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Closed Form Formula for WENO Smoothness Indicators with Arbitrary Order

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Abstract

In this article, we present a general closed form formula for computing lower order local smoothness indicators used in forming nonlinear WENO weights of arbitrary orders $2r-1$. The closed form formula of the smoothness indicators is given in a convenient matrix form. The smoothness indicators can be derived for any order of the WENO scheme through a recursive formula. A Maple code used in deriving the necessary coefficients of the lower order polynomials are also presented so that users can regenerate the necessary coefficients on their own.

Keywords
Weighted Essentially Non-Oscillatory, WENO-Z, Smoothness Indicators, Smoothness Influence Matrix

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1 Introduction

We start by briefly describe the $(2r-1)$_th order weighted essentially non-oscillatory conservative finite difference scheme when applied to hyperbolic conservation laws in the form of

$$\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{F}(\mathbf{u}) = 0. \tag{1}$$

Without loss of generality, we will restrict our discussion to the one dimensional scalar case.

Consider an uniform grid defined by the points $x_i = i\Delta x, i = 0, \ldots, N$, which are also called cell centers, with cell boundaries given by $x_{i+\frac{1}{2}} = x_i + \frac{\Delta x}{2}$, where Δx is the uniform grid spacing. The semi-discretized

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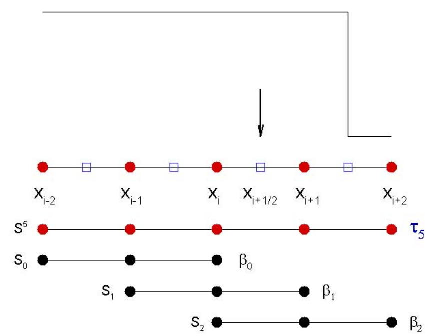

Figure 1: The computational uniform grid x_i and the 5-points stencil S^5, composed of three 3-points stencils S_0, S_1, S_2, used for the fifth-order ($r = 3$) WENO reconstruction step.

form of (1), by the method of lines, yields a system of ordinary differential equations

$$\frac{du_i(t)}{dt} = - \left.\frac{\partial f}{\partial x}\right|_{x=x_i}, \quad i = 0, \ldots, N, \tag{2}$$

where $u_i(t)$ is a numerical approximation to the point value $u(x_i, t)$.

A conservative finite-difference formulation for hyperbolic conservation laws requires high-order consistent numerical fluxes at the cell boundaries in order to form the flux differences across the uniformly-spaced cells. The conservative property of the spatial discretization is obtained by implicitly defining the numerical flux function $h(x)$ as

$$f(x) = \frac{1}{\Delta x} \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} h(\xi) d\xi, \tag{3}$$

such that the spatial derivative in (2) is exactly approximated by a conservative finite difference formula at the cell boundaries,

$$\frac{du_i(t)}{dt} = \frac{1}{\Delta x} \left(h_{i+\frac{1}{2}} - h_{i-\frac{1}{2}} \right), \tag{4}$$

where $h_{i\pm\frac{1}{2}} = h(x_{i\pm\frac{1}{2}})$.

High order polynomial interpolations to $h_{i\pm\frac{1}{2}}$ are computed using known grid values of f, $f_i = f(x_i)$. For example, the classical fifth-order WENO scheme uses a 5-points stencil, hereafter named S^5, which is subdivided into three 3-points stencils $\{S_0, S_1, S_2\}$, as shown in Fig. 1. The r_th degree polynomial approximation $\hat{f}_{i\pm\frac{1}{2}} = h_{i\pm\frac{1}{2}} + O(\Delta x^{2r-1})$ is built through the convex combination of the interpolated values $\hat{f}^k(x_{i\pm\frac{1}{2}})$, in which $\hat{f}^k(x)$ is the r degree polynomial below, defined in each one of the stencils $S_k, k =$

$0, \ldots, r-1$:

$$\hat{f}_{i\pm\frac{1}{2}} = \sum_{k=0}^{r-1} \omega_k \hat{f}^k(x_{i\pm\frac{1}{2}}), \tag{5}$$

where

$$\hat{f}^k(x_{i+\frac{1}{2}}) = \hat{f}^k_{i+\frac{1}{2}} = \sum_{j=0}^{r-1} c_{kj} f_{i-k+j}, \qquad \hat{f}^k(x_{i-\frac{1}{2}}) = \hat{f}^k_{i-\frac{1}{2}} = \sum_{j=0}^{r-1} \tilde{c}_{kj} f_{i-k-j}, \quad i = 0, \ldots, N. \tag{6}$$

The c_{kj} and $\tilde{c}_{jk}$ are Lagrangian interpolation coefficients (see [2]), which depend on the left-shift parameter $k = 0, \ldots, r-1$, but not on the values f_i.

The un-normalized nonlinear weights α_k and normalized nonlinear weights ω_k defined in [3] are, respectively,

$$\alpha_k = d_k \left(1 + \left(\frac{\tau_{2r-1}}{\beta_k + \epsilon} \right)^p \right), \qquad \omega_k = \frac{\alpha_k}{\sum_{l=0}^{r-1} \alpha_l}, \tag{7}$$

where τ_{2r-1} is the global higher order smoothness indicator formed by a linear combination of lower order local smoothness indicator β_k and yields higher order information about the underlining function being measured [3]. The coefficients $d_k, k = 0, \ldots, r-1$ are called the ideal weights since they generate a $(2r-1)$_th order central upwinding scheme at $x_{i+\frac{1}{2}}$ based on the $2r-1$ points stencil S^{2r-1}. The parameter ϵ is used to avoid the division by zero in the denominator and p is chosen to increase the difference of scales of distinct weights at non-smooth parts of the solution.

The success of the WENO method is largely based on the classical lower order local smoothness indicators

$$\beta_k = \sum_{n=1}^{r-1} \Delta x^{2n-1} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(\frac{d^n}{dx^n} \hat{f}^k(x) \right)^2 dx, \qquad k = 0, \ldots, r-1. \tag{8}$$

They measure the sum of normalized regularity in L^2 norm of the $(r-1)$_th degree polynomial approximation $\hat{f}^k(x)$ and its derivatives up to $r-1$ order at the grid cell center x_i in the stencil $S_k, \quad k = 0, \ldots, r-1$.

For the rest of the paper, instead of providing the explicit but often unreadable and error prone numerical values of various coefficients used in the definition of the WENO scheme of a given order $2r-1$ (see [1, 4] for example), we will describe a general yet simple procedure which yields a compact closed form formula for generalization of the lower order local smoothness indicators $\beta_k, k = 0, \ldots, r-1$ to $(2r-1)$_th order WENO scheme with arbitrary r.

2 Classical Lower Order Local Smoothness Indicators β_k

In the following discussion, we will develop a general framework for the derivation of the lower order local smoothness indicators β_k of arbitrary order in a compact bilinear form and discuss some of its properties. We shall suppress the shifting parameter k from the discussion below for clarity reason. It will be noted explicitly when and if situation warranted. From hereon, for simplicity, we shall refer the lower order local smoothness indicators β_k as smoothness indicators β_k unless stated otherwise.

Consider a given set of stencils S_k (see figure 1) consisting of the functional values

$$\{f_{k-r+1}, f_{k-r+2}, \ldots, f_k\} \quad \text{at} \quad \{x_{k-r+1}, x_{k-r+2}, \ldots, x_k\}, \quad k = 0, \ldots, r-1, \tag{9}$$

where $f_i = f(x_i), x_i = i\Delta x$ is the grid point with uniform grid spacing Δx, a polynomial of degree $r-1$ that interpolate a function $f(x)$ (without loss of generality) at $x_0 = 0$ can be expressed as

$$p_k(x) = \sum_{l=0}^{r-1} a_l x^l, \tag{10}$$

where a_l are the coefficients of the polynomial $p_k(x)$.

The coefficient a_l is a linear combination of $\{f_{k-r+1}, f_{k-r+2}, \ldots, f_k\}$ for a given shifting parameter k. (It is important for the reader to note that a_l are different for different shifting parameter k.)

We define the polynomial coefficients vector of order n,

$$\mathbf{a}_n = \{a_n, a_{n+1}, ..., a_{r-1}\}_k^T, \tag{11}$$

where a_n are the coefficients of $p_k(x)$ in (10). Each coefficient vector of order n has $r-n$ elements. We shall also define $\mathbf{a} = \mathbf{a}_1$ for the discussion from here on.

The n_th derivative of $p_k(x)$, is

$$p_k^{(n)}(x) \quad = \quad \sum_{l=n}^{r-1} \frac{l!}{(l-n)!} a_l x^{l-n}, \quad 1 \leq n \leq r-1. \tag{12}$$

Therefore,

$$\Delta x^{2n-1} \int_{-\frac{1}{2}\Delta x}^{\frac{1}{2}\Delta x} \left(p_k^{(n)}(x)\right)^2 dx = \sum_{l=n}^{r-1} \sum_{\substack{j=n \\ l+j \text{ even}}}^{r-1} a_l a_j C_{lj}^n = \sum_{l=n}^{r-1} a_l^2 C_{ll}^n + 2 \sum_{l=n}^{\frac{r-1}{2}} \sum_{\substack{j=l+2 \\ l+j \text{ even}}}^{r-1} a_l a_j C_{lj}^n, \tag{13}$$

where

$$C_{lj}^n = \begin{cases} \frac{l!j!}{(l-n)!(j-n)!} \frac{2^{-(l+j-2n)}}{(l+j-2n+1)} \Delta x^{l+j} & \text{mod } (l+j, 2) = 0 \\ 0 & \text{mod } (l+j, 2) = 1 \end{cases}, \quad j, l = 0, \ldots, r-1. \tag{14}$$

is an element of a $(r-n) \times (r-n)$ matrix denoted as $\mathbf{C}^n$. Using this matrix, (13) can be recasted in a bilinear inner product form as

$$\Delta x^{2n-1} \int_{-\frac{1}{2}\Delta x}^{\frac{1}{2}\Delta x} \left(p_k^{(n)}(x)\right)^2 dx = < \mathbf{a}_n, \mathbf{C}^n \mathbf{a}_n > . \tag{15}$$

Once all the matrices $\mathbf{C}^n$, $n = 1, \ldots, r-1$ are defined, the smoothness indicators β_k can be found as, after exchanging the summations,

$$\beta_k = \sum_{n=1}^{r-1} < \mathbf{a}_n, \mathbf{C}^n \mathbf{a}_n > \quad = \quad \sum_{l=1}^{r-1} a_l^2 \sum_{n=1}^{l} C_{ll}^n + 2 \sum_{l=1}^{\frac{r-1}{2}} \sum_{\substack{j=l+2 \\ l+j \text{ even}}}^{r-1} a_l a_j \sum_{n=1}^{l} C_{lj}^n, \tag{16}$$

where the first term on the right hand side is the sum of diagonal terms involving C_{ll}^n and the second term on the right hand side is the sum of the upper-triangular terms involving $C_{lj}^n, j = l+2, \ldots, r-1$ and $l+j$ is even (see example below).

Equation (16) can be simplified further as

$$\beta_k =< \mathbf{a}, \mathbf{A}\mathbf{a} > \;=\; \sum_{l=1}^{r-1} \sum_{\substack{j=1 \\ l+j \text{ even}}}^{r-1} a_l a_j A_{lj} = \sum_{l=1}^{r-1} a_l^2 A_{ll} + 2 \sum_{l=1}^{\frac{r-1}{2}} \sum_{\substack{j=l+2 \\ l+j \text{ even}}}^{r-1} a_l a_j A_{lj}, \tag{17}$$

where $\mathbf{A}$ is a $(r-1) \times (r-1)$ matrix with element

$$A_{lj} = \sum_{n=1}^{l} C_{lj}^n. \tag{18}$$

The matrix $\mathbf{A}$ is called *Smoothness Influence Matrix* and have the following properties:

- It depends on the degree of polynomial $(r-1)$ but independent of the number of derivative (n) and the shifting parameter (k).
- It has zero off-diagonal elements with odd sum of row and column index above and below the diagonal elements. In the other words, the matrix has a structure as illustrated below.

$$\mathbf{A} = \begin{pmatrix} \times & 0 & \times & 0 & \times \\ 0 & \times & 0 & \times & 0 \\ \times & 0 & \times & 0 & \times \\ 0 & \times & 0 & \times & 0 \\ \times & 0 & \times & 0 & \times \end{pmatrix},$$

  where $\times$ denotes a non-zero positive rational number.
- The element A_{lj} is of $O(\Delta x^{l+j})$.
- $\mathbf{A}$ is a positive-definite symmetric real matrix.
- If we denote $\mathbf{A}^q$ as $\mathbf{A}$ associated with the case where polynomial is of degree $1 < q \leq n-1$, then $\mathbf{A}_{lj}^q = \mathbf{A}_{lj}^n, 1 \leq l, j \leq q, q < n$. In the other words, the matrix of $\mathbf{A}$ of degree $1 \leq q \leq, n-1$ is the submatrix of $\mathbf{A}$ of degree n as illustrated in (21) below.
- The diagonal elements of $\mathbf{A}$ $(j = l)$ has a simple form

$$A_{ll} = \sum_{n=1}^{l} \frac{(l!)^2}{(2(l-n)+1)4^{l-n}((l-n)!)^2} \Delta x^{2l}. \tag{19}$$

For example, $\{\mathbf{C}^n, n=1,\ldots,4\}$ can be computed via (14),

$$\mathbf{C}^1 = \begin{pmatrix} \Delta x^2 & 0 & \frac{1}{4}\Delta x^4 & 0 \\ 0 & \frac{1}{3}\Delta x^4 & 0 & \frac{1}{10}\Delta x^6 \\ \frac{1}{4}\Delta x^4 & 0 & \frac{9}{80}\Delta x^6 & 0 \\ 0 & \frac{1}{10}\Delta x^6 & 0 & \frac{1}{28}\Delta x^8 \end{pmatrix}, \quad \mathbf{C}^2 = \begin{pmatrix} 4\Delta x^4 & 0 & 2\Delta x^6 \\ 0 & 3\Delta x^6 & 0 \\ 2\Delta x^6 & 0 & \frac{9}{5}\Delta x^8 \end{pmatrix},$$

$$\mathbf{C}^3 = \begin{pmatrix} 36\Delta x^6 & 0 \\ 0 & 48\Delta x^8 \end{pmatrix}, \quad \mathbf{C}^4 = \begin{pmatrix} 576\Delta x^8 \end{pmatrix}, \tag{20}$$

and the Smoothness Influence Matrix $\mathbf{A}$ is computed via (18),

$$\mathbf{A} = \begin{pmatrix} \Delta x^2 & 0 & \frac{1}{2}\Delta x^4 & 0 \\ 0 & \frac{13}{3}\Delta x^4 & 0 & \frac{21}{5}\Delta x^6 \\ \frac{1}{2}\Delta x^4 & 0 & \frac{3129}{80}\Delta x^6 & 0 \\ 0 & \frac{21}{5}\Delta x^6 & 0 & \frac{87617}{140}\Delta x^8 \end{pmatrix}. \tag{21}$$

Then, the smoothness indicators $\beta_k =< \mathbf{a}, \mathbf{A}\mathbf{a} >$ for a given shifting parameter $k = 0,\ldots,r-1$ (Note that $\mathbf{a}$ depends on k) becomes

- for $r = 3$ ($p_k(x)$ is a polynomial of degree 2),

$$\begin{aligned} \beta_k &= a_1^2 A_{11} + a_2^2 A_{22} \\ &= a_1^2 \Delta x^2 + \frac{13}{3} a_2^2 \Delta x^4. \end{aligned}$$

- for $r = 5$ ($p_k(x)$ is a polynomial of degree 4),

$$\begin{aligned} \beta_k &= a_1^2 A_{11} + a_2^2 A_{22} + a_3^2 A_{33} + a_4^2 A_{44} + 2(a_1 a_3 A_{13} + a_2 a_4 A_{24}) \\ &= a_1^2 \Delta x^2 + \frac{13}{3} a_2^2 \Delta x^4 + \frac{3129}{80} a_3^2 \Delta x^6 + \frac{87617}{140} a_4^2 \Delta x^8 + 2\left(\frac{1}{2} a_1 a_3 \Delta x^4 + \frac{21}{5} a_2 a_4 \Delta x^6\right). \end{aligned}$$

We remark that

- The Δx^{l+j} factor in the matrix A will be canceled out by those contained in the product of $a_l a_j$.
- Due to the fact that the smoothness indicators β_k are independent of grid spacing Δx, one can simply take $\Delta x = 1$ and keep track of the order of the Δx in each term by summing the indexes up.
- The computation of β_k only requires the elements of the upper triangular elements of the Smoothness Influence matrix $\mathbf{A}$ and only every other off-diagonal elements in each row.

  Therefore, the number of distinct terms in the form of $a_l a_j$ involved in the computation of β_k is

$$\begin{cases} \displaystyle\sum_{n=1}^{\frac{r-1}{2}} (2n) = \frac{r-1}{2}\left(\frac{r-1}{2}+1\right) & \mod(r-1,2) = 0 \\ \displaystyle\sum_{n=0}^{\frac{r}{2}-1} (2n+1) = \left(\frac{r}{2}\right)^2 & \mod(r-1,2) = 1 \end{cases}. \tag{22}$$

- It is easy to see that the matrix $\mathbf{A}$ is a positive definite symmetric real matrix since, for any $\mathbf{a} \neq 0$,

$$0 < \beta_k =< \mathbf{a}, \mathbf{A}\mathbf{a} > . \tag{23}$$

  Therefore, $\mathbf{A} = \mathbf{S}\Lambda\mathbf{S}^T$ where Λ is the diagonal matrix containing the eigenvalues of $\mathbf{A}$, λ and $\mathbf{S}$ is the orthogonal transformation matrix whose columns are the eigenvectors of $\mathbf{A}$. Moreover,

$$\beta_k = \sum_{l=1}^{r-1} \lambda_l \mathbf{w}_l \mathbf{w}_l^T, \tag{24}$$

  where the vector $\mathbf{w} = \mathbf{S}^T\mathbf{a}$.

2.1 Smoothness Influence Matrix A

The smoothness influence matrix $\mathbf{A}_{r-1}$ in the case of $r = 8$, after taking $\Delta x = 1$,

$$\mathbf{A}_{r-1} = \begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 & \frac{1}{16} & 0 & \frac{1}{64} & 0 \\ 0 & \frac{13}{3} & 0 & \frac{21}{10} & 0 & \frac{87}{112} & 0 & \frac{37}{144} \\ \frac{1}{4} & 0 & \frac{3129}{80} & 0 & \frac{14127}{448} & 0 & \frac{12535}{768} & 0 \\ 0 & \frac{21}{10} & 0 & \frac{87617}{140} & 0 & \frac{508579}{672} & 0 & \frac{551543}{1056} \\ \frac{1}{16} & 0 & \frac{14127}{448} & 0 & \frac{252337135}{16128} & 0 & \frac{895099145}{33792} & 0 \\ 0 & \frac{87}{112} & 0 & \frac{508579}{672} & 0 & \frac{11102834003}{19712} & 0 & \frac{46545155573}{36608} \\ \frac{1}{64} & 0 & \frac{12535}{768} & 0 & \frac{895099145}{33792} & 0 & \frac{16165726308907}{585728} & 0 \\ 0 & \frac{37}{144} & 0 & \frac{551543}{1056} & 0 & \frac{46545155573}{36608} & 0 & \frac{969943578534563}{549120} \end{bmatrix} . \tag{25}$$

To find the smoothness influence matrix of order $q \leq r-1$, $\mathbf{A}_q$, one can simply take the upper left $q \times q$ submatrix of $\mathbf{A}_{r-1}$ above.

The following Maple[1] code can be used to generate the matrix above for any given order $(2r-1)$.

```
#>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>
r := 4 ;
#-------------------------------------------------------------------------------
  Smoothness_Influence_Matrix := proc (NN, mm, C) local Nm, l,  j, ll, jj:
    Nm := N-m+1 ;
    for l from 1 to Nm do ; for j from 1 to Nm do ;
      ll := l+mm-1 ; jj := j+mm-1;
      if (mods(ll+jj,2) = 0) then ;
        C[l,j] :=  (ll!*jj!)/((ll-mm)!*(jj-mm)!) /2^(ll+jj-2*mm)/(ll+jj-2*mm+1);
```

[1]Maple is a registered trademark of Maplesoft, a division of Waterloo Maple Inc. 2010.

```
      end if;
    end do ; end do ;
  end proc:
#--------------------------------------------------------------------------------
N := r-1: C := Array(1..N,1..N,1..N) : A := Matrix(1..N,1..N):

for m from 1 to N do :
  Nm := N-m+1 ; CC := Matrix(1..Nm);
  Smoothness_Influence_Matrix (N, m, CC); m, CC;
  C[m,m..N,m..N] := CC; print(C[m,1..N,1..N]);
end do:

for l from 1 to N do ; for j from 1 to N do ; for m from 1 to l do ;
  A[l,j] := A[l,j] + C[m,l,j];
end do ; end do ; end do;
A:=A;
#>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>
```

3 Coefficients of the polynomial $\mathbf{a}_k$

In order to find the smoothness indicators β_k, the coefficients of the polynomial approximation $a_l, l = 1, \ldots, r-1$ (10) or (11) are needed for each shifting parameter $k = 0, \ldots, r-1$ of $(2r-1)$_th order WENO scheme ($r = 2, \ldots, 6$ are given here). A Maple code used to generate the coefficients can be found at the end of the section.

As shown earlier, the smoothness indicators β_k of a given order $(2r-1)$ will be computed as

$$\beta_k = < \mathbf{a}_k, \mathbf{A}_{r-1}\mathbf{a}_k >, \qquad k = 0, \ldots, r-1. \tag{26}$$

where the polynomial coefficients vector $\mathbf{a}_k$, can be expressed via the following formula, with a matrix $\mathbf{G}^k$ multiply with a vector of data points $\{f_{k-r+1}, f_{k-r+2}, \ldots, f_k\}$,

$$\mathbf{a}_k = [a_1, a_2, \ldots, a_{r-1}]_k^T = \mathbf{G}^k \mathbf{f}_k, \quad \text{with} \quad \mathbf{f}_k = [f_{k-r+1}, f_{k-r+2}, \ldots, f_k]^T. \tag{27}$$

For example,

- $r = 2$,

$$\mathbf{G}^0 = \begin{bmatrix} -1 & 1 \end{bmatrix}, \tag{28}$$

- $r = 3$,

$$\mathbf{G}^0 = \begin{bmatrix} \frac{1}{2} & -2 & \frac{3}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \qquad \mathbf{G}^1 = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}, \tag{29}$$

- $r = 4$,

$$\mathbf{G}^0 = \begin{bmatrix} -\frac{7}{24} & \frac{11}{8} & -\frac{23}{8} & \frac{43}{24} \\ -\frac{1}{2} & 2 & -\frac{5}{2} & 1 \\ -\frac{1}{6} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix} \quad \mathbf{G}^1 = \begin{bmatrix} \frac{5}{24} & -\frac{9}{8} & \frac{5}{8} & \frac{7}{24} \\ 0 & \frac{1}{2} & -1 & \frac{1}{2} \\ -\frac{1}{6} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix}, \tag{30}$$

- $r = 5$,

$$\mathbf{G}^0 = \begin{bmatrix} \frac{3}{16} & -\frac{25}{24} & \frac{5}{2} & -\frac{29}{8} & \frac{95}{48} \\ \frac{7}{16} & -\frac{9}{4} & \frac{37}{8} & -\frac{17}{4} & \frac{23}{16} \\ \frac{1}{4} & -\frac{7}{6} & 2 & -\frac{3}{2} & \frac{5}{12} \\ \frac{1}{24} & -\frac{1}{6} & \frac{1}{4} & -\frac{1}{6} & \frac{1}{24} \end{bmatrix} \quad \mathbf{G}^1 = \begin{bmatrix} -\frac{5}{48} & \frac{5}{8} & -\frac{7}{4} & \frac{25}{24} & \frac{3}{16} \\ -\frac{1}{16} & \frac{1}{4} & \frac{1}{8} & -\frac{3}{4} & \frac{7}{16} \\ \frac{1}{12} & -\frac{1}{2} & 1 & -\frac{5}{6} & \frac{1}{4} \\ \frac{1}{24} & -\frac{1}{6} & \frac{1}{4} & -\frac{1}{6} & \frac{1}{24} \end{bmatrix}$$

$$\mathbf{G}^2 = \begin{bmatrix} \frac{5}{48} & -\frac{17}{24} & 0 & \frac{17}{24} & -\frac{5}{48} \\ -\frac{1}{16} & \frac{3}{4} & -\frac{11}{8} & \frac{3}{4} & -\frac{1}{16} \\ -\frac{1}{12} & \frac{1}{6} & 0 & -\frac{1}{6} & \frac{1}{12} \\ \frac{1}{24} & -\frac{1}{6} & \frac{1}{4} & -\frac{1}{6} & \frac{1}{24} \end{bmatrix}, \tag{31}$$

- $r = 6$,

$$\mathbf{G}^0 = \begin{bmatrix} -\frac{739}{5760} & \frac{955}{1152} & -\frac{1339}{576} & \frac{2179}{576} & -\frac{4915}{1152} & \frac{12139}{5760} \\ -\frac{3}{8} & \frac{37}{16} & -6 & \frac{67}{8} & -\frac{49}{8} & \frac{29}{16} \\ -\frac{41}{144} & \frac{241}{144} & -\frac{289}{72} & \frac{349}{72} & -\frac{421}{144} & \frac{101}{144} \\ -\frac{1}{12} & \frac{11}{24} & -1 & \frac{13}{12} & -\frac{7}{12} & \frac{1}{8} \\ -\frac{1}{120} & \frac{1}{24} & -\frac{1}{12} & \frac{1}{12} & -\frac{1}{24} & \frac{1}{120} \end{bmatrix}$$

$$\mathbf{G}^1 = \begin{bmatrix} \frac{341}{5760} & -\frac{461}{1152} & \frac{701}{576} & -\frac{1349}{576} & \frac{1541}{1152} & \frac{739}{5760} \\ \frac{1}{16} & -\frac{3}{8} & \frac{7}{8} & -\frac{1}{2} & -\frac{7}{16} & \frac{3}{8} \\ -\frac{5}{144} & \frac{37}{144} & -\frac{61}{72} & \frac{97}{72} & -\frac{145}{144} & \frac{41}{144} \\ -\frac{1}{24} & \frac{1}{4} & -\frac{7}{12} & \frac{2}{3} & -\frac{3}{8} & \frac{1}{12} \\ -\frac{1}{120} & \frac{1}{24} & -\frac{1}{12} & \frac{1}{12} & -\frac{1}{24} & \frac{1}{120} \end{bmatrix}$$

$$\mathbf{G}^2 = \begin{bmatrix} -\frac{259}{5760} & \frac{379}{1152} & -\frac{667}{576} & \frac{259}{576} & \frac{557}{1152} & -\frac{341}{5760} \\ 0 & -\frac{1}{16} & \frac{3}{4} & -\frac{11}{8} & \frac{3}{4} & -\frac{1}{16} \\ \frac{7}{144} & -\frac{47}{144} & \frac{47}{72} & -\frac{35}{72} & \frac{11}{144} & \frac{5}{144} \\ 0 & \frac{1}{24} & -\frac{1}{6} & \frac{1}{4} & -\frac{1}{6} & \frac{1}{24} \\ -\frac{1}{120} & \frac{1}{24} & -\frac{1}{12} & \frac{1}{12} & -\frac{1}{24} & \frac{1}{120} \end{bmatrix}. \tag{32}$$

By symmetry, for $k = 0, \dots, \frac{r-1}{2}$, one has

$$G_{ij}^{r-1-k} = (-1)^i G_{i(r-j)}^k, \quad i = 1, \dots, r-1, \quad j = 1, \dots, r. \tag{33}$$

4 Ideal Weights d_k

We shall denote $\mathbf{d}_r = (d_0, \dots, d_r)$ a vector containing the $r-1$ ideal weights for the $(2r-1)$_th order Upwind-Central WENO scheme.

r	$2r-1$	$\mathbf{d}_r$
2	3	$(\frac{1}{3}, \frac{2}{3})$
3	5	$(\frac{1}{10}, \frac{3}{5}, \frac{3}{10})$
4	7	$(\frac{1}{35}, \frac{12}{35}, \frac{18}{35}, \frac{4}{35})$
5	9	$(\frac{1}{126}, \frac{10}{63}, \frac{10}{21}, \frac{20}{63}, \frac{5}{126})$
6	11	$(\frac{1}{462}, \frac{5}{77}, \frac{25}{77}, \frac{100}{231}, \frac{25}{154}, \frac{1}{77})$

The following Maple code will compute the coefficients of the polynomial $\mathbf{a}_k$ in the matrix form $\mathbf{G}$ and the Ideal weights d_k for any given WENO scheme of order $(2r-1)$.

```
#>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>
restart :
#=========================================================================================
# = INPUT r for THE 2r-1 ORDER WENO SCHEME ========
r := 4 :

#==============================   PACKAGES/INITIALIZATIONS   ============================
with (linalg, inverse, multiply):
#==============================  PROCEDURES/FUNCTIONS  ==================================
  Polynomial_in_x := proc (Degree) local P, k :
    P := add(a[k]*x^k, k=0..Degree); end proc :
#---------------------------------------------------------------------------------
  Coefficient_Matrix := proc (N, dx) local j, k; global A, P, a, r;
    for j from 1 to N do; for k from 1 to N do;
      A[j,k] := coeff( int( P, x=((j-1)-m+r-1/2)*dx..((j-1)-m+r+1/2)*dx ), a[N-k] )/dx;
    end do; end do; end proc :
#---------------------------------------------------------------------------------
  Polynomial_p := proc (N, v) local p_r, k;
    p_r := add(v[k]*x^(N-k), k=1..N); end proc :
#---------------------------------------------------------------------------------
  Ideal_Weight_r := proc (N)
```

```
    local  r, k, mm, ii, H, bb, H_Term, Convex_Linear_Weights;
    global Ideal_Weights;

    mm := r1-r0+1; H := Matrix(1..mm,1..mm): bb := Vector(1..mm) :

    Convex_Linear_Weights := expand(add(d[r]*p[r],r=r0..r1));
    for r from r0 to r1 do;
         ii  := r-r0+1 :
      bb[ii] := coeff(Higher_Order_Approximation,f[-N+ii]);
      H_Term := coeff(Convex_Linear_Weights     ,f[-N+ii]);
      for k from r0 to r1 do; H[ii,k-r0+1] := coeff(H_Term,d[k]); end do;
    end do;

    Ideal_Weights := multiply(inverse(H),bb); end proc :
#=======================END OF PROCEDURES/FUNCTIONS==================================
N := r-1 :                m := (N-1)/2 : r0 := -m   : r1 := m : Order := 2*N-1 :
if ((N mod 2) = 0) then : m :=  N   /2 : r0 := -m+1 : r1 := m : Order := 2*N-1 : end if :
#===================================================================================
A := Matrix(1..N,1..N) : u := Vector(1..N) :

for r from r0 to r1 do :
  P := Polynomial_in_x (N-1) :
  for j from 1 to N do; u[j] := f[r-m+j-1]; end do;
  Coefficient_Matrix (N, dx);
  p_r[r] := Polynomial_p (N, multiply(inverse(A),u));
end do :

for r from r0 to r1 do; for k from 1 to N do;
  A[r-r0+1,k] := sort( collect(simplify(coeff(p_r[r],x,k-1)), dx), L);
end do; end do;

for r from r0 to r1 do;
  G[r-r0] := subs(dx=1,A[r-r0+1,2..N]);
end do;

for r from r0 to r1 do: p[r] := simplify(subs(x=dx/2,p_r[r])): end do:

#===================================================================================
A := Matrix(1..Order,1..Order) : u := Vector(1..Order) : v := Vector[row](1..Order) :

r := 0 : m := N-1 : P := Polynomial_in_x (Order) :
for j from 1 to Order do : u[j] := f[-N+j] : end do :
Coefficient_Matrix (Order, dx) :
                         B := multiply(inverse(A),u)):
Higher_Order_Approximation := simplify(subs(x=dx/2,Polynomial_p (Order, B))):
for k from -N+1 to N-1 do:
  a[k] := coeff(Higher_Order_Approximation,f[k]): v[k+N] := a[k]:
end do:
Higher_Order_Coefficients  := v; Ideal_Weights := Ideal_Weight_r (N);
#>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>
```

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