Institute for Computational Mathematics
Hong Kong Baptist University

ICM Research Report
08-16

Numerical soliton solutions for a discrete sine-Gordon system

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13 November 2008

Abstract:

In this paper we use an analytical-numerical approach to find, in a systematic way, new 1-soliton solutions for a discrete sine-Gordon system in one spatial dimension. Since the spatial domain is unbounded, the numerical scheme employed to generate these soliton solutions is based on the artificial boundary method. A large selection of numerical examples provides much insight into the possible shapes of these new 1-solitons.

2000 Mathematics Subject Classification: 65M06, 65L10, 35Q53, 35Q51.

Keywords: Sine-Gordon equation, soliton solutions, numerical single solitons, artificial boundary method.

1 Introduction

The sine-Gordon equation,

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sin(u) = 0 \tag{1.1}$$

is a semilinear hyperbolic equation in $1+1$ dimensions. This PDE has its origin in the 19th century where it arose in the study of surfaces of constant negative curvature (cf. [4]). In the second half of the 20th century the sine-Gordon equation has attracted considerable attention, owing to its importance in the mathematical modeling of various physical phenomena, for example in nonlinear optics (propagation of pulses in resonant media); superconductivity (wave propagation in a Josephson transmission line; condensed matter physics (charge density waves in periodic pinning potentials); and in solid state physics (propagation of a dislocation in a crystal). Details and additional examples can be found in [1, 2, 6].

A very important property of the sine-Gordon equation (1.1) is the existence of soliton solutions. Many of these special solutions have been obtained in closed form, by using analytical methods such as Bäcklund transformations ([7]), the nonlinear separation of variables method ([1]); see also [2] (Chapter 6). The known soliton solutions of (1.1) mainly can be classified as follows:

1. 1-soliton solutions : Two 1-soliton solutions are given by ([8, 5])

$$u(x,t) = 4 \arctan \left\{ \exp\left(\pm \frac{x - \mu t - x_0}{\sqrt{1-\mu^2}} \right) \right\}, \quad \mu^2 < 1, \tag{1.2}$$

   and

$$u(x,t) = -\pi + 4 \arctan \left\{ \exp\left(\pm \frac{x - \mu t - x_0}{\sqrt{\mu^2 - 1}} \right) \right\}, \quad \mu^2 > 1. \tag{1.3}$$

   Here, $x_0, \ \mu \in \mathbb{R}$ and $|\mu| \neq 1$.

2. Breather solutions: Two breather solutions to (1.1) are given by ([8, 5])

$$u(x,t) = 4 \arctan \left\{ \frac{\mu \sinh(kx + A)}{k \cosh(\mu t + B)} \right\}, \quad \mu^2 = k + 1, \tag{1.4}$$

   and

$$u(x,t) = 4 \arctan \left\{ \frac{\mu \sin(kx + A)}{k \cosh(\mu t + B)} \right\}, \quad \mu^2 = 1 - k^2 > 0. \tag{1.5}$$

   Here, A and B are arbitrary (real) constants, and the real numbers μ and k are related by the conditions in (1.4) and (1.5), respectively.

3. N-soliton solutions:
   An N-soliton solution for (1.1) is given by

$$u(x,t) = x + \arccos \left\{ 1 - 2 \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) \ln F(x,t) \right\}, \tag{1.6}$$

with

$$F(x,t) := \det[\, M_{ij} \,], \quad M_{ij} := \frac{2}{a_i + a_j} \cosh\left(\frac{z_i + z_j}{2}\right),$$
$$z_i := \pm \frac{x - \mu_i t + C_i}{\sqrt{1-\mu_i^2}}, \quad a_i := \pm\sqrt{\frac{1-\mu_i^2}{1+\mu_i^2}}.$$

Here, C_i and μ denote arbitrary constants and the (real) constants $\{\mu_i\}$ satisfy $\mu_i^2 < 1$.

Numerical methods and numerical simulation have played an important role in the development of soliton theory. In 1965 Zabusky and Kruskal ([10]) studied the interaction of two solitary waves of the KdV equation. They made the important discovery that, in spite of the highly nonlinear interaction (collision) process the two solitary waves retained their velocities and their "shapes" after the collision – there was no break-up of the two waves. After 1965, the numerical method gradually became a very powerful tool in the studying on the nonlinear wave problem in many areas. Much of the subsequent research on soliton interaction, also in other nonlinear PDEs, appeared [12, 13, 15, 16, 17, 18]. Furthermore the numerical method has been applied to find new solitons, for example, recently Han and Xu ([19]) used the numerical approach to find new soliton solutions for the generalized KdV equations: it appears that these solitons do not possess the "closed-form" representations.

In the present paper we focus on 1-soliton solutions of the sine-Gordon equation and a discrete sine-Gordon system. While the definition of solitons in the literature is often loose (especially in higher spatial dimensions), for the sine-Gordon equation in one spatial dimension we have the following definition.

Definition 1.1:
If a solution $u = u(x,t)$ of (1.1) satisfies the two conditions

(1) $u(x,t)$ is a traveling wave; that is, $u(x,t) = v(x - \mu t)$ where the constant μ is the wave speed;

(2) $v(\xi)$ is not a constant on $\mathbb{R}$; and

(3) there exist two constants v_- and v_+ so that

$$v(\xi) \longrightarrow \begin{cases} v_- & \text{when } \xi \to -\infty \\ v_+ & \text{when } \xi \to +\infty; \end{cases} \tag{1.7}$$

then $u(x,t)$ is called a 1-soliton solution of the sine-Gordon equation (1.1).

Obviously the functions $u(x,t)$ given in (1.2) and (1.3) are 1-soliton solutions of the sine-Gordon equation.

Suppose that $u(x,t)$ is a 1-soliton solution of (1.1), and let $\xi := x - \mu t$. Then $v(\xi)$ satisfies

$$(\mu^2 - 1)v_{\xi\xi} + \sin(v) = 0, \quad \xi \in \mathbb{R}, \tag{1.8}$$
$$v(\xi) \to v_-, \quad \xi \to -\infty, \tag{1.9}$$
$$v(\xi) \to v_+, \quad \xi \to +\infty. \tag{1.10}$$

Case (i): $\mu = \pm 1$. The problem (1.8)-(1.10) reduces to

$$\sin(v) = 0, \quad \xi \in \mathbb{R}, \tag{1.11}$$
$$v(\xi) \to v_-, \quad \xi \to -\infty, \tag{1.12}$$
$$v(\xi) \to v_+, \quad \xi \to +\infty. \tag{1.13}$$

For any real numbers v_- and v_+ the problem has no non-constant solution. Thus, the sine-Gordon equation has no 1-soliton solutions when the wave speed $\mu = \pm 1$.
Case (ii): $\mu^2 < 1$. Setting

$$\eta := \frac{\xi}{\sqrt{1-\mu^2}},$$

we may write the problem (1.8)-(1.10) in the form

$$-v_{\eta\eta} + \sin(\eta) = 0, \quad \eta \in \mathbb{R}, \tag{1.14}$$
$$v(\eta) \to v_-, \quad \eta \to -\infty, \tag{1.15}$$
$$v(\eta) \to v_+, \quad \eta \to +\infty. \tag{1.16}$$

Case (iii): $\mu^2 > 1$. In analogy to Case (ii) we set

$$\eta := \frac{\xi}{\sqrt{\mu^2-1}}.$$

Problem (1.8)-(1.10) then becomes

$$v_{\eta\eta} + \sin(\eta) = 0, \quad \eta \in \mathbb{R}, \tag{1.17}$$
$$v(\eta) \to v_-, \quad \eta \to -\infty, \tag{1.18}$$
$$v(\eta) \to v_+, \quad \eta \to +\infty. \tag{1.19}$$

Suppose that $v^-(\eta)$ is a non-constant solution of problem (1.14)-(1.16), and $v^+(\eta)$ is a non-constant solution of (1.17)-(1.19). The following properties are readily verified.

1. The function $v^-(\eta)+\pi$ is a non-constant solution of (1.17)-(1.19), and the function $v^+(\eta) + \pi$ is a non-constant solution of (1.14)-(1.16).

2. For arbitrary constant x_0 the functions

$$u(x,t) = v^-\left(\frac{x - \mu t - x_0}{\sqrt{1-\mu^2}}\right), \quad \mu^2 < 1, \tag{1.20}$$
$$u(x,t) = v^+\left(\frac{x - \mu t - x_0}{\sqrt{\mu^2-1}}\right), \quad \mu^2 > 1, \tag{1.21}$$

   are 1-soliton solutions of the sine-Gordon equation.

In particular, from (1.3) we obtain a function

$$v^0(\eta) := 4 \arctan\left(\exp(\eta)\right)$$

which is a non-constant solution of problem (1.14)-(1.16) with $v_- = 0$ and $v_+ = 2\pi$. Therefore, if we can find all non-constant solutions of the nonlinear problem (1.14)-(1.16) for $v_-, v_+ \in \mathbb{R}$ then we can obtain *all* 1-soliton solutions of the sine-Gordon equation (1.1).

The outline of the paper is as follows. In Section 2, we analyze the solutions of the problem (1.14)-(1.16). In Section 3, we reduce the sine-Gordon equation to a discrete sine-Gordon system by the finite difference method and discuss 1-soliton solutions of the discrete sine-Gordon system. In section 4, a detailed description of our numerical approach for systematically generating 1-soliton solutions of the discrete sine-Gordon system is given. Section 5 contains a representative selection of numerical 1-soliton solutions of the discrete sine-Gordon system . We end the paper by pointing some future work that will grow out of the present study.

2 A nonlinear boundary-value problem on $\mathbb{R}$

For any given real numbers v_- and v_+ we consider the non-constant solutions of the nonlinear problem (1.14)-(1.16). Suppose $v(\eta)$ is a non-constant solution of problem (1.14)-(1.16). From the nonlinear differential equation

$$-v_{\eta\eta} + \sin(v) = 0, \quad \eta \in \mathbb{R},$$

we obtain the first integral

$$(v_\eta)^2 = 2(C - \cos(v)), \tag{2.1}$$

with constant of integration C. The boundary conditions (1.5) and (1.6) yield

$$\lim_{\eta\to-\infty} (v_\eta)^2 = 2(C - \cos(v_-)) = 0, \tag{2.2}$$

$$\lim_{\eta\to+\infty} (v_\eta)^2 = 2(C - \cos(v_+)) = 0, \tag{2.3}$$

and hence

$$C = \cos(v_-) = \cos(v_+). \tag{2.4}$$

This implies that we must have

$$v_- = v_+ + 2k\pi, \quad k = 0, \pm 1, \dots\ . \tag{2.5}$$

Case (i): For $k = 0$ we have $v_- = v_+$. Since $v(\eta)$ is a non-constant solution of (1.4)-(1.6), there exists $\eta_0 \in \mathbb{R}$ such that

$$v_\eta(\eta_0) = 0, \quad v(\eta_0) =: v^* \neq v_-.$$

At the point $\eta = \eta_0$,

$$(v^*_\eta(\eta_0))^2 = 2(C - \cos(v^*)) = 0,$$

and hence

$$\cos(v^*) = C = \cos(v_-), \tag{2.6}$$

implying that

$$v^* = v_- + 2j\pi, \quad j = \pm 1, \pm 2, \dots\ . \tag{2.7}$$

By (2.1) we know that $C - \cos(v(\eta)) \geq 0$ with η between v^* and v_-, and by (2.6) we find

$$C = 1. \tag{2.8}$$

Therefore, we obtain

$$1 = \cos(v_-)$$

and

$$v_- = 2j\pi \ \ (j = 0, \pm 1, \ldots) \quad \text{and} \quad v^* = 2\ell\pi \ \ (\ell = 0, \pm 1, \ldots).$$

Case (ii): $v_- = v_+ + 2k\pi$ $(k \neq 0)$.
Since $(v_\eta)^2 \geq 0$, we have

$$C - \cos(v(\eta)) \geq 0, \quad \eta \in \mathbb{R},$$

and

$$C \geq 1. \tag{2.9}$$

Combining (2.4) and (2.9) we arrive at the equation

$$\cos(v_-) = \cos(v_+) = 1, \tag{2.10}$$

and thus obtain

$$v_- = 2j\pi, \quad j = 0, \pm 1, \ldots\ , \tag{2.11}$$
$$v_+ = 2k\pi, \quad k = 0, \pm 1, \ldots\ . \tag{2.12}$$

From above analysis we know that if the nonlinear boundary-value problem (1.14)-(1.16)possesses a non-constant solution, then (2.11) and (2.12) must hold. Furthermore, if $v(\eta)$ is a non-constant solution of (1.14)-(1.16), then $v(\eta) - 2j\pi$ $(j = \pm 1, \pm 2, \ldots)$ is also a non-constant solution of the boundary-value problem. Therefore, we now need to consider only the nonlinear boundary-value problem on $\mathbb{R}$

$$(\mathrm{P}_k) \qquad \begin{cases} -v_{\eta\eta} + \sin(\eta) = 0, & \eta \in \mathbb{R}, \\ v(\eta) \to 0, & \eta \to -\infty, \\ v(\eta) \to 2k\pi, & \eta \to +\infty. \end{cases} \tag{2.13}$$

for integers $k = 0, \pm 1, \ldots$.
If we can obtain all nonzero solutions of the problem (P_k) for any given integer k, then we can also obtain all the nonzero solutions of (1.14)-(1.16), and hence all 1-soliton solutions of the sine-Gordon equation (1.1). For $k = 1$,we have known from equation (1.2) that the function

$$v(\eta) = 4\arctan(\exp(\eta))$$

is a nonzero solution of problem (P_k) with $k = 1$.

3 A discrete sine-Gordon system

Let Δx and Δt denote the mesh size of variable x and the mesh size of time t. Furthermore let

$$x_j = j\Delta x, \quad j = 0, \ \pm 1, \ \pm 2, \cdots, \tag{3.1}$$
$$t_m = m\Delta t, \quad m = 0, \ 1, \ 2, \cdots. \tag{3.2}$$

Then we can obtain the following finite difference scheme of sine-Gordon equation:

$$\left(u_j^{m+1} - 2u_j^m + u_j^{m-1}\right)/\Delta t^2 - \left(u_{j+1}^m - 2u_j^m + u_{j-1}^m\right)/\Delta x^2 + \sin u_j^m = 0, \qquad (3.3)$$
$$j = 0,\ \pm 1,\ \pm 2, \cdots,\ m = 0,\ 1,\ 2, \cdots,$$

which is called discrete sine-Gordon system. We now consider the traveling solutions of the discrete sine-Gordon system (3.3) with speed μ, in which mesh size Δ is fixed, but mesh size Δt will be determined later. Namely

$$u_j^m = v(x_j - \mu t_m), \qquad (3.4)$$

with $\Delta t = \Delta x/|\mu|$ and $v(x)$ is an unknown function defined on $\{x_j | j = 0,\ \pm 1,\ \pm 2, \cdots\}$.

i. When $\mu > 0$

$$u_j^{m+1} = v(x_j - \mu t_{m+1}) = v\left((x_j - \mu \Delta t) - \mu t_m\right) = u_{j-1}^m,$$
$$u_j^{m-1} = v(x_j - \mu t_{m-1}) = u_{j+1}^m.$$

ii. When $\mu < 0$

$$u_j^{m+1} = v(x_j - \mu t_{m+1}) = u_{j+1}^m,$$
$$u_j^{m-1} = v(x_j - \mu t_{m-1}) = u_{j-1}^m.$$

If $\{u_j^m\}$ satisfies (3.4), from (3.3) we obtain:

$$\frac{u_{j+1}^m - 2u_j^m + u_{j-1}^m}{\Delta t^2} - \frac{u_{j+1}^m - 2u_j^m + u_{j-1}^m}{\Delta x^2} + \sin u_j^m = 0,$$

namely

$$(\mu^2 - 1)(u_{j+1}^m - 2u_j^m + u_{j-1}^m)/\Delta x^2 + \sin u_j^m = 0, \qquad (3.5)$$
$$j = 0,\ \pm 1,\ \pm 2, \cdots,\ m = 0,\ 1,\ 2, \cdots.$$

(i) When $0 < \mu^2 < 1$, let $h^2 = \Delta x^2/(1 - \mu^2)$, from (3.5) we obtain the following system of nonlinear algebraic equations contained infinitely many unknowns:

$$-(v_{j+1} - 2v_j + v_{j-1})/h^2 + \sin v_j = 0, \quad j = 0,\ \pm 1,\ \pm 2, \cdots. \qquad (3.6)$$

(ii) When $\mu^2 > 1$, let $h^2 = \Delta x^2/(\mu^2 - 1)$, we obtain

$$(v_{j+1} - 2v_j + v_{j-1})/h^2 + \sin v_j = 0, \quad j = 0,\ \pm 1,\ \pm 2, \cdots. \qquad (3.7)$$

We now consider the following boundary value problem

$$(\mathrm{P}_k^h) \qquad \begin{cases} -(v_{j+1} - 2v_j + v_{j-1})/h^2 + \sin v_j = 0, \\ v_j \to 0, \qquad \text{when } \ j \to -\infty, \\ v_j \to 2k\pi, \qquad \text{when } \ j \to +\infty. \end{cases} \qquad (3.8)$$

Problem (P_k^h) also can be understood as finite difference approximation of problem (P_k). If we can find a nonzero solution $\{v_j,\ j = 0,\ \pm 1, \cdots\}$ of problem (P_k^h), then we can obtain a traveling solution of the discrete sine-Gordon system with the boundary conditions in (3.8), which is called 1-soliton solution of the discrete sine-Gordon system (3.3).

4 Numerical solution of problem (P_k^h)

There are three essential difficulties in finding accurate numerical solutions of problem (P_k^h):

1. Problem (P_k^h) contains infinitely many unknowns.
2. Problem (P_k^h) is *nonlinear.*
3. The solution of problem (P_k^h) is *not unique.*

In order to overcome the first difficulty, we choose the artificial boundary method ([20],[11]) to reduce problem (P_k^h) to a problem contained only finite unknowns. Thus, introducing two larger integer N and $-N$ (N¿0), when $j \le -N+1$, $v_j \approx 0$, therefore we obtain an approximation of the equation in (3.8)

$$-\frac{v_{j+1} - 2v_j + v_{j-1}}{h^2} + v_j = 0, \quad -\infty < j \le -N+1, \tag{4.1}$$

which is a finite difference equation with constant coefficients. The general solution of (4.1) is given by

$$v_j = c_1(\mu_1)^j + c_2(\mu_2)^j, \quad -\infty < j \le -N+1,$$

with constants c_1, c_2 and

$$\mu_1 = (1 + h^2/2) - h\sqrt{1 + h^2/4} = \frac{1}{(1 + h^2/2) + h\sqrt{1 + h^2/4}},$$

$$\mu_2 = (1 + h^2/2) + h\sqrt{1 + h^2/4}.$$

For $h > 0$, we know that

$$0 < \mu_1 < 1, \quad \text{and} \quad \mu_2 > 1. \tag{4.2}$$

By the boundary condition $v_j \to 0,$ when $j \to -\infty$, we obtain

$$v_j = c_1(\mu_1)^j, \quad -\infty < j \le -N+1,$$

namely

$$v_{-N} = \frac{1}{\mu_1} v_{-N+1}. \tag{4.3}$$

Similarly when $j \ge N-1$, $v_j \approx 2k\pi$, we obtain an approximation of equation (3.8)

$$-\frac{v_{j+1} - 2v_j + v_{j-1}}{h^2} + v_j - 2\pi = 0, \quad N-1 \le j < +\infty, \tag{4.4}$$

$$v_j \to 2k\pi, \quad j \to +\infty. \tag{4.5}$$

The general solution of problem (4.4) and (4.5) is given by

$$v_j - 2k\pi = C(\mu_1)^j, \qquad N-1 \le j < +\infty.$$

Therefore we have

$$v_N - 2k\pi = \mu_1(v_{N-1} - 2k\pi) \tag{4.6}$$

Using the boundary conditions (4.3) and (4.6), the problem (P_k^h) is reduced to

$$(\mathrm{P}_k^h(b)) \qquad \begin{cases} -\left(v_{j-1}-2v_j+v_{j+1}\right)/h^2+\sin v_j=0, \\ \qquad\qquad\qquad\qquad -N+1\le j\le N-1, \\ v_{-N}=\frac{1}{\mu_1}v_{-N+1}, \\ v_N=\mu_1 v_{N-1}+2(1-\mu_1)k\pi. \end{cases} \tag{4.7}$$

where $b=Nh$. The problem $(\mathrm{P}_k^h(b))$ is a system of nonlinear algebraic equations, which only contains finite unknowns $\{v_j, -N\le j\le N\}$.

In order to solve the nonlinear system $(\mathrm{P}_k^h(b))$, we choose an appropriate iteration method with carefully selected initial data. This method is described as follows:

1. For a given initial function $v^0(\eta)$ we choose the initial data by

$$v_j^0 := v^0(\eta_j), \quad -N\le j\le N \tag{4.8}$$

   with $\eta_j = jh$.

2. For known data $\{v_j^{n-1}:\ -N\le j\le N\}$, solve the linear algebraic system

$$\begin{cases} -\frac{v_{j-1}^n-2v_j^n+v_{j+1}^n}{h^2}+v_j^n=v_j^{n-1}-\sin(v_j^{n-1}), \\ \qquad\qquad\qquad\qquad -N+1\le j\le N+1, \\ v_{-N}=\frac{1}{\mu_1}v_{-N+1}, \\ v_N=\mu_1 v_{N-1}+2(1-\mu_1)k\pi. \end{cases} \tag{4.9}$$

3. The stopping rule is

$$\max_{-N\le j\le N}|v_j^n-v_j^{n-1}|\le\varepsilon:=10^{-11}. \tag{4.10}$$

When the condition (4.10) is satisfied, the iteration will stop and we obtain a numerical approximation solution of problem (P_k^h). Let $v_h^n(\eta)$ denote by this numerical approximation, namely, $v_h^n(\eta)$ is a function defined on $\{\eta_j,\ -N\le j\le N\}$ and $v_h^n(\eta_j)=v_j^n,\ -N\le j\le N$. From our computation we understand that in general the problem $\mathrm{P}_k^h(b)$ exists many solutions. For different choice of initial functions $v^0(\eta)$, the different numerical solutions can be obtained.

5 New 1-soliton solutions of the discrete sine-Gordon system

Firstly we consider the problems $(\mathrm{P}_k)$and $\mathrm{P}_k^h)$ with $k=1$ as a test problem. The problem (P_1) admits a soliton solution

$$v(\eta)=4\arctan\{\exp(\eta)\},\quad \eta\in\mathbb{R},$$

which approaches 2π as $\eta\to+\infty$ and 0 as $\eta\to-\infty$. Thus we can use this reference solution to test the performance of our numerical approach for systematically generating 1-soliton solutions. In the calculation, we take mesh size $h=1/4$, Figure 1 shows

Table 1: L_1-norm and order in interval [-8,8].

	$h=\frac{1}{4}$	order	$h=\frac{1}{8}$	order	$h=\frac{1}{16}$	order	$h=\frac{1}{32}$	order
Error	8.739e-4	–	2.204e-4	1.987	5.534e-5	1.994	1.38e-5	1.997

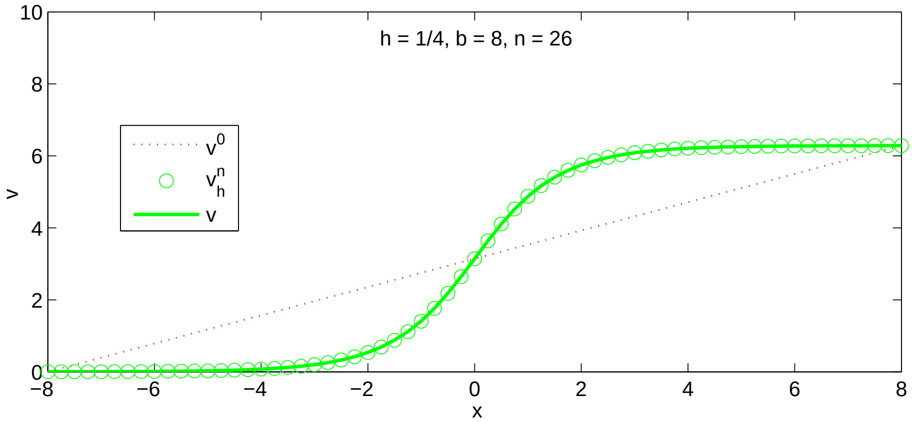

Figure 1: Monotone 1-soliton (cf. (1.2)) ((P_k^h) with k = 1)

the numerical solution to match the exact solution perfectly with the initial piece-wise function

$$v^0 = \frac{\pi}{b}\eta + \pi,$$

where $b = Nh$ is used to characterize the length of the computational domain. In fact, for this example, we also can choose the other nonzero piece-wise initial functions to obtain the same numerical solution in figure 1. Furthermore, for the given tolerance error $\varepsilon = 10^{-11}$, this numerical scheme can convergent to the numerical solution under iteration times $n = 26$. To evaluate the convergence rate of the numerical solution accurately, one error estimation is defined by

$$E_1 = ||v_h^n(\cdot) - v_{exa}(\cdot)||_1. \tag{5.1}$$

Table 1 shows the accuracy of our numerical scheme under L_1-norm (5.1) with mesh size $h = 1/4,\ 1/8,\ 1/16,\ 1/32$. One can see that the numerical scheme is of second order.

Next, we present a selection of the many new 1-soliton solutions of the discrete sine-Gordon system, which are obtained by numerical method given in section 4 and we call them "numerical solitons". In the each numerical result, we firstly list the basic parameter k, h, $b = Nh$, the times of iteration n and the initial function $v^0(\eta)$, which can be founded in the figures for each numerical soliton. Secondly, let $v_h^n(\eta)$ denote the corresponding numerical solutions of problem (P_k^h). For the monotone soliton solutions, one can refer to figure 2-4. For the tower-like soliton solutions, one can see the different level plots in figure 4-6. For some special soliton solutions, which are showed in figure 7-11.

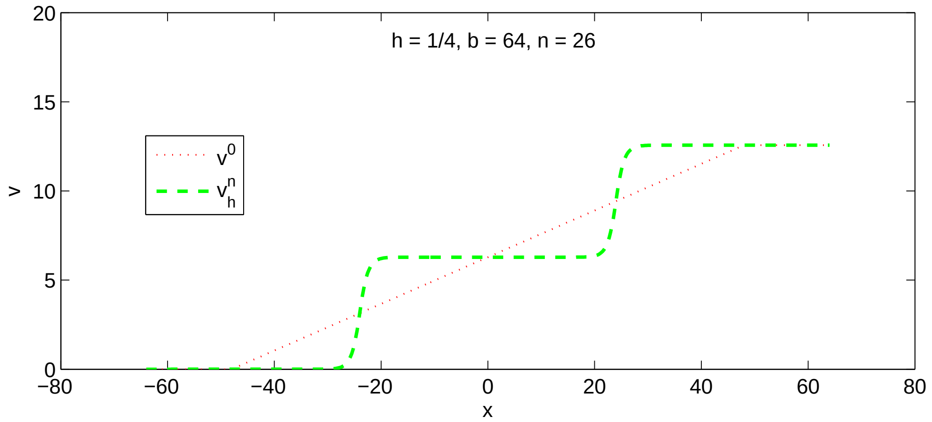

Figure 2: Monotone 2-step 1-soliton ((P_k^h) with k = 2)

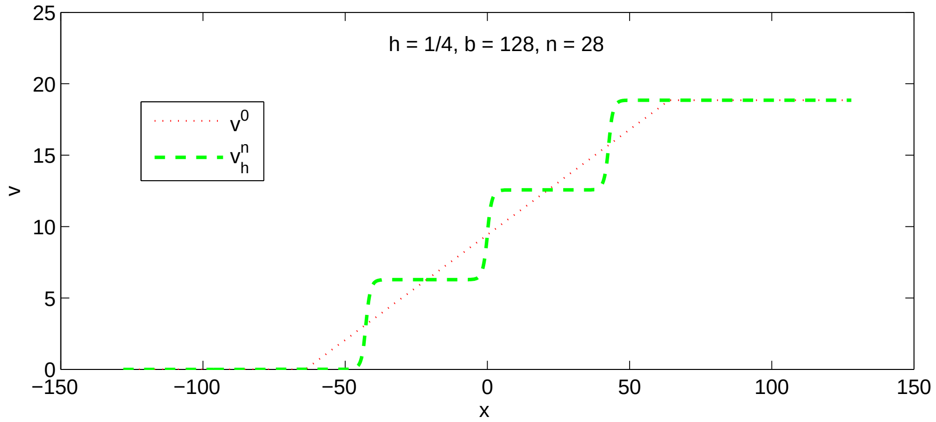

Figure 3: Monotone 3-step 1-soliton ((P_k^h) with k = 3)

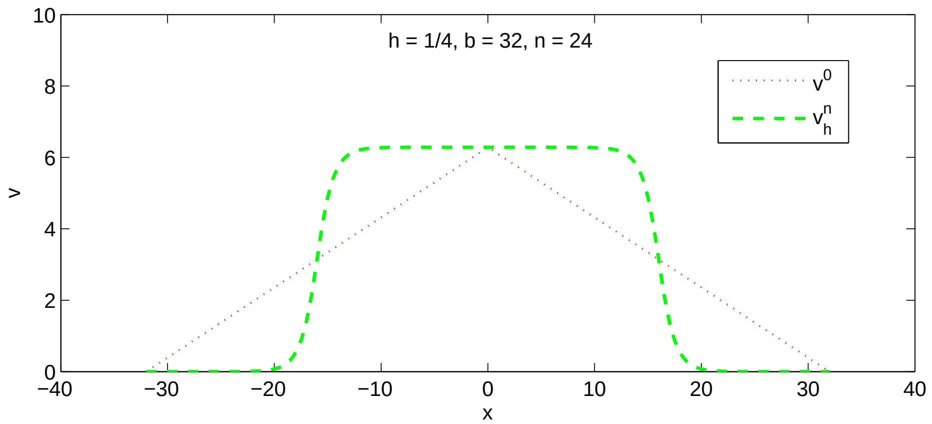

Figure 4: symmetric 1-level 1-soliton ((P_k^h) with k = 0)

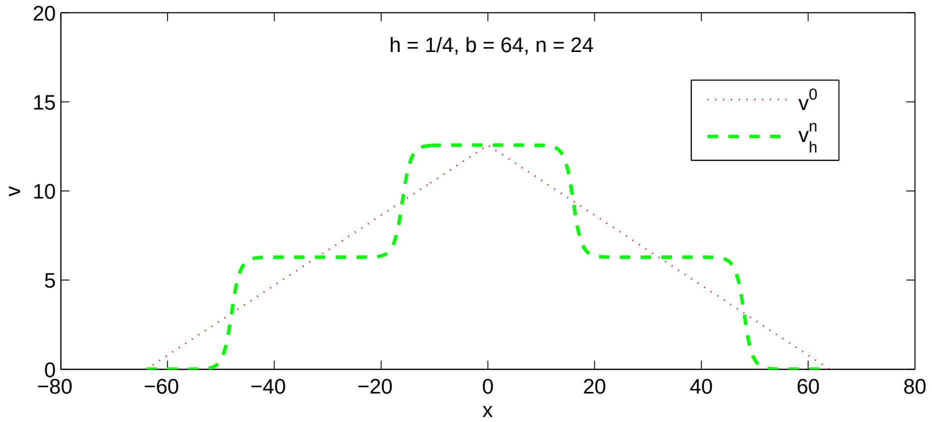

Figure 5: symmetric 2-level 1-soliton $((P_k^h)$ with k $= 0)$

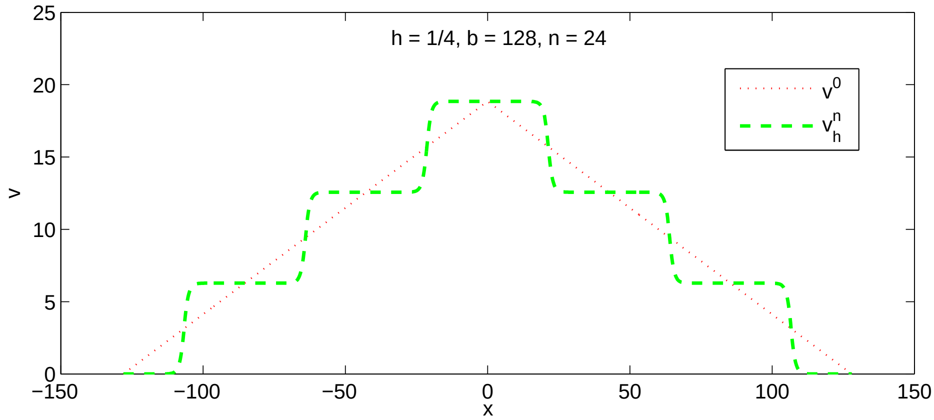

Figure 6: symmetric 3-level 1-soliton $((P_k^h)$ with k $= 0)$

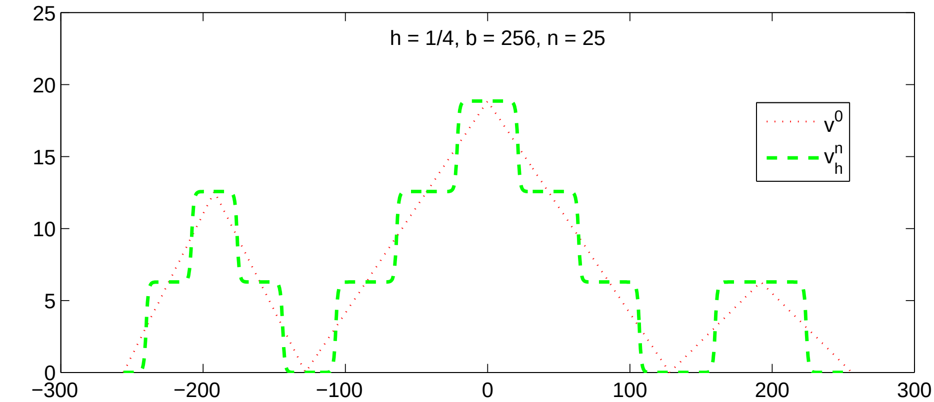

Figure 7: Non-symmetric 1-soliton with three symmetric towers $((P_k^h)$ with k $= 0)$

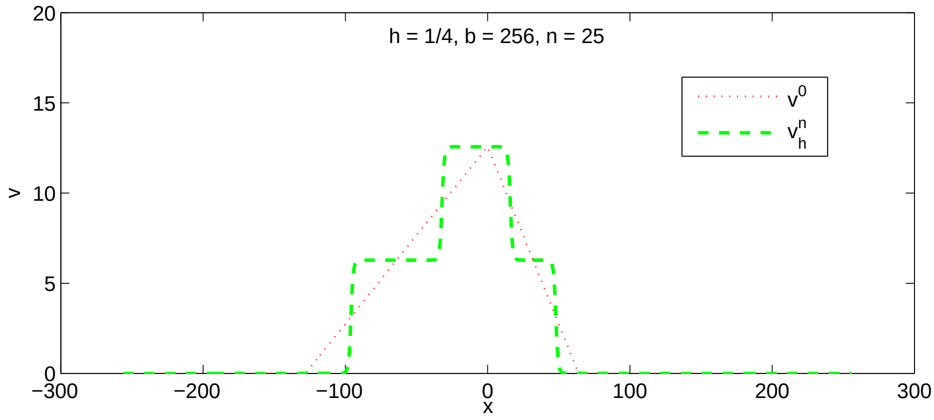

Figure 8: Non-symmetric 1-tower 1-soliton $((P_k^h)$ with k $= 0)$

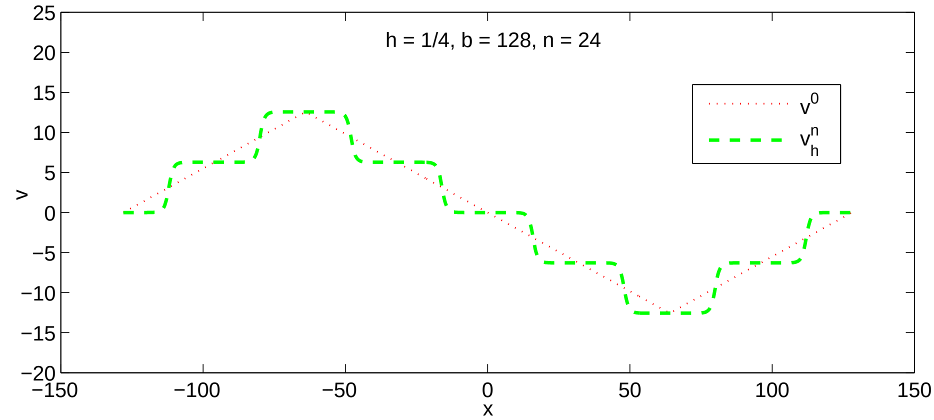

Figure 9: General nonsymmetric 1-soliton $((P_k^h)$ with k $= 0)$

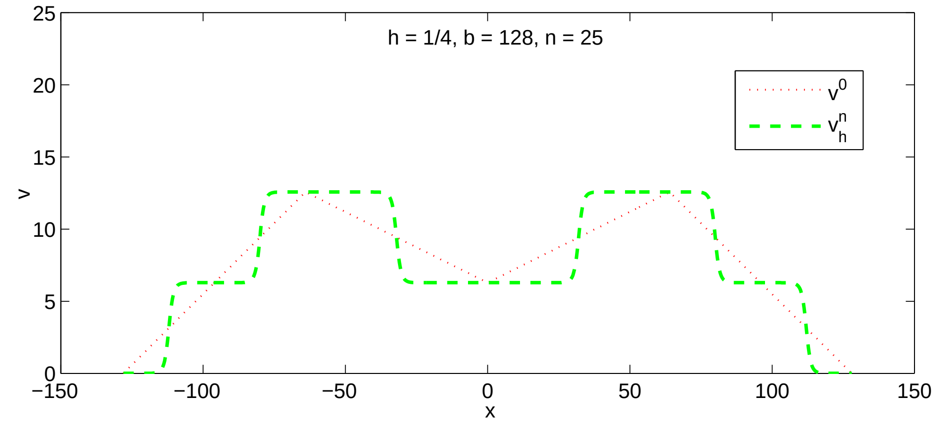

Figure 10: symmetric 2-tower 1-soliton $((P_k^h)$ with k $= 0)$

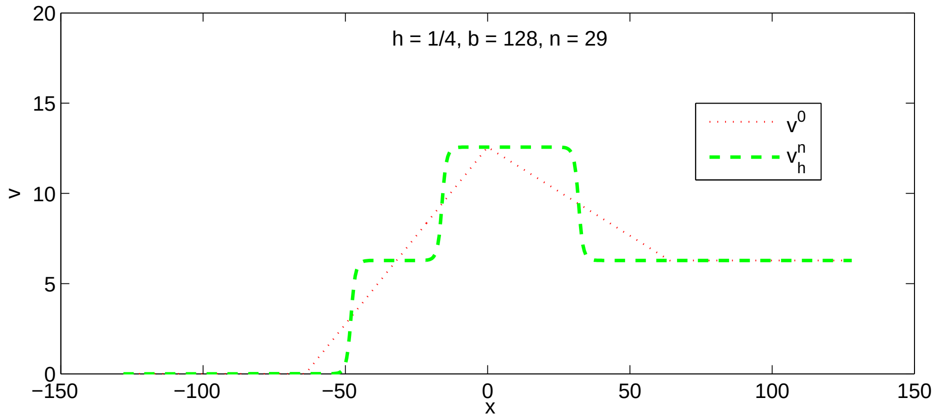

Figure 11: Non-symmetric 2-level 1-soliton $((P_k^h)$ with k = 1)

6 Concluding remarks

We have shown that combining analytical techniques with an appropriate numerical approach can form the basis of a systematic numerical search for new 1-soliton solutions of the discrete (spatially one-dimensional) sine-Gordon system. This approach provides considerable insight into the properties of possible 1-solitons for this PDE.

Those results will form the basis for the numerical study on soliton interaction for the sine-Gordon equation.

Acknowledgements

The research was supported in part by the Natural Sciences and Engineering Research Council (NSERC) of Canada, by Hong Kong Research Grants Council and FRG of Hong Kong Baptist University.
This paper was written during a recent two-month research visit by Professor Houde Han to Hong Kong Baptist University. He gratefully acknowledges the hospitality extended to him by the Department of Mathematics and the financial support by HKRGC.

References

[1] R.K. Dold, J.C. Eibeck, J.D. Gibbon and H.C. Morris, *Solitons in Nonlinear Wave Equations*, Academic Press, New York, 1982.

[2] P.G. Drazin, *Solitons*, London Mathematical Society Lecture Note Series, 85, Cambridge University Press, Cambridge, 1983.

[3] P.G. Drazin and R.S. Johnson, *Solitons: An Introduction*, Cambridge University Press, Cambridge, 1989.

[4] L.P. Eisenhart, *A Treatise on the Differential Geometry of Curves and Surfaces*, Dover, New York, 1960.

[5] E. Infeld and G. Rowlands, *Nonlinear waves, Solitons and Chaos*, Second Edition, Cambridge University Press, Cambridge, 2000.

[6] A.C. Newell, *Solitons in mathematics and Physics*, Society for Industrial and Applied Mathematics, Philadelphia, 1985.

[7] C. Rogers and W.K. Schief, *Bäcklund and Darboux Transformations*, Cambridge University Press, Cambridge, 2002.

[8] M. Tabor, *Chaos and Integrability in Nonlinear Dynamics. An Introduction*, Wiley, New York, 1989.

[9] N.J. Zabrusky, Computational synergetics and mathematical innovation, *J. Comput. Phys.*, 43 (1981), 195–249.

[10] N.J. Zabusky and M.D. Kruskal, Interaction of solitons in a collisionless plasma and recurrence of initial states, *Phys. Rev. Lett.*, 15 (1965), 240–243.

[11] H. Han, The artificial boundary method - numerical solutions of partial differential equations in unbounded domains, in *Frontiers and Prospents of Contemporary Applied Mathematics*, edited by T. Li, P. Zhang, Higher Education Press and World Scientific, (2005), 33-58.

[12] M.J. Ablowitz, B.M. Herbst, C. Schober, On the numerical solution of the sine-Gordon equation: I. Integrable discretizations and homoclinic monifolds, *J. Comput. Phys.,* 126 (1996), 299–314.

[13] M.J. Ablowitz, B.M. Herbst, C. Schober, On the numerical solution of the sine-Gordon equation: II. Performance of numerical schemes, *J. Comput. Phys.,* 131 (1997), 354–367.

[14] F. Rus, F.R. Villatoro, Padé numerical method for the Rosenau-Hyman compacton equation, *Math. Comput. Simul.* 76 (1-3) (2007), 188-192.

[15] P. Rosenau, J.M. Hyman, M. Staley, Multidimensional compactons, *Phys. Rev. Lett.,* 98 (2007), 024101.

[16] F. Rus, F.R. Villatoro, Padé numerical method for the Rosenau-Hyman compacton equation, *Math. Comput. Simul.* 76 (1-3) (2007), 188-192.

[17] M.S. Ismail, T.R. Taha, A numerical study of compactons, *Math. Comput. Simul.,* 47(6) (1998), 519-530.

[18] F. Rus, F.R. Villatoro, Self-similar radiation from numerical Rosenau-Hyman compactons, *J. Comput. Phys.*, 227 (2007), 440-454.

[19] H.D. Han, Z.L. Xu, Numerical solitons of generalized Korteweg-de Vries equations, *Appl. Math. Comput.*, 186 (2007), 483–487.

[20] H.D. Han, X.N. Wu, Approximation of infinite boundary condition and its applications to finite element methods, *J. Comput. Math.* 3 (1985), 179–192.