_h), y(u_h)-y) \le 0. \tag{4.6}$$

It follows from the Schwarz inequality that

$$\begin{aligned}
(B^*(p(u_h) - p_h), u_h - u_I)_U &\le C\|B^*(p(u_h) - p_h)\|_{0,\Omega_U}\|u_h - u_I\|_{0,\Omega_U} \\
&\le C\|p(u_h) - p_h\|^2_{0,\Omega} + C\delta\|u_h - u_I\|^2_{0,\Omega_U},
\end{aligned} \tag{4.7}$$

where δ is an arbitrary small positive constant. Then, (4.2)-(4.7) lead to

$$\begin{aligned}\alpha\|u_h - u_I\|^2_{0,\Omega_U} &\leq (\alpha u + B^*p, u_I - u)_U + Ch_U^2(h_U + \|u_I - u_h\|_{0,\Omega_U})\\ &\quad + C\|p(u_h) - p_h\|^2_{0,\Omega} + C\delta\|u_I - u_h\|^2_{0,\Omega_U}\\ &\leq (\alpha u + B^*p, u_I - u)_U + Ch_U^3 + C\|p(u_h) - p_h\|^2_{0,\Omega}\\ &\quad + C\delta\|u_I - u_h\|^2_{0,\Omega_U}.\end{aligned}$$

Hence,

$$\|u_h - u_I\|^2_{0,\Omega_U} \leq C(\alpha u + B^*p, u_I - u)_U + Ch_U^3 + C\|p(u_h) - p_h\|^2_{0,\Omega}. \tag{4.8}$$

Using the standard finite element analysis technique (see, i.e., [**6**]), it can be proved that

$$\|p(u_h) - p_h\|_{0,\Omega} \leq Ch^2. \tag{4.9}$$

Note that

$$\begin{aligned}(\alpha u + B^*p, u_I - u)_U &= \int_{\Omega_U^+}(\alpha u + B^*p)(u_I - u) + \int_{\Omega_U^0}(\alpha u + B^*p)(u_I - u)\\ &\quad + \int_{\Omega_U^b}(\alpha u + B^*p)(u_I - u),\end{aligned}$$

and

$$(\alpha u + B^*p)|_{\Omega_U^+} = 0, \quad (u_I - u)|_{\Omega_U^0}.$$

Then,

$$\begin{aligned}(\alpha u + B^*p, u_I - u)_U &= \int_{\Omega_U^b}(\alpha u + B^*p)(u_I - u)\\ &= \sum_{\tau_U\subset\Omega_U^b}\int_{\tau_U}\Big(\alpha u + B^*p - \pi^c(\alpha u + B^*p)\Big)(\pi^c u - u)\\ &\leq C\sum_{\tau_U\subset\Omega_U^b} h_{\tau_U}^2|\alpha u + B^*p|_{1,\tau_U}|u|_{1,\tau_U}\\ &\leq Ch_U^2(\|u\|^2_{1,\infty,\Omega_U} + \|p\|^2_{1,\infty,\Omega})meas(\Omega_U^b) \leq Ch_U^3.\end{aligned} \tag{4.10}$$

Therefore, (4.1) follows from (4.8), (4.9) and (4.10).

Lemma 4.2. *Suppose all conditions of Lemma 4.1 are valid. Moreover, assume that* $u \in W^{1,\infty}(\Omega_U) \cap H^2(\Omega_U^+ \cup \Omega_U^0)$. *Then,*

$$\|R_h u - u\|_{0,\Omega_U} \leq Ch_U^{\frac{3}{2}}, \tag{4.11}$$

where R_h *is the recovery operator defined in the beginning of section 3.*

Proof. Let

$$\Omega_h^{++} = \{\cup\tau_U : \omega_z \subset \Omega_U^+, \forall z \in \bar{\tau}_U\}, \quad \Omega_h^{00} = \{\cup\tau_U : \omega_z \subset \Omega_U^0, \forall z \in \bar{\tau}_U\}.$$

Then,

$$R_h u(x) = u(x) = 0 \quad \forall x \in \Omega_U^{00}. \tag{4.12}$$

It can be proved by the standard technique (see, e.g., [**6**]) that

$$\|R_h u - u\|_{0,\Omega_U^{++}} \leq Ch_U^2\|u\|_{2,\Omega_U^+}, \tag{4.13}$$

and

$$\|R_h u - u\|^2_{0,\Omega_U\setminus(\Omega_U^{++}\cup\Omega_U^{00})} \le Ch_U^2\|u\|^2_{1,\Omega_U^{bb}} \le Ch_U^2\|u\|^2_{1,\infty,\Omega_U} meas(\Omega_U^{bb}),$$

where

$$\Omega_U^{bb} = \Omega_U \setminus (\Omega_U^{++} \cup \Omega_U^{00}).$$

Note that $meas(\Omega_U^b) = O(h_U)$ and hence $meas(\Omega_U^{bb}) = O(h_U)$. We have that

$$\|R_h u - u\|_{0,\Omega_U\setminus(\Omega_U^{++}\cup\Omega_U^{00})} \le Ch_U^{\frac{3}{2}}. \tag{4.14}$$

Therefore, it follows from (4.12)-(4.14) that

$$\begin{aligned}\|R_h u - u\|^2_{0,\Omega_U} &= \|R_h u - u\|^2_{0,\Omega_U^{++}} + \|R_h u - u\|^2_{0,\Omega_U^{00}} + \|R_h u - u\|^2_{0,\Omega_U\setminus(\Omega_U^{++}\cup\Omega_U^{00})}\\ &\le Ch_U^4 + 0 + Ch_U^3 \le Ch_U^3.\end{aligned}$$

This proves (4.11).

Theorem 4.1. *Suppose all conditions of Lemma 4.1 and 4.2 are valid. Then,*

$$\|R_h u_h - u\|_{0,\Omega_U} \le C(h_U^{\frac{3}{2}} + h^2). \tag{4.15}$$

Proof. Let u_I be defined in Lemma 4.1. Then,

$$\|R_h u_h - u\|_{0,\Omega_U} \le \|u - R_h u\|_{0,\Omega_U} + \|R_h u - R_h u_I\|_{0,\Omega_U} + \|R_h u_I - R_h u_h\|_{0,\Omega_U}. \tag{4.16}$$

Moreover, Lemma 4.2 implies that

$$\|u - R_h u\|_{0,\Omega_U} \le Ch_U^{\frac{3}{2}}. \tag{4.17}$$

Noting the definition of R_h, we have

$$R_h u = R_h u_I, \quad \|R_h u_I - R_h u_h\|_{0,\Omega_U} \le C\|u_I - u_h\|_{0,\Omega_U}. \tag{4.18}$$

It has been proved in Lemma 4.1 that

$$\|u_I - u_h\|_{0,\Omega_U} \le C(h_U^{\frac{3}{2}} + h^2). \tag{4.19}$$

Therefore, (4.15) follows from (4.16)-(4.19).

Corollary 4.1. *Let u and u_h be the solutions of (2.9) and (2.15), respectively. Assume that all conditions of Lemma 4.1 and 4.2 are valid. Then,*

$$\|u - u_h\|_{-1,\Omega_U} \le C(h_U^{\frac{3}{2}} + h^2). \tag{4.20}$$

Proof. For any function $\phi \in H^1(\Omega_U)$, let $\phi_I \in U^h$ be defined as in Lemma 4.1. Then,

$$(u - u_h, \phi)_U = (u - u_h, \phi - \phi_I)_U + (u - u_h, \phi_I)_U. \tag{4.21}$$

Note that

$$(u - u_h, \phi - \phi_I)_U \le \|u - u_h\|_{0,\Omega_U}\|\phi - \phi_I\|_{0,\Omega_U} \le C(h_U + h)h_U\|\phi\|_{1,\Omega_U}. \tag{4.22}$$

Moreover, Lemma 4.1 leads to

$$\begin{aligned}(u - u_h, \phi_I)_U = (u_I - u_h, \phi_I)_U &\le \|u_I - u_h\|_{0,\Omega_U}\|\phi_I\|_{0,\Omega_U}\\ &\le C(h_U^{\frac{3}{2}} + h^2)\|\phi\|_{1,\Omega_U}.\end{aligned} \tag{4.23}$$

Therefore, it follows from (4.21)-(4.23) that

$$\|u - u_h\|_{-1,\Omega_U} = \sup_{\phi\in H^1(\Omega_U)} \frac{(u - u_h, \phi)_U}{\|\phi\|_{1,\Omega_U}} \le C(h_U^{\frac{3}{2}} + h^2).$$

This proves the Corollary.

Using the standard superconvergence analysis technique (see, i.e., [**10**], [**7**]), the following superclose result can be proved, where "superclose" means that the distance between the finite element solution and the interpolation of the exact solution is much smaller than the distance between the finite element solution and the exact solution (see [**15**] for details).

LEMMA 4.3. *Let y, p be the solutions of the equations (2.7) and (2.8), and y_h, p_h be the solutions of the equations (2.13) and (2.14). Let y_I and p_I be the piecewise linear Lagrange interpolations of y and p. Assume that all the conditions in Lemma 4.1 are valid. Moreover, assume that the mesh T^h is uniform, and $y, p \in H^3(\Omega)$. Then,*

$$\|y_h - y_I\|_{1,\Omega} + \|p_h - p_I\|_{1,\Omega} \le C(h^2 + h_U^{\frac{3}{2}}). \tag{4.24}$$

Proof. Note that (2.10) is still valid for the finite element space V^h. It follows from (2.10), (2.7) and (2.13) that there exist a constant c and a function $w_h \in V^h \subset H_0^1(\Omega)$, $w_h \neq 0$, such that

$$\begin{aligned} & c\|y_h - y_I\|_{1,\Omega}\|w_h\|_{1,\Omega} \le a(y_h - y_I, w_h) - g(y_h - y_I, w_h) \\ = \quad & a(y - y_I, w_h) - g(y - y_I, w_h) + a(y_h - y, w_h) - g(y_h - y, w_h) \\ = \quad & a(y - y_I, w_h) - g(y - y_I, w_h) + (B(u_h - u), w_h). \end{aligned} \tag{4.25}$$

Note that when the mesh T^h is uniform, it can be proved that (using element analysis technique and Green's formula, see, e.g., [**10**], [**7**], [**11**] for details)

$$|a(y - y_I, w_h)| \le Ch^2|y|_{3,\Omega}|w_h|_{1,\Omega}. \tag{4.26}$$

It is easy to show that

$$|g(y - y_I, w_h)| \le C\|u - u_I\|_{0,\Omega}\|w_h\|_{0,\Omega} \le Ch^2|y|_{2,\Omega}\|w_h\|_{1,\Omega}. \tag{4.27}$$

Moreover, Corollary 4.1 leads to

$$|(B(u_h - u), w_h)| \le C\|u_h - u\|_{-1,\Omega_U}\|B^* w_h\|_{1,\Omega_U} \le C(h_U^{\frac{3}{2}} + h^2)\|w_h\|_{1,\Omega}. \tag{4.28}$$

Summing up, (4.25)-(4.28) imply that

$$\|y_h - y_I\|_{1,\Omega} \le C(h_U^{\frac{3}{2}} + h^2). \tag{4.29}$$

Similarly, it can be proved that

$$\|p_h - p_I\|_{1,\Omega} \le C(h_U^{\frac{3}{2}} + h^2). \tag{4.30}$$

Then, (4.24) follows from (4.29) and (4.30).

Based on above Lemma, we have following global superconvergence results using the recovery operator defined in Section 3.

THEOREM 4.2. *Suppose all conditions in Lemmas 4.1, 4.2 and 4.3 are valid. Then,*

$$\|G_h y_h - \nabla y\|_{0,\Omega} + \|G_h p_h - \nabla p\|_{0,\Omega} \le C(h^2 + h_U^{\frac{3}{2}}). \tag{4.31}$$

Proof. Note that

$$\|G_h y_h - \nabla y\|_{0,\Omega} \le \|G_h y_h - G_h y_I\|_{0,\Omega} + \|G_h y_I - \nabla y\|_{0,\Omega}. \tag{4.32}$$

It follows from Lemma 4.3 that

$$\|G_h y_h - G_h y_I\|_{0,\Omega} \le C\|\nabla(y_h - y_I)\|_{0,\Omega} \le C(h^2 + h_U^{\frac{3}{2}}). \tag{4.33}$$

It has been proved in [**17**] (Remark 3.2 and Theorem 3.2) that $G_h v_I = \nabla v$ on τ_U if v is a quadratic function on the neighborhood of τ_U ($\cup_{\bar{\tau}'_U \cap \bar{\tau}_U \neq \emptyset}\{\tau'_U\}$). Then the standard interpolation error estimate technique (see, e.g., [**6**]) can be used to show that

$$\|G_h y_I - \nabla y\|_{0,\Omega} \leq Ch^2 |y|_{3,\Omega}. \tag{4.34}$$

Therefore, (4.32)-(4.34) imply that

$$\|G_h y_h - \nabla y\|_{0,\Omega} \leq C(h^2 + h_U^{\frac{3}{2}}). \tag{4.35}$$

Similarly, it can be proved that

$$\|G_h p_h - \nabla p\|_{0,\Omega} \leq C(h^2 + h_U^{\frac{3}{2}}). \tag{4.36}$$

Therefore, (4.31) follows from (4.35) and (4.36).

Based on the supercovergence analysis, we have the following results for the recovery type a posteriori error estimator.

THEOREM 4.3. *Suppose all conditions of Theorems 4.1 and 4.2 are valid. Then,*

$$\|R_h u_h - u_h\|_{0,\Omega_U} = \|u - u_h\|_{0,\Omega_U} + O(h_U^{\frac{3}{2}} + h^2), \tag{4.37}$$

$$\|G_h y_h - \nabla y_h\|_{0,\Omega} = \|\nabla(y - y_h)\|_{0,\Omega} + O(h_U^{\frac{3}{2}} + h^2), \tag{4.38}$$

$$\|G_h p_h - \nabla p_h\|_{0,\Omega} = \|\nabla(p - p_h)\|_{0,\Omega} + O(h_U^{\frac{3}{2}} + h^2). \tag{4.39}$$

That is

$$\eta^2 = \|u - u_h\|^2_{0,\Omega_U} + \|\nabla(y - y_h)\|^2_{0,\Omega} + \|\nabla(p - p_h)\|^2_{0,\Omega} + o(h_U^3 + h^4), \tag{4.40}$$

where η is defined in (3.1).

Proof. Note that

$$\left| \|R_h u_h - u_h\|_{0,\Omega_U} - \|u - u_h\|_{0,\Omega_U} \right| \leq \|R_h u_h - u\|_{0,\Omega_U}.$$

It follows from Theorem 4.1 that

$$\|R_h u_h - u\|_{0,\Omega_U} \leq C(h_U^{\frac{3}{2}} + h^2).$$

Hence,

$$\left| \|R_h u_h - u_h\|_{0,\Omega_U} - \|u - u_h\|_{0,\Omega_U} \right| \leq C(h_U^{\frac{3}{2}} + h^2).$$

This proves (4.37). (4.38) and (4.39) can be proved from Theorem 4.2 similarly. (4.40) is the direct result of (4.37)-(4.39).

REMARK 4.1. In Theorem 4.3, it is shown that under some strong conditions, the recovery type a posteriori error estimator η is not only equivalent as shown in the last section, but also asymptotically exact under the superconvergence conditions.

REMARK 4.2. We investigate a posteriori error estimate of recovery type for a kind of optimal control problem governed by integral-differential equations. There are many important issues still to be addressed in this area. Especially, many computational issues and applications have to be studied, and we will provide numerical examples in the coming paper.

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Institute of System Sciences, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China

E-mail address: `yan@staff.iss.ac.cn`