Final year projects with Dr. Amy Pang, Fall 2017

These projects are in the field of *algebraic combinatorics* - the use of linear algebra and abstract algebra to study combinatorial objects (e.g. graphs, permutations, trees). Although some of the projects are about Markov chains (which is a type of random process), they involve much more algebra and combinatorics than probability. (A quick overview of the connection between the algebra and the probability is at amypang.github.io/2207/week14_print.pdf , but this doesn't show you the type of calculation you will be doing - for that, see the references in the descriptions below.)

It is much easier to explain these projects face-to-face than in writing. These descriptions probably make them sound more complicated than they are. So you are strongly encouraged to speak with Dr. Pang if you are interested in the projects (email: amypang@hkbu.edu.hk).

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1. **A remove-and-reattach Markov chain on planar binary trees**
2. **A vertex-deletion Markov chain on ordered trees**
3. **A vertex-deletion Markov chain on graphs**

A *combinatorial Hopf algebra* is a set of linear transformations describing how combinatorial objects break and combine. Your supervisor's preprint [5, Sec. 3] gives a general method to make a family of Markov chains out of each such set of linear transformations. These chains describe breaking an object randomly into many parts, then combining the parts together again, potentially in a different way. [5, Sec. 4] contains very general formulas for the eigenvectors of these linear transformations, and there is a standard way to use these eigenvectors to obtain the expected values of certain functions of the Markov chain. (You are **not** expected to learn this entirely on your own from the papers; your supervisor will explain the main idea to you, and the papers will give you the details.)

But all this general theory is not very useful unless we can apply it to specific examples. So your goal will be to compute explicitly (by substituting into the general theory) the chain and eigenvectors for respectively the Loday-Ronco algebra of planar binary trees [2, p.9], the Grossman-Larson algebra of ordered trees [1, p.8], and a variant on the Hopf algebra of graphs [4, Ex. 4.1.3], for example [3, Sec. 3]. The chains should describe removing a vertex from the tree or graph, and then possibly reattaching it elsewhere. You will also look for "interesting" functions whose expected values you can compute using your eigenvectors. (Since the linear transformations will contain many variables and not just numbers, you will have to use your supervisor's formula to compute the eigenvectors. Methods from linear algebra classes will not be effective.)

Prerequisites: A strong background in linear algebra, and preferably some knowledge of abstract algebra. No probability or graph theory knowledge is necessary, though if you know something in these areas, you might be able to take the project in an exciting new direction. Also, you need to be quick in learning new ideas.

It is strongly preferred that you make substantial progress on the project during the summer.

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4. **The Catalan Hopf algebra**

A *combinatorial Hopf algebra* is a set of linear transformations describing how combinatorial objects (e.g. graphs, trees) break and combine. One can construct different Hopf algebras out of the same set of combinatorial objects, by defining different breaking and combining rules.

The Cartier-Milnor-Moore theorem says that, under certain conditions, all Hopf algebras constructed out of the same set of combinatorial objects are isomorphic - i.e. there is a change of basis that will change one set of linear transformations into the other. But the theorem does not tell us what this change of basis is. Your supervisor is interested in explicitly finding this change of basis for what are called *Catalan objects*, from which 4 Hopf algebras have already been built. (You can think of this as: 4 people have given 4 different answers to the same question, and we want to know how these 4 answers are related.)

This is probably a hard problem, so the goal of this project is not to solve it, but to simply gather some preliminary information. You will compute some small examples by hand to make some conjectures about what the change of basis might look like. You can see an example of these computations in [1, p.8].

Prerequisites: A strong background in linear algebra, and preferably also advanced linear algebra or abstract algebra. Also, you need to be quick in learning new ideas.

It is strongly preferred that you make substantial progress on the project during the summer.

References

[1] Marcelo Aguiar and Frank Sottile, Cocommutative Hopf algebras of permutations and trees, In: J. Algebraic Combin. 22.4 (2005), pp. 451–470. http://www.math.cornell.edu/ maguiar/GL.pdf

[2] Marcelo Aguiar and Frank Sottile, Structure of the Loday-Ronco Hopf algebra of trees, In: J. Algebra 295.2 (2006), pp. 473–511. http://www.math.cornell.edu/ maguiar/Loday.pdf

[3] Mathieu Guay-Paquet, A second proof of the Shareshian–Wachs conjecture, by way of a new Hopf algebra, In: Arxiv e-prints. https://arxiv.org/pdf/1601.05498

[4] C.Y. Amy Pang, Hopf Algebras and Markov chains, In: Arxiv e-prints. https://arxiv.org/pdf/1412.8221

[5] C.Y. Amy Pang, Markov chains from descent operators on combinatorial Hopf algebras, In: Arxiv e-prints. https://arxiv.org/pdf/1609.04312v1.pdf