Tridiagonal matrices

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Similarity

Let

$$T_k = \begin{pmatrix} \alpha_1 & \omega_1 & & & \\ \beta_1 & \alpha_2 & \omega_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{k-2} & \alpha_{k-1} & \omega_{k-1} \\ & & & \beta_{k-1} & \alpha_k \end{pmatrix}$$

and $\beta_i \neq \omega_i,\ i = 1, \ldots, k-1$

Proposition

Assume that the coefficients $\omega_j,\ j = 1, \ldots, k-1$ *are different from zero and the products* $\beta_j\, \omega_j$ *are positive. Then, the matrix* T_k *is similar to a symmetric tridiagonal matrix. Therefore, its eigenvalues are real.*

Proof.

Consider $D_k^{-1} T_k D_k$ which is similar to T_k, D_k diagonal matrix with diag. el. δ_j

Take

$$\delta_1 = 1, \quad \delta_j^2 = \frac{\beta_{j-1} \cdots \beta_1}{\omega_{j-1} \cdots \omega_1}, \ j = 2, \ldots k$$

□

Let

$$J_k = \begin{pmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{k-2} & \alpha_{k-1} & \beta_{k-1} \\ & & & \beta_{k-1} & \alpha_k \end{pmatrix}$$

where the values $\beta_j, j = 1, \ldots, k-1$ are assumed to be nonzero

Proposition

$$\det(J_{k+1}) = \alpha_{k+1} \det(J_k) - \beta_k^2 \det(J_{k-1})$$

with initial conditions

$$\det(J_1) = \alpha_1, \quad \det(J_2) = \alpha_1\alpha_2 - \beta_1^2.$$

The eigenvalues of J_k are the zeros of $\det(J_k - \lambda I)$

The zeros do not depend on the signs of the coefficients β_j, $j = 1, \dots, k-1$

We suppose $\beta_j > 0$ and we have a Jacobi matrix

Cholesky factorizations

Let Δ_k be a diagonal matrix with diagonal elements δ_j, $j = 1, \ldots, k$ and

$$L_k = \begin{pmatrix} 1 & & & & \\ l_1 & 1 & & & \\ & \ddots & \ddots & & \\ & & l_{k-2} & 1 & \\ & & & l_{k-1} & 1 \end{pmatrix}$$

$$J_k = L_k \Delta_k L_k^T$$

$$\delta_1 = \alpha_1, \quad l_1 = \beta_1 / \delta_1$$

$$\delta_j = \alpha_j - \frac{\beta_{j-1}^2}{\delta_{j-1}}, j = 2, \ldots, k, \quad l_j = \beta_j / \delta_j, j = 2, \ldots, k-1$$

The factorization can be completed if no δ_j is zero for $j = 1, \ldots, k-1$

This does not happen if the matrix J_k is positive definite, all the elements δ_j are positive and the genuine Cholesky factorization can be obtained from Δ_k

$$J_k = L_k^C (L_k^C)^T$$

with $L_k^C = L_k \Delta_k^{1/2}$ which is

$$L_k^C = \begin{pmatrix} \sqrt{\delta_1} & & & & \\ \frac{\beta_1}{\sqrt{\delta_1}} & \sqrt{\delta_2} & & & \\ & \ddots & \ddots & & \\ & & \frac{\beta_{k-2}}{\sqrt{\delta_{k-2}}} & \sqrt{\delta_{k-1}} & \\ & & & \frac{\beta_{k-1}}{\sqrt{\delta_{k-1}}} & \sqrt{\delta_k} \end{pmatrix}$$

The factorization can also be written as

$$J_k = L_k^D \Delta_k^{-1} (L_k^D)^T$$

with

$$L_k^D = \begin{pmatrix} \delta_1 & & & & \\ \beta_1 & \delta_2 & & & \\ & \ddots & \ddots & & \\ & & \beta_{k-2} & \delta_{k-1} & \\ & & & \beta_{k-1} & \delta_k \end{pmatrix}$$

Clearly, the only elements we have to compute and store are the diagonal elements δ_j, $j = 1, \dots, k$

To solve a linear system $J_k x = c$, we successively solve

$$L_k^D y = c, \quad (L_k^D)^T x = \Delta_k y$$

The previous factorizations proceed from top to bottom (LU)
We can also proceed from bottom to top (UL)

$$J_k = \bar{L}_k^T D_k^{-1} \bar{L}_k$$

with

$$\bar{L}_k = \begin{pmatrix} d_1^{(k)} & & & & \\ \beta_1 & d_2^{(k)} & & & \\ & \ddots & \ddots & & \\ & & \beta_{k-2} & d_{k-1}^{(k)} & \\ & & & \beta_{k-1} & d_k^{(k)} \end{pmatrix}$$

and D_k a diagonal matrix with elements $d_j^{(k)}$

$$d_k^{(k)} = \alpha_k, \quad d_j^{(k)} = \alpha_j - \frac{\beta_j^2}{d_{j+1}^{(k)}}, \, j = k-1, \ldots, 1$$

From LU and UL factorizations we can obtain all the so-called "twisted" factorizations of J_k

$$J_k = M_k \Omega_k M_k^T$$

M_k is lower bidiagonal at the top for rows with index smaller than l and upper bidiagonal at the bottom for rows with index larger than l

$$\omega_1 = \alpha_1, \quad \omega_j = \alpha_j - \frac{\beta_{j-1}^2}{\omega_{j-1}}, j = 2, \ldots, l-1$$

$$\omega_k = \alpha_k, \quad \omega_j = \alpha_j - \frac{\beta_j^2}{\omega_{j+1}}, j = k-1, \ldots, l+1$$

$$\omega_l = \alpha_l - \frac{\beta_{l-1}^2}{\omega_{l-1}} - \frac{\beta_l^2}{\omega_{l+1}}$$

Eigenvalues

The eigenvalues of J_k are the zeros of $\det(J_k - \lambda I)$

$$\det(J_k - \lambda I) = \delta_1(\lambda) \cdots \delta_k(\lambda) = d_1^{(k)}(\lambda) \cdots d_k^{(k)}(\lambda)$$

This shows that

$$\delta_k(\lambda) = \frac{\det(J_k - \lambda I)}{\det(J_{k-1} - \lambda I)}, \quad d_1^{(k)}(\lambda) = \frac{\det(J_k - \lambda I)}{\det(J_{2,k} - \lambda I)}$$

Theorem

The eigenvalues $\theta_i^{(k+1)}$ of J_{k+1} strictly interlace the eigenvalues of J_k

$$\theta_1^{(k+1)} < \theta_1^{(k)} < \theta_2^{(k+1)} < \theta_2^{(k)} < \cdots < \theta_k^{(k)} < \theta_{k+1}^{(k+1)}$$

(Cauchy interlacing theorem)

Proof.

Eigenvector $x = (y\ \zeta)^T$ of J_{k+1} corresponding to θ

$$J_k y + \beta_k \zeta e^k = \theta y$$

$$\beta_k y_k + \alpha_{k+1}\zeta = \theta\zeta$$

Eliminating y from these relations, we obtain

$$(\alpha_{k+1} - \beta_k^2((e^k)^T(J_k - \theta I)^{-1}e^k))\zeta = \theta\zeta$$

$$\alpha_{k+1} - \beta_k^2 \sum_{j=1}^{k} \frac{\xi_j^2}{\theta_j^{(k)} - \theta} - \theta = 0$$

where ξ_j is the last component of the jth eigenvector of J_k

The zeros of this function interlace the poles $\theta_j^{(k)}$ □

Inverse

Theorem

There exist two sequences of numbers $\{u_i\}, \{v_i\}, i = 1, \dots, k$ *such that*

$$J_k^{-1} = \begin{pmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 & \dots & u_1 v_k \\ u_1 v_2 & u_2 v_2 & u_2 v_3 & \dots & u_2 v_k \\ u_1 v_3 & u_2 v_3 & u_3 v_3 & \dots & u_3 v_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_1 v_k & u_2 v_k & u_3 v_k & \dots & u_k v_k \end{pmatrix}$$

Moreover, u_1 *can be chosen arbitrarily, for instance* $u_1 = 1$

see Baranger and Duc-Jacquet; Meurant

Hence, we just have to compute $J_k^{-1} e^1$ and $J_k^{-1} e^k$

$$J_k v = e^1$$

Use UL factorization of J_k

$$v_1 = \frac{1}{d_1^{(k)}}, \quad v_j = (-1)^{j-1} \frac{\beta_1 \cdots \beta_{j-1}}{d_1^{(k)} \cdots d_j^{(k)}}, j = 2, \ldots, k$$

For

$$v_k J_k u = e^k$$

use LU factorization

$$u_k = \frac{1}{\delta_k v_k}, \quad u_{k-j} = (-1)^j \frac{\beta_{k-j} \cdots \beta_{k-1}}{\delta_{k-j} \cdots \delta_k v_k}, j = 1, \ldots, k-1$$

Theorem

The inverse of the symmetric tridiagonal matrix J_k is characterized as

$$(J_k^{-1})_{i,j} = (-1)^{j-i}\beta_i \cdots \beta_{j-1} \frac{d_{j+1}^{(k)} \cdots d_k^{(k)}}{\delta_i \cdots \delta_k}, \forall i, \forall j > i$$

$$(J_k^{-1})_{i,i} = \frac{d_{i+1}^{(k)} \cdots d_k^{(k)}}{\delta_i \cdots \delta_k}, \forall i$$

Proof.

$$u_i = (-1)^{-(i+1)} \frac{1}{\beta_1 \cdots \beta_{i-1}} \frac{d_1^{(k)} \cdots d_k^{(k)}}{\delta_i \cdots \delta_k}$$

□

The diagonal elements of J_k^{-1} can also be obtained using twisted factorizations

Theorem

Let l be a fixed index and ω_j the diagonal elements of the corresponding twisted factorization
Then

$$(J_k^{-1})_{l,l} = \frac{1}{\omega_l}$$

In the sequel we will be interested in $(J_k^{-1})_{1,1}$

$$(J_k^{-1})_{1,1} = \frac{1}{d_1^{(k)}}$$

Can we compute $(J_k^{-1})_{1,1}$ incrementally?

Theorem

$$(J_{k+1}^{-1})_{1,1} = (J_k^{-1})_{1,1} + \frac{(\beta_1 \cdots \beta_k)^2}{(\delta_1 \cdots \delta_k)^2 \delta_{k+1}}$$

Proof.

$$J_{k+1} = \begin{pmatrix} J_k & \beta_k e^k \\ \beta_k (e^k)^T & \alpha_{k+1} \end{pmatrix}$$

The upper left block of J_{k+1}^{-1} is the inverse of the Schur complement

$$\left(J_k - \frac{\beta_k^2}{\alpha_{k+1}} e^k (e^k)^T \right)^{-1}$$

Inverse of a rank-1 modification of J_k

Use the Sherman–Morrison formula

$$(A + \alpha xy^T)^{-1} = A^{-1} - \alpha \frac{A^{-1}xy^T A^{-1}}{1 + \alpha y^T A^{-1} x}$$

This gives

$$\left(J_k - \frac{\beta_k^2}{\alpha_{k+1}} e^k (e^k)^T \right)^{-1} = J_k^{-1} + \frac{(J_k^{-1} e^k)((e^k)^T J_k^{-1})}{\frac{\alpha_{k+1}}{\beta_k^2} - (e^k)^T J_k^{-1} e^k}$$

Let $l^k = J_k^{-1} e^k$

$$(J_{k+1}^{-1})_{1,1} = (J_k^{-1})_{1,1} + \frac{\beta_k^2 (l_1^k)^2}{\alpha_{k+1} - \beta_k^2 l_k^k}$$

$$l_1^k = (-1)^{k-1}\frac{\beta_1\cdots\beta_{k-1}}{\delta_1\cdots\delta_k}, \quad l_k^k = \frac{1}{\delta_k}$$

To simplify the formulas, we note that

$$\alpha_{k+1} - \beta_k^2 l_k^k = \alpha_{k+1} - \frac{\beta_k^2}{\delta_k} = \delta_{k+1}$$

□

We start with $(J_1^{-1})_{1,1} = \pi_1 = 1/\alpha_1$ and $c_1 = 1$

$$t = \beta_k^2\,\pi_k, \quad \delta_{k+1} = \alpha_{k+1} - t, \quad \pi_{k+1} = \frac{1}{\delta_{k+1}}, \quad c_{k+1} = t\,c_k\,\pi_k$$

This gives

$$(J_{k+1}^{-1})_{1,1} = (J_k^{-1})_{1,1} + c_{k+1}\pi_{k+1}$$

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