<u>CASE STUDIES IN MODELLING</u>

<u>Problem Solving</u>

1. There are N horses taking part in a horse racing event. Two betting companies independently estimated the horses' chances of winning as P_i and Q_i, respectively, $i = 1, 2, ..., N$, $(\sum_{i=1}^{N} P_i = \sum_{i=1}^{N} Q_i = 1)$. For each bet of one dollar, each company will payout the amount $\frac{\alpha}{P_i}$ and $\frac{\alpha}{Q_i}$ respectively, assuming the *i*'th horse wins (α is a positive constant less than 1). For now, we will suppose that the payout <u>already</u> includes the one dollar capital.

   (a) You wish to bet a total sum of M dollars in this racing event. Find the betting plan that will guarantee a fixed maximal payout, W, whichever horse wins, and obtain an expression for W.

   (7 marks)

   (b) Obtain an expression for $\sum_{i=1}^{N} |P_i - Q_i|$ in terms of *W*. Hence prove that the betting plan in (a) is profitable if and only if

   $$\sum_{i=1}^{N} |P_i - Q_i| > 2(1 - \alpha)$$

   (13 marks)

2. Two ice cream salesmen operate on a long narrow beach which is uniformly populated with tourists. The beach can be modelled as one dimensional and of unit length. We shall denote the **sales position** as (*x*, *y*), where *x* denotes the operating location of the first salesman, and *y* the location for the second salesman. Initially, the sales position is (0,1), i.e. the two salesman operate at the two far ends of the beach. However, they will always move to a better location if it improves sales.

   (a) <u>consumer model</u>: Customers will always choose to buy from the nearest salesman. Assuming this model, prove that the sales position will eventually settle down to the equilibrium position (0.5, 0.5). Find the customers' average distance from the nearest salesman at this equilibrium sales position.

   (10 marks)

   (b) It has been observed that the equilibrium sales position has changed to (0.25, 0.75). What is the significance of this new sales position? What modifications of the consumer model in (a) will explain this new equilibrium sales position?

   (10 marks)

(20 marks)

3. (a) Consider the 2x2 Lloyd's puzzle in figure 1. The bottom right corner is a blank space (B) into which adjacent pieces can slide in or out. Find a sequence of moves (or, using group theory notation, transpositions) which will change the arrangement in figure 1 to that shown in figure 2. Note that the blank space B must return to the bottom right corner at the end of the sequence of moves.

1	2
3	B

figure 1

3	1
2	B

figure 2

(6 marks)

(b) List all possible arrangements into which figure 1 can be changed to. Briefly comment on your answer.

(4 marks)

(c) Consider Lloyd's 4x4 puzzle shown in figure 3. Is it possible to rearrange the pieces so that the pieces are as in figure 4? Justify your answer.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	B

figure 3

15	14	13	12
11	10	9	8
7	6	5	4
3	2	1	B

figure 4

(10 marks)

(20 marks)

4. The following four dice are fair.

	0	
4	0	4
	4	
	4	

Die 1

	3	
3	3	3
	3	
	3	

Die 2

	5	
1	1	1
	5	
	5	

Die 3

	2	
2	2	2
	6	
	6	

Die 4

Consider the following game. Player *A* chooses one of these 4 dice and player *B* one of the remaining three. The dice are thrown independently, and whoever gets the larger outcome wins $1 from the other.

(a) Which player has the advantage and why?

(5 mks)

(b) Give an/the optimal strategy for both players.

(15 mks)