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Derivative Superconvergence of Equilateral Triangular Finite Elements

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Abstract. Derivative superconvergent points under locally equilateral triangular mesh for both the Poisson and Laplace equations are reported. Our results are conclusive. For the Poisson equation, symmetry points are only superconvergent points for cubic and higher order elements. However, for the Laplace equation, most of superconvergent points are not symmetry points, which are reported for the first time in the literature.

1. INTRODUCTION

We consider the numerical solution to Poisson's equation

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with boundary $\partial\Omega$, f and g are sufficiently smooth. If $f \equiv 0$, (1.1) reduces to Laplace's equation.

Let $\{\tau_h\}$ be a sequence of finite element (FE) meshes of the domain Ω with mesh size $h \to 0$. Moreover, we assume that Ω is locally partitioned into uniform equilateral triangles. For each h, let $V_{n,h}(\Omega)$ be the corresponding nth order *polynomial FE space* defined in Ω, and $u_{n,h}$ be the nth order FE solution of (1.1). To simplify the notation, we shall write V_n for $V_{n,h}$ and u_h for $u_{n,h}$ respectively when no confusing may arise.

In this paper we study *natural superconvergence* property of u_h under local equilateral triangular mesh. Here "natural superconvergence" means that the superconvergent points are obtained without applying any recovery or averaging techniques in the FE solutions. For the literature in superconvergence, see e.g. [1, 3, 6–9, 13, 14, 18] and references therein.

In [5], Blum, Lin, and Rannacher showed that, for linear FE with a uniform equilateral triangular mesh, the error $(u - u_h)$ is of order $O(h^4)$ at nodal points, which is 2 orders higher than the optimal global convergent rate. It is also known that for equilateral (as well as regular) quadratic element, the two Gaussian points on element edge are superconvergent points for the tangential derivative [2, 18], and

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the mid-point on element edge as well as vertices are superconvergent points for function values [14]. It seems that the Gaussian points and Lobatto points are superconvengent points. However, Zhang and Naga have demonstrated recently, that Lobatto and Gaussian points are not natural superconvergent points for the cubic and quartic FE under an equilateral triangle mesh [17]. We are motivated by a naturally related question: What are superconvergent points for high order equilateral triangular finite elements?

Some superconvergent results of equilateral triangular elements can be obtained from the *symmetry theory* due to Schatz, Sloan, and Wahlbin [12]. The theory predicts that superconvergence occurs at local mesh symmetry points. More precisely, for odd order (linear, cubic, etc.) elements, the symmetry points are derivative superconvergent points; for even order (quadratic, quartic, and so on) elements, the symmetry points are function value superconvergent points. The theory is applicable to equilateral elements. Nevertheless, it does not say that there is no other superconvergent points; and therefore, it is not conclusive.

In mid 90s, a systematic approach of finding superconvergent points is developed by Babuška and Strouboulis et al. They reduced the problem of finding natural superconvergent points to the problem of finding intersections of certain polynomial contours in the *master cell* (a 2-cell). The actual superconvergent points are determined by computer programs without explicitly constructing those polynomials. Therefore, the method is called "computer-based" proof. In [4], the authors considered derivative superconvergence of isosceles right triangular elements in four mesh patterns.

Later, Zhang proposed an analytic approach through an orthogonal decomposition [15, 16]. He constructed explicitly the polynomials, which are needed for determining the superconvergent points in the "computer-based" method, and studied derivative superconvergence of several families of rectangular and brick finite elements. Using this approach, Lin and Zhang studied natural superconvergence of derivatives and function values under the four mesh patterns of isosceles right triangles [10]. The results confirm all superconvergent points for the Poisson and Laplace equations reported in [4]. Moreover, many new superconvergence results are obtained.

We shall apply the analytic approach to equilateral triangular elements in the present article. We would like to point out that this case was not considered in the "computer-based" proof. Nevertheless, it is the most important case from a practical view point. Since an automatic mesh generator based on the Delaunay triangulation produces near equilateral triangles on most part of the domain. See, for instance, example meshes generated by *DistMesh* [11] (Figure 1 and Figure 2).

Here is the outline of the paper. In Section 2, some preliminaries are provided to set a base for the analytic approach. In Section 3, we illustrate the process by using an example. Superconvergence results are summarized in Section 4. Finally, numerical examples are provided in Section 5 to verify the computational efficiency of the proposed points.

2. PRELIMINARIES

Consider the superconvergence in the local equilateral triangular mesh illustrated in Figure 3 (a), where we assume that pollution effects have been properly controlled. We note that the grid is locally translation invariant with respect to the

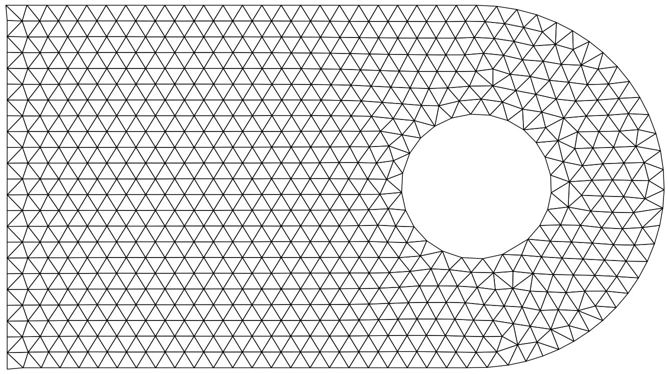


FIGURE 1. Example meshes generated by DistMesh 1

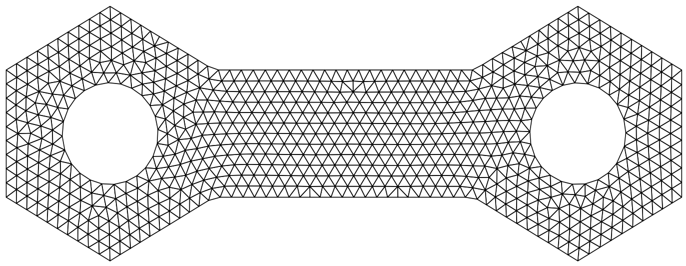
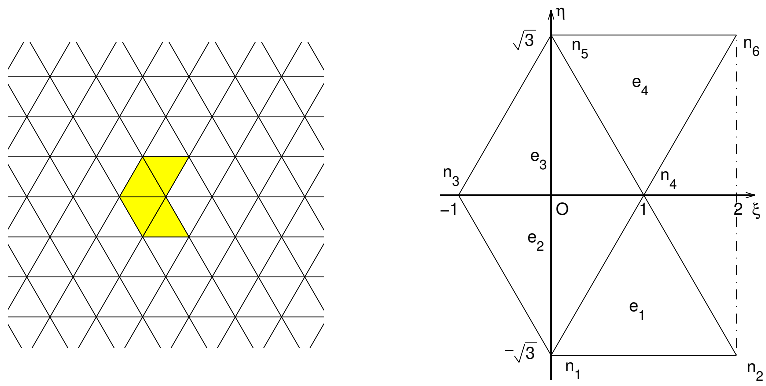


FIGURE 2. Example meshes generated by DistMesh 2

master patch, which is not a 2-*cell*. The main theorems in [4] are also applicable to this patch. Therefore, the study of local superconvergence in an equilateral triangular mesh can be focused on the master patch, or equivalently on the *reference patch* K; see Figure 3 (b). We shall not prove this statement, since the proof is analogous to that for isosceles right triangular elements. See also [10, 14–16] for more details.

It is necessary to specify what a periodic function is. A function g is said to be *periodic with respect to the master patch* if it is always true that $g(x+h, y) = g(x, y)$ and $g(x, y + \sqrt{3}\, h) = g(x, y)$ in the local mesh. In contrast, a function γ is said to be *periodic with respect to the reference patch* if $\gamma = B \circ g$, where B is the mapping which maps the master patch to the reference patch. We will simply say that the function is periodic in both cases. Here, and in the context, (x, y) and (ξ, η) are



(a) Local Mesh (b) Reference Patch K

FIGURE 3. In the equilateral local mesh (a), the shaded patch is the *master patch*.

used for the standard Euclidean coordinates in the local mesh and the reference patch, respectively.

Let us denote by $n_1 : (0, -\sqrt{3})$, $n_2 : (2, -\sqrt{3})$, $n_3 : (-1, 0)$, $n_4 : (1, 0)$, $n_5 : (0, \sqrt{3})$, and $n_6 : (2, \sqrt{3})$ the six vertices of K with the corresponding coordinates, as indicated in Figure 3 (b). Also denote by $e_1 : \triangle_{124}$, $e_2 : \triangle_{314}$, $e_3 : \triangle_{345}$, and $e_4 : \triangle_{546}$ the four triangular elements in K with the corresponding vertices.

Let $V_n(K)$ be the local FE space defined on K. On each element it contains complete polynomials of degrees up to n. For a monomial v of total degree $(n+1)$ in two variables, we define an interpolation operator I_n such that,

(i) $I_n v \in V_n(K)$, and at each vertex n_p of K,

$$I_n v(n_p) = v(n_p); \tag{2.1}$$

(ii) Along each edge l of K,

$$\int_l (v - I_n v) r^j \, \mathrm{d}r = 0, \quad j = 0, 1, \cdots, n-2; \tag{2.2}$$

(iii) In each triangle T of K,

$$\int_T (v - I_n v) r^j s^k \, \mathrm{d}r \, \mathrm{d}s = 0, \quad j, k \geq 0, \quad j + k = 0, 1, \cdots, n-3. \tag{2.3}$$

We shall note that $v - I_n v$ is periodic in K [14]. Thus, a periodic FE approximation z_v^π of $v - I_n v$ is obtained by solving

$$\begin{aligned} \int_K \nabla(v - I_n v - z_v^\pi) \cdot \nabla w = 0, \quad \forall w \in V_n^\pi(K); \\ \int_K (v - I_n v - z_v^\pi) = 0, \end{aligned} \tag{2.4}$$

where $V_n^\pi(K)$ is the nth order *periodic FE space* defined on K.

Denote $\psi_v = v - I_n v - z_v^\pi$. Let v run through all monomials of degree $(n+1)$. Then $\left\{\psi_{\xi^{n+1-i}\eta^i}\right\}_{i=1}^{n+1}$ spans a space $\Phi_{n+1}(K)$, which is a subset of $V_{n+1}^\pi(K)$ and satisfying

$$\int_K \nabla\psi\cdot\nabla w = 0,\ \forall w \in V_n^\pi(K); \qquad \int_K \psi = 0. \tag{2.5}$$

The family of Φ s defines an orthogonal decomposition of the periodic FE space.

It has been shown that [10,15]

THEOREM 2.1. (*i*) *When $n \geq 1$, derivative superconvergent points of $V_n(K)$ along the ξ-direction for the* **Poisson** *equation are the intersections of the contours*

$$\{\frac{\partial\psi}{\partial\xi} = 0 \mid \psi \in \Phi_{n+1}(K)\}.$$

(*ii*) *When $n \geq 1$, derivative superconvergent points of $V_n(K)$ along the ξ-direction for the* **Laplace** *equation are the intersections of the contours*

$$\frac{\partial\psi_{n+1}^{Re}}{\partial\xi} = 0 \quad \text{and} \quad \frac{\partial\psi_{n+1}^{Im}}{\partial\xi} = 0$$

with

$$\psi_{n+1}^{Re} = Re(z^{n+1}) - r_n, \quad \psi_{n+1}^{Im} = Im(z^{n+1}) - s_n,$$

where

$$\psi_{n+1}^{Re},\ \psi_{n+1}^{Im} \in \Phi_{n+1}(K); \quad r_n, s_n \in V_n(K); \quad z = \xi + i\eta.$$

Similar result holds for the η-direction derivative.

3. DETERMINING SUPERCONVERGENT POINTS; AN EXAMPLE

From Theorem 2.1, it is clear that to determine the nth order superconvergent points one needs first to find a basis of $\Phi_{n+1}(K)$. Recall that $\left\{\psi_{\xi^{n+1-i}\eta^i}\right\}_{i=1}^{n+1}$ is a basis, which can be determined by using the approach described in Section 2.

For instance, we study superconvergence of cubic elements. In this case, $n = 3$ and we need to consider all monomials of degree 4. We first take monomial ξ^4. By (2.1) - (2.3) one may get the interpolation of ξ^4 as

$$I_n(\xi^4) = \begin{cases} 4\xi^3 - \frac{2}{3}\sqrt{3}\xi^2\eta - \frac{34}{5}\xi^2 + \frac{4}{3}\sqrt{3}\xi\eta - \frac{2}{15}\eta^2 + \frac{28}{5}\xi - \frac{11}{15}\sqrt{3}\eta - \frac{9}{5} & \text{in } e_1 \\ +\frac{2}{3}\sqrt{3}\xi^2\eta + \frac{6}{5}\xi^2 - \frac{2}{15}\eta^2 - \frac{1}{5}\sqrt{3}\eta - \frac{1}{5} & \text{in } e_2 \\ -\frac{2}{3}\sqrt{3}\xi^2\eta + \frac{6}{5}\xi^2 - \frac{2}{15}\eta^2 + \frac{1}{5}\sqrt{3}\eta - \frac{1}{5} & \text{in } e_3 \\ 4\xi^3 + \frac{2}{3}\sqrt{3}\xi^2\eta - \frac{34}{5}\xi^2 - \frac{4}{3}\sqrt{3}\xi\eta - \frac{2}{15}\eta^2 + \frac{28}{5}\xi + \frac{11}{15}\sqrt{3}\eta - \frac{9}{5} & \text{in } e_4 \end{cases}. \tag{3.1}$$

Then we solve (2.4) and obtain the correction term

$$z_{\xi^4}^\pi = \begin{cases} +\frac{3}{14}\sqrt{3}\xi^2\eta - \frac{1}{14}\sqrt{3}\eta^3 + \frac{18}{35}\xi^2 - \frac{3}{7}\sqrt{3}\xi\eta - \frac{12}{35}\eta^2 - \frac{36}{35}\xi + \frac{3}{70}\sqrt{3}\eta + \frac{3}{7} & \text{in } e_1 \\ -\frac{3}{14}\sqrt{3}\xi^2\eta + \frac{1}{14}\sqrt{3}\eta^3 - \frac{9}{70}\xi^2 + \frac{3}{10}\eta^2 + \frac{9}{70}\sqrt{3}\eta + \frac{3}{70} & \text{in } e_2 \\ +\frac{3}{14}\sqrt{3}\xi^2\eta - \frac{1}{14}\sqrt{3}\eta^3 - \frac{9}{70}\xi^2 + \frac{3}{10}\eta^2 - \frac{9}{70}\sqrt{3}\eta + \frac{3}{70} & \text{in } e_3 \\ -\frac{3}{14}\sqrt{3}\xi^2\eta + \frac{1}{14}\sqrt{3}\eta^3 + \frac{18}{35}\xi^2 + \frac{3}{7}\sqrt{3}\xi\eta - \frac{12}{35}\eta^2 - \frac{36}{35}\xi - \frac{3}{70}\sqrt{3}\eta + \frac{3}{7} & \text{in } e_4 \end{cases}. \tag{3.2}$$

Hence, $\psi_{\xi^4} = \xi^4 - I_n(\xi^4) - z_{\xi^4}^\pi$ is the basis function in $\Phi_4(K)$ associated with ξ^4.

By a similar approach, we can find a basis of $\Phi_4(K)$. It is worthy to point out that the expression of each piecewisely defined basis function is symmetric. For example,

$$\begin{aligned} \psi_{\xi^4}|_{e_2}(\xi,\eta) &= \psi_{\xi^4}|_{e_1}(\xi+1,-\eta-\sqrt{3}),\\ \psi_{\xi^4}|_{e_3}(\xi,\eta) &= \psi_{\xi^4}|_{e_1}(\xi+1,\eta-\sqrt{3}),\\ \psi_{\xi^4}|_{e_4}(\xi,\eta) &= \psi_{\xi^4}|_{e_1}(\xi,-\eta). \end{aligned} \tag{3.3}$$

Moreover, the locations of the superconvergent points in different elements are also symmetric.

Here, we provide the expressions of the basis functions in one element, say e_3. The expressions in the other elements are obtained by symmetry.

$$\begin{aligned} \psi_{\xi^4} &= \xi^4+\tfrac{19}{42}\sqrt{3}\xi^2\eta+\tfrac{1}{14}\sqrt{3}\eta^3-\tfrac{15}{14}\xi^2-\tfrac{1}{6}\eta^2-\tfrac{1}{14}\sqrt{3}\eta+\tfrac{11}{70},\\ \psi_{\xi^3\eta} &= \xi^3\eta+\tfrac{1}{3}\sqrt{3}\xi\eta^2-\tfrac{4}{5}\xi\eta,\\ \psi_{\xi^2\eta^2} &= \xi^2\eta^2-\tfrac{17}{42}\sqrt{3}\xi^2\eta+\tfrac{17}{126}\sqrt{3}\eta^3+\tfrac{3}{70}\xi^2-\tfrac{17}{30}\eta^2+\tfrac{37}{210}\sqrt{3}\eta-\tfrac{1}{70},\\ \psi_{\xi\eta^3} &= \xi\eta^3-\sqrt{3}\xi\eta^2+\tfrac{3}{5}\xi\eta,\\ \psi_{\eta^4} &= \eta^4-\tfrac{3}{14}\sqrt{3}\xi^2\eta-\tfrac{27}{14}\sqrt{3}\eta^3+\tfrac{9}{70}\xi^2+\tfrac{33}{10}\eta^2-\tfrac{33}{70}\sqrt{3}\eta-\tfrac{3}{70}. \end{aligned} \tag{3.4}$$

Now we are in the position to determine superconvergent points of cubic equilateral triangular finite elements. According to Theorem 2.1, the ξ-derivative superconvergent points of the Poisson equation in element e_3 are common zeros of

$$\begin{aligned} \tfrac{\partial\psi_{\xi^4}}{\partial\xi} &= 4\xi^3+\tfrac{19}{21}\sqrt{3}\xi\eta-\tfrac{15}{7}\xi,\\ \tfrac{\partial\psi_{\xi^3\eta}}{\partial\xi} &= 3\xi^2\eta+\tfrac{1}{3}\sqrt{3}\eta^2-\tfrac{4}{5}\eta,\\ \tfrac{\partial\psi_{\xi^2\eta^2}}{\partial\xi} &= 2\xi\eta^2-\tfrac{17}{21}\sqrt{3}\xi\eta+\tfrac{3}{35}\xi,\\ \tfrac{\partial\psi_{\xi\eta^3}}{\partial\xi} &= \eta^3-\sqrt{3}\eta^2+\tfrac{3}{5}\eta,\\ \tfrac{\partial\psi_{\eta^4}}{\partial\xi} &= -\tfrac{3}{7}\sqrt{3}\xi\eta+\tfrac{9}{35}\xi, \end{aligned} \tag{3.5}$$

which turns out to be the midpoint $(0,0)$ of the ξ-direction side 3-4. This is a local symmetry point, which is predictable by the symmetry theory [12, 14].

For the Laplace equation, we need to consider the two harmonic compositions of the monomials of degree 4 [10]. In particular, the superconvergent points are common zeros of

$$\begin{aligned} \tfrac{\partial}{\partial\xi}(\psi_{\xi^4}-6\psi_{\xi^2\eta^2}+\psi_{\eta^4}) &= 4\xi^3-12\xi\eta^2+\tfrac{16}{3}\sqrt{3}\xi\eta-\tfrac{12}{5}\xi,\\ \tfrac{\partial}{\partial\xi}(\psi_{\xi^3\eta}-\psi_{\xi\eta^3}) &= 3\xi^2\eta-\eta^3+\tfrac{4}{3}\sqrt{3}\eta^2-\tfrac{7}{5}\eta. \end{aligned} \tag{3.6}$$

It is straightforward to solve the equation system and get 7 points

$$\begin{gathered} (0,0),\ (\pm\sqrt{\tfrac{3}{5}},0),\\ (\pm\tfrac{\sqrt{255-30\sqrt{10}}}{30},\tfrac{1}{6}\sqrt{3}+\tfrac{1}{30}\sqrt{30}),\\ (\pm\tfrac{\sqrt{255+30\sqrt{10}}}{30},\tfrac{1}{6}\sqrt{3}-\tfrac{1}{30}\sqrt{30}). \end{gathered} \tag{3.7}$$

Note that except $(0,0)$, the other six are not symmetry points. Moreover, the first three points are the cubic Gaussian points on the horizontal side 3-4.

4. SUPERCONVERGENCE RESULTS

As demonstrated in Section 3, we can determine superconvergent points in element of higher order. In this section, we collect derivative superconvergence results of equilateral triangular finite elements of orders up to 7.

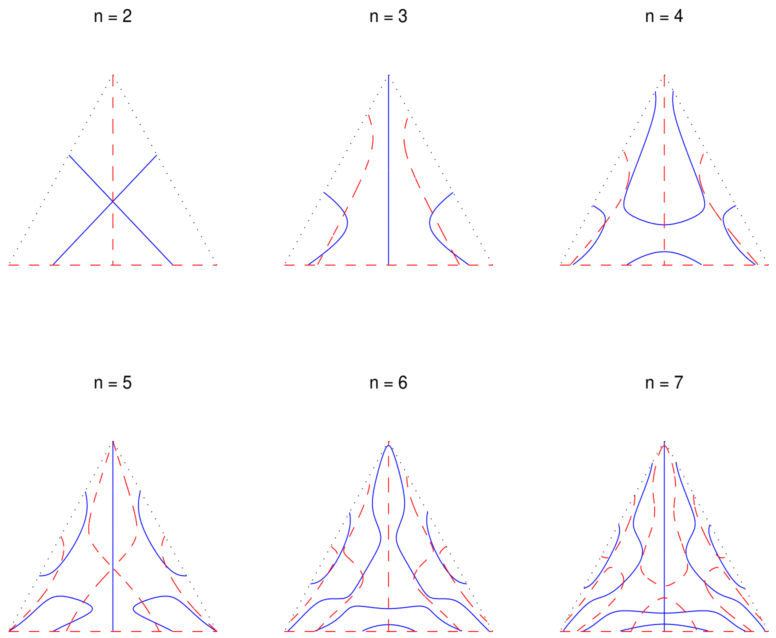


FIGURE 4. Contours of $\partial\psi^{Re}_{n+1}/\partial\xi = 0$ (solid curves) and $\partial\psi^{Im}_{n+1}/\partial\xi = 0$ (dashed curves) in e_3 for $n = 2, \ldots, 7$.

We present ξ-derivative superconvergence results in one element e_3. Superconvergent points in the other elements can be obtained by symmetry. Similar results for derivatives in other tangential directions of the triangle can be obtained by rotation.

PROPOSITION 4.1. *Consider ξ-derivative superconvergent points of the Poisson equation in element e_3.*
$\diamond$ *When n is odd, the mid-points $(0, 0)$ of the horizontal side 3-4 is a superconvergent point;*
$\diamond$ *When $n = 2$, the two Gaussian points $(\pm\sqrt{1/3}, 0)$ of the horizontal side 3-4 are superconvergent points;*
$\diamond$ *When $n > 2$ is even, there is no superconvergent point.*

We note that the results of equilateral triangular elements for the Poisson equation are the same as those of isosceles right triangular elements in the regular pattern [10]. The symmetry theory also predicts these results [14].

For the Laplace equation, as shown in cubic element, more superconvergent points are expected. The contours of $\partial\psi^{Re}_{n+1}/\partial\xi = 0$ and $\partial\psi^{Im}_{n+1}/\partial\xi = 0$ in e_3 for $n = 2, \ldots, 7$ are given in Figure 4. We collect the results for elements of orders $n = 1, \ldots, 7$ in Table 1 and Table 2, where each pair of decimals represents the coordinates of a point.

Clearly, most of the superconvergent points are not local symmetry points. The points are symmetrically distributed in e_3 at the axis $\xi = 0$. It is worthy to note

TABLE 1. ξ-derivative Superconvergent Points in e_3

	$n = 1$	$n = 2$	$n = 3$	$n = 4$
1	0.0000000000000000 0.0000000000000000	−0.5773502691896258 0.0000000000000000	−0.7745966692414834 0.0000000000000000	−0.8811276252944361 0.0000000000000000
2		0.0000000000000000 0.5773502691896258	−0.6234922522953107 0.1061009487597575	−0.5709960541651958 0.3781667469585688
3		0.5773502691896258 0.0000000000000000	−0.4218104763924046 0.4712493204298683	−0.3642869981675120 0.0000000000000000
4			0.0000000000000000 0.0000000000000000	0.0000000000000000 0.1221000311804541
5			0.4218104763924046 0.4712493204298683	0.0000000000000000 0.3664629360032316
6			0.6234922522953107 0.1061009487597575	0.3642869981675120 0.0000000000000000
7			0.7745966692414834 0.0000000000000000	0.5709960541651958 0.3781667469585688
8				0.8811276252944361 0.0000000000000000

TABLE 2. ξ-derivative Superconvergent Points in e_3 (continued)

	$n = 5$	$n = 6$	$n = 7$
1	−1.0000000000000000 0.0000000000000000	−0.9695969660896559 0.0000000000000000	−0.9699527677597880 0.0000000000000000
2	−0.5773502691896258 0.0000000000000000	−0.7177506654157521 0.0000000000000000	−0.7882338752251438 0.0000000000000000
3	−0.5052492296133585 0.6034929575234453	−0.6783922160080868 0.01503711799833967	−0.5207373436961605 0.6793735012901207
4	−0.4035575400403984 0.08044059062630058	−0.5826214618734005 0.5252759964089793	−0.4383926913046738 0.0000000000000000
5	−0.2800992336704002 0.2681720530398249	−0.2577883552058981 0.0000000000000000	−0.3914869140578662 0.4431235658033960
6	0.0000000000000000 0.0000000000000000	−0.2476077858672690 0.5063407538357728	−0.3033615049114394 0.03091972628864010
7	0.0000000000000000 0.5773502691896258	0.0000000000000000 0.06188594646206987	−0.2024117254333288 0.1777771900258029
8	0.0000000000000000 1.7320508075688773	0.0000000000000000 0.2072673366424771	0.0000000000000000 0.3050925027164925
9	0.2800992336704002 0.2681720530398249	0.0000000000000000 1.6879915572409531	0.0000000000000000 0.4122428914463635
10	0.4035575400403984 0.08044059062630058	0.2476077858672690 0.5063407538357728	0.0000000000000000 1.6862444740709656
11	0.5052492296133585 0.6034929575234453	0.2577883552058981 0.0000000000000000	0.2024117254333288 0.1777771900258029
12	0.5773502691896258 0.0000000000000000	0.5826214618734005 0.5252759964089793	0.3033615049114394 0.03091972628864010
13	1.0000000000000000 0.0000000000000000	0.6783922160080868 0.01503711799833967	0.3914869140578662 0.4431235658033960
14		0.7177506654157521 0.0000000000000000	0.4383926913046738 0.0000000000000000
15		0.9695969660896559 0.0000000000000000	0.5207373436961605 0.6793735012901207
16			0.7882338752251438 0.0000000000000000
17			0.9699527677597880 0.0000000000000000

that the Gaussian points of the horizontal side 3-4 are superconvergent points in e_3 for linear, quadratic, and cubic elements (i.e. $n = 1, 2, 3$); But the case is not true for elements of order $n \geq 4$. Moreover, the superconvergent results here are

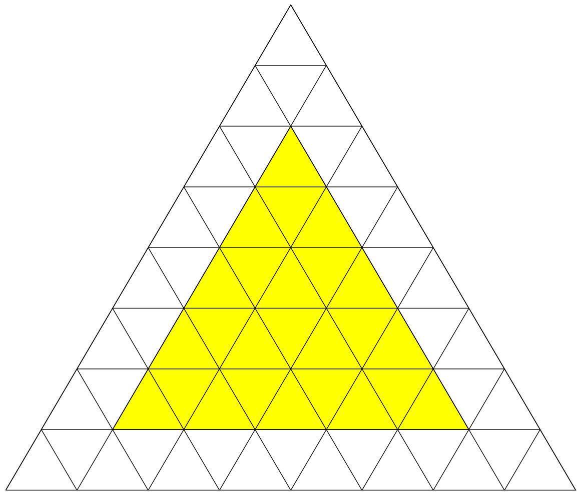


FIGURE 5. Domain Ω (with initial triangulation) and the subdomain Ω_0 (shaded region)

very different from those of isosceles right triangular elements in any pattern [10]. At least the numbers of the superconvergent points are not the same.

5. NUMERICAL TESTS

To verify the above superconvergence results, we solve the model problem (1.1) in an equilateral triangular domain Ω, which is enclosed by three straight lines $y = 0$, $y = \sqrt{3}x$, and $y = \sqrt{3}(1-x)$.

Figure 5 shows the initial triangulation corresponding to mesh size $h = 1/8$. The refined triangulations are constructed by successive decomposition of each triangle into four congruent subtriangles of half size. We choose

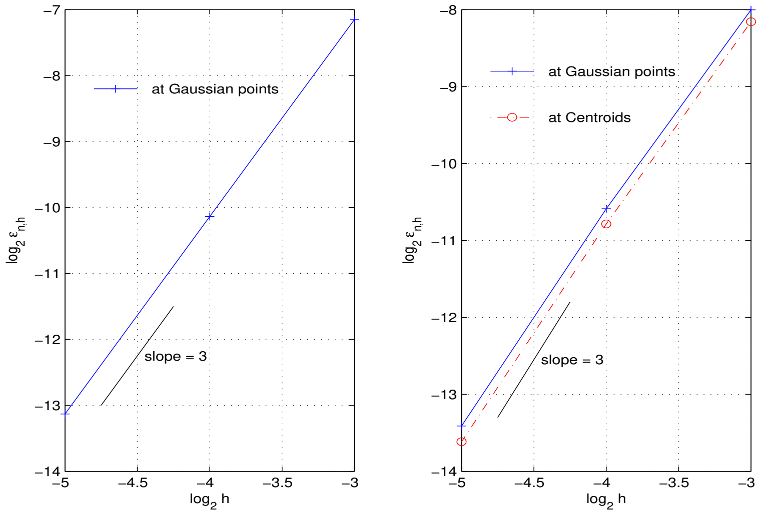
$$u(x, y) = 16^2(x^2 - x)^2(y^2 - y)^2$$

for the Poisson equation and

$$u(x, y) = x^8 - 28x^6y^2 + 70x^4y^4 - 28x^2y^6 + y^8$$

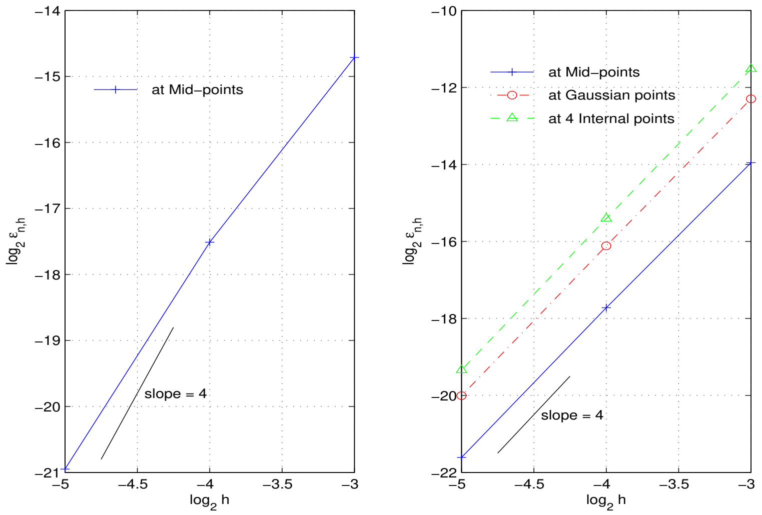
for the Laplace equation. f and g are determined by (1.1). Although our numerical tests will not be exhaustive, they are certainly representative for the indicated case. The derivative error $\varepsilon_{n,h} = |\partial(u - u_{n,h})/\partial\xi|$ will be measured at some specific points in a fixed interior subdomain $\Omega_0 = \{p = (x, y) \mid p \in \Omega,\ \mathrm{dist}(p, \partial\Omega) \geq \sqrt{3}/16\} \subset \Omega$; see Figure 5. We present numerical tests for quadratic, cubic, and quartic elements in this paper.

Test I confirms the local $O(h^3)$ superconvergence of the derivative error for quadratic element. In particular, for the Poisson equation, superconvergence is observed at the two Gaussian points of the horizontal side; for the Laplace equation, in addition to those two Gaussian points, the centroid of each element is also a superconvergent point. See Figure 6. Note that the centroid is not a local symmetry point of an equilateral triangle.



(a) Poisson Equation (b) Laplace Equation

FIGURE 6. Superconvergence of Quadratic Element



(a) Poisson Equation (b) Laplace Equation

FIGURE 7. Superconvergence of Cubic Element

Test II supports the results discussed in Section 3. As we mentioned, most superconvergent points for the Laplace equation are not symmetry points. The three Gaussian points of the horizontal side are superconvergent points of the Laplace equation, not for the Poisson equation, as indicated in Figure 7.

Finally, test III is designed to verify quartic element's superconvergent points of the Laplace equation. We can see from Figure 8 that, at the specified points, the convergent rate is $O(h^5)$ instead of the global optimal rate $O(h^4)$. Here the

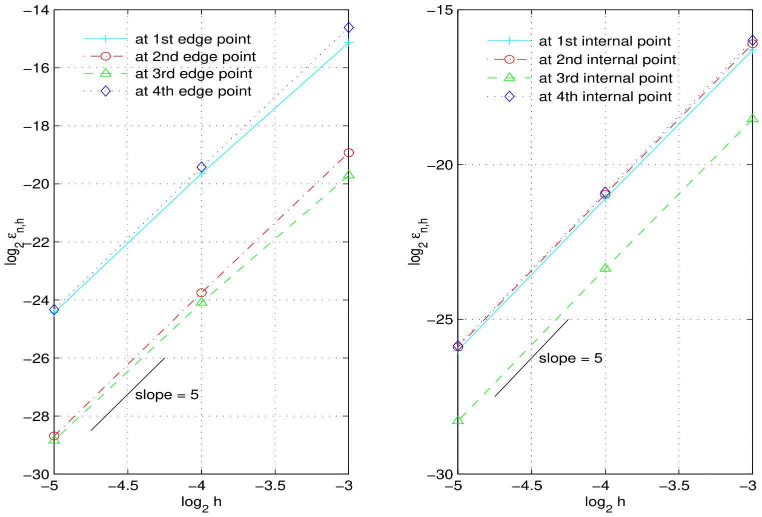


FIGURE 8. Superconvergence of Quartic Element for the Laplace Equation

edge points and internal points are enumerated as in Table 1. In particular, the first to forth edge points are the points of number 1, 3, 6, and 8 in the table; the four internal ones are the points of number 2, 4, 5, and 7. It is clear that none of the points are symmetry points. Moreover, the four edge points are not Gaussian points any more.

A final remark. We have reported in this paper, some new superconvergent points for equilateral triangles, the most important case in practice. Nevertheless, our results can only be used in an interior region of a domain where the periodic pattern of equilateral triangulation is valid and the pollution error due to corner singularity is properly controlled.

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