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On the generalized pantograph functional-differential equation

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The generalized pantograph equation \( y'(t) = Ay(t) + By(qt) + Cy'(qt), y(0) = y_0 \), where \( q \in (0, 1) \), has numerous applications, as well as being a useful paradigm for more general functional-differential equations with monotone delay. Although many special cases have been already investigated extensively, a general theory for this equation is lacking – its development and exposition is the purpose of the present paper. After deducing conditions on \( A, B, C \in \mathbb{C}^{d \times d} \) that are equivalent to well-posedness, we investigate the expansion of \( y \) in Dirichlet series. This provides a very fruitful form for the investigation of asymptotic behaviour, and we duly derive conditions for \( \lim_{t \to \infty} y(t) = 0 \). The behaviour on the stability boundary possesses no comprehensive explanation, but we are able to prove that, along an important portion of that boundary, \( y \) is almost periodic and, provided that \( q \) is rational, it is almost rotationally symmetric. The paper also addresses itself to a detailed analysis of the scalar equation \( y'(t) = by(qt), y(0) = 1 \), to high-order pantograph equations, to a phenomenon, similar to resonance, that occurs for specific configurations of eigenvalues of \( A \), and to the equation \( Y'(t) = AY(t) + Y(qt) B, Y(0) = Y_0 \).

1 Introduction

The theme of this paper is the generalized pantograph equation (GPE)

\[
y'(t) = Ay(t) + By(qt) + Cy'(qt), \quad y(0) = y_0, \tag{1.1}
\]

where \( q \in (0, 1) \), \( A, B \) and \( C \) are \( d \times d \) complex matrices, and \( y_0 \) is a column vector in \( \mathbb{C}^d \). Many special cases of the differential delay equation (1.1) can be encountered in applications: absorption of light by interstellar matter [1], analytic number theory [29], collection of current by the pantograph of an electric locomotive [32], nonlinear dynamical systems [11], probability theory on algebraic structures [36], Cherenkov radiation [31], continuum mechanics [35], and the theory of dielectric materials [4]. Moreover, (1.1) is an interesting example of a functional equation with a variable delay: sufficiently complicated to provide a clue to the behaviour of more general classes of such equations, but also simple enough to be tractable by relatively straightforward means.

The equation (1.1) differs markedly from the more familiar differential delay equation

\[
y'(t) = Ay(t) + By(t-\tau), \tag{1.2}
\]

\[
y(t) = \phi(t), \quad -\tau < t \leq 0,
\]

where \( \tau > 0 \) is a constant delay. The latter is comprehensively understood, and various aspects of its behaviour – well-posedness, continuity, asymptotic boundedness – are well-

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1 This is the origin of the ‘pantograph’ in ‘generalized pantograph equation’.
known [3, 21]. This is not the case with the generalized pantograph equation (1.1), not even in the ‘pure delay’ form \( C = O \). Special forms of the equation have been extensively investigated [5–7, 9–17, 26, 28, 30, 32, 36]. In particular, Carr and Dyson [8] investigated (employing techniques different from the present work) equation (1.1) with \( C = O \), and reported conditions for its asymptotic stability.

Much intuition is available from the study of ordinary differential equations (ODE). The first – and most important – step in the analysis of (1.1) is to abandon this intuition. Functional equations are not ordinary differential equations! In §2 we demonstrate this point by examining the scalar equation \( y'(t) = by(qt), b \in \mathbb{C}\setminus\{0\}, q \in (0, 1) \), with the initial condition \( y(0) = 1 \). This equation has been already investigated by Morris et al. [30], Derfel [9], and Kuang & Feldstein [28]. Our conclusion, namely that \( \lim \sup_{t \to -\infty} |y(t)| = \infty \), regardless of the value of \( b \) and \( q \), is not new, although our technique is different from those of Morris et al. [30] and Derfel [9]. The main justification of the inclusion of this section, except for completeness of exposition, is in its demonstration how standard function-theoretic concepts – the order of an entire function, multiplier sequences, etc. – lead to nontrivial results, and expose features of the pantograph equation that are counterintuitive in the familiar ODE setting.

Existence and uniqueness of the solution of (1.1) is discussed in §3. This is a far from trivial question, since the initial-value problem (1.1) is infinite-dimensional. Indeed, it is easy to observe that any other finite starting point except for the origin is inconsistent with a single initial condition. As we show, existence and well-posedness of the solution are assured, subject to a simple algebraic condition, namely that the spectrum of \( C \) contains no points of the form \( q^{-\ell}, \ell \in \mathbb{Z}^+ \). The last condition is necessary and sufficient: if a ‘forbidden’ eigenvalue exists, then there is a nontrivial solution (hence, rescaling by a constant, an infinity of trivial solutions) to (1.1) with the initial condition \( y = 0 \).

In §4 we explore asymptotic stability. What are the conditions for \( \lim_{t \to -\infty} y(t) = 0 \)? In the scalar, ‘pure delay’ form \( y'(t) = ay(t) + by(qt), y(0) = 1 \), Kato & McLeod [26] proved that \( a \in \mathbb{R}, a < 0, |b| + a < 0 \), implies that \( y(t) \to 0 \). The goal of §4 is much more comprehensive, and it is realized by implicitly expanding the solution of (1.1) in series of the form

\[
\sum_{\ell=0}^{\infty} D_\ell e^{\ell t a} V, \quad t \geq 0,
\]

where \( D_0, D_1, \ldots \) and \( V \) are constant matrices. As long as the eigenvalues of \( A \) are distinct, each component in the above expansion is a Dirichlet series (a linear combination of functions of the form \( e^{\omega t} \) for different \( \omega \in \mathbb{C} \) [37]). Subject to minor restrictions, we prove that stability takes place for all initial values if and only if \( A \) is invertible, \( \rho(A^{-1} B) < 1 \) and all the eigenvalues of \( A \) have a negative real part.

The behaviour on the boundary of the ‘stability set’ is the subject matter of §5. There are two portions of this set, \( \rho(A^{-1} B) = 1 \) and \( \max \text{Re } \sigma(A) = 0 \), which are characterized by different dynamics. In the first case, the solution either tends to a limit point or it asymptotically lies on a well-defined manifold. In the second case, \( y \) is almost-periodic [27]. Almost-periodicity of solutions of specific ‘pantograph’ equations, e.g. \( y'(t) = y(t) + by(qt) \), has been already studied by Derfel & Shevako [13]. Here we treat the subject matter with greater generality, as well as deriving the size of the ‘almost-period’ when \( q \) is rational.

Although the classical linear ODE theory is best expressible in a matrix formalism, much
useful information can be obtained by studying high-order scalar equations. This is also the case with the GPE

\[ \mathcal{A} y(t) = \mathcal{B} y(qt), \]  
(1.3)

where

\[ \mathcal{A} = \sum_{\ell=0}^{n} a_{\ell} \frac{d^{\ell}}{dt^{\ell}} \]

and

\[ \mathcal{B} = \sum_{\ell=0}^{n} b_{\ell} \frac{d^{\ell}}{dt^{\ell}} \]

are two linear differential operators with constant coefficients. The initial conditions are \( y^{(i)}(0) = y_{i}, i = 0, 1, \ldots, n-1 \). Subject to minor restrictions, we derive explicitly in §6 \( n \) linearly independent solutions of (1.3) in terms of Dirichlet series. An important tool in our analysis is the theory of \( q \)-hypergeometric functions [19].

We have already twice used the phrase ‘subject to minor restrictions’. Specifically, there are two restrictions. Firstly, certain eigenvalues of \( C \) are forbidden (as already stated), otherwise (1.1) is not well-posed. The second restriction is, however, an artifact of our method of proof: unless certain relationships are forbidden between eigenvalues of \( A \) and \( B \), the solution of (1.1) cannot be written as Dirichlet series. This is due to the presence of polynomial resonating terms and we report an initial discussion of this phenomenon in §7.

Finally, in §8 we examine the solution of the generalized pantograph equation of the second kind

\[ Y'(t) = AY(t) + Y(qt) B, \quad Y(0) = Y_0, \]  
(1.4)

The function \( Y \) is itself a \( d \times d \) matrix. Of course, unless \( A \), \( B \) and \( Y_0 \) all belong to the same commutative Banach algebra, (1.4) is different from the matrix counterpart of (1.1). As it turns out, (1.4) is, in fact, considerably easier, and we are able to express its Dirichlet series solution explicitly.

Many important question with regard to the GPE (1.1) are still open, and we mention them throughout the paper. Important clues are frequent obtainable from numerically-derived solutions, and we display these in §5, to illustrate a conjecture on the presence of ‘almost-self-similarity’ in the almost-periodic solution with rational value of \( q \).

Future papers will address themselves to the advanced case \( q > 1 \) [24], and to stability and asymptotic stability in the presence of nonlinearities and of general nonconstant delays [14, 25].

2 The scalar equation \( y'(t) = by(qt) \)

In the present section we consider the solution of the scalar equation

\[ y'(t) = by(qt), \quad y(0) = 1, \]  
(2.1)

where \( b \in \mathbb{C} \) and \( q \in (0, 1) \). A naive expectation is that the solution of (2.1) with \( b = -1 \) and \( q = 0.99999999999 \), say, should be ‘similar’ to \( e^{-t} \). In fact, global features of \( y \) are markedly different. There is not much new material in this section, since (2.1) has been already analysed elsewhere [30, 9]. Although a few important steps in the proofs are somewhat different, the main reasons for the inclusion of the present material are completeness of exposition and a demonstration how some very old (and sometimes not well-known) techniques from the theory of functions can be used to derive a great deal of information.
It is trivial to verify by substitution that the solution of (2.1) exists, and that it possesses
the Taylor expansion
\[ y(t) = \sum_{k=0}^{\infty} \frac{q(k-1)q}{k!} (bt)^k. \]  
(2.2)

Moreover, this solution is unique, since the differential operator is linear and it is easy to
verify by repeated differentiation that the solution of \( y'(t) = by(qt) \) with the initial condition
\( y(0) = 0 \) is necessarily \( y(t) \equiv 0 \).

The coefficients of (2.2) contain all the information that is required in the present section.
Let
\[ f(z) = \sum_{k=0}^{\infty} \frac{q(k-1)q}{k!} z^k, \]
thus \( y(t) = f(bt) \). It follows at once (e.g. by the M-test) that \( f \) is an entire function. Recall
that the order of an entire function \( g \) is the nonnegative number \( \rho \) such that
\[ \rho = \lim_{r \to \infty} \frac{\ln \ln M(r)}{\ln r} \]
where \( M(r) = \limsup_{|z|=r} |g(z)| \).

We proceed to evaluate the order of \( f \). According to a classical formula [22], the order of
\( g(z) = \sum_{k=0}^{\infty} g_k z^k \) equals
\[ \limsup_{k \to \infty} \frac{k \ln k}{\ln |g_k|}. \]
It follows at once that, in our case, \( \rho = 0 \).

Functions of order zero have an important property. According to Ahlfors' theorem on
asymptotic values [20], an entire function \( g \) of order \( \rho \) has at most \( 2\rho \) finite asymptotes
at \( \infty \). Thus, \( \rho = 0 \) means that \( f \) has no finite asymptotes there. In other words, given
any continuous curve \( \gamma, \sigma \in [0,1] \), such that \( \gamma_0 = 0 \), say, and \( \gamma_1 = \infty \), it is true that
\( \limsup_{\tau \to 1} |f(\gamma_\tau)| = \infty \).

**Theorem 1** [30] The solution of (2.1) cannot be uniformly bounded for \( t \geq 0 \), irrespective of
the value of \( b \in \mathbb{C} \setminus \{0\} \) and \( q \in (0,1) \).

**Proof** An immediate consequence of \( y(t) = f(bt) \) and the Ahlfors theorem. \( \square \)

The last theorem has been originally proved by Morris *et al.* [30] by using the
Phragmén–Lindelöf principle. Since one is interested only in asymptotes that are straight
lines, that principle is sufficient to deduce that they do not exist. The theorem has been
extended to the multi-lag equation
\[ y'(t) = \sum_{j=1}^{M} b_j y(q_j t), \quad y(0) = 1, \]
where \( q_1, q_2, \ldots, q_M \in (0,1) \), by Derfel [9] by using the Wiman–Valiron theorem.

We note the first break with ODE intuition: even if \( \text{Re} \ b < 0 \), the solution of (2.1) cannot
be uniformly bounded, no matter how close \( q \) is to 1 and the equation to the standard linear
ODE \( y'(t) = by(t) \).
Table 1 The first 12 zeros for $q = \frac{1}{4}, \frac{1}{5}, \frac{3}{4}$, correct to six decimal places

<table>
<thead>
<tr>
<th>$q = \frac{1}{4}$</th>
<th>$q = \frac{1}{5}$</th>
<th>$q = \frac{3}{4}$</th>
</tr>
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<tbody>
<tr>
<td>$-1.165771$</td>
<td>$-1.488079$</td>
<td>$-2.400127$</td>
</tr>
<tr>
<td>$-8.526040$</td>
<td>$-4.881141$</td>
<td>$-4.530587$</td>
</tr>
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<td>$-261.750113$</td>
<td>$-34.775316$</td>
<td>$-12.527513$</td>
</tr>
<tr>
<td>$-1299.776943$</td>
<td>$-84.977290$</td>
<td>$-19.661505$</td>
</tr>
<tr>
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<td>$-201.002876$</td>
<td>$-30.168801$</td>
</tr>
<tr>
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<td>$-45.516576$</td>
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<td>$-99.853247$</td>
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<td>$-145.838717$</td>
</tr>
<tr>
<td>$-11578032.195886$</td>
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<td>$-50493904.523165$</td>
<td>$-24937.603801$</td>
<td>$-304.673863$</td>
</tr>
</tbody>
</table>

According to the Hadamard factorization theorem [22], $p_f = 0$ and $f(0) = 1$ implies that either $f$ is a polynomial (which, clearly, is not the case here), or that it can be represented in the form

$$f(z) = \prod_{j=1}^{\infty} \left(1 - \frac{z}{\delta_j}\right).$$
(2.3)

Hence, $f$ has an infinity of zeros in the complex plane. The function $y$ corresponds to a single ray of $f$ and it is of interest to identify directions $\arg b$ such that $y$ possesses zeros. These can be characterized completely by using multiplier sequences [30].

The real sequence $\{\alpha_k\}_{k=0}^{\infty}$ is called a multiplier sequence (of the first kind) [33] if, given any polynomial $\sum_{k=0}^{\infty} \alpha_k z^k$ with all zeros real, also the zeros of the polynomial $\sum_{k=0}^{\infty} \alpha_k b^k z^k$ are real. An example of a multiplier sequence with direct relevance to our analysis is $\{q^{k^2}\}_{k=0}^{\infty}$, $q \in (0, 1)$ [33].

Multiplier sequences retain real zeros when acting on the Taylor coefficients of certain entire functions. Following [22], we say that $g$ belongs to the class $A$ if

$$g(z) = e^{\beta + \sum_{j=1}^{N} \left(1 - \frac{z}{v_j}\right)},$$

where $N \in \mathbb{Z}^+$, $\beta \geq 0$, $\gamma \in \mathbb{R}$ and $v_j > 0$, $j = 1, 2, \ldots$. According to a classical result of Laguerre [22], if $g(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ is in $A$ then so is $g^*(z) = \sum_{k=0}^{\infty} \alpha_k b^k z^k$, provided that $\{\alpha_k\}_{k=0}^{\infty}$ is a multiplier sequence.

**Theorem 2** [30] If $b < 0$ then the solution of (2.1) possesses an infinity of zeros, otherwise $y$ cannot vanish for $t \geq 0$.

**Proof** We let $g(z) = \exp(z/q^2)$. This is clearly a function in the class $A$. Since $\{q^{k^2}\}_{k=0}^{\infty}$ is a multiplier sequence, also $f \equiv g^*$ is in $A$. Thus, all the zeros of $f$ are real. Moreover, since

\[\textit{An empty product is allowed.}\]
all Taylor coefficients of $f$ are positive, the zeros of $f$ are all in $(-\infty, 0)$. The proof follows at once from (2.2) and (2.3).

The original proof in Morris et al. [30] contains a modest gap, since it applies multiplier sequences, as defined in the polynomial setting, directly to $g$, without invoking Laguerre’s results on class $A$ functions. An alternative proof of the last theorem, based on geometric considerations and due to Hahn, is described in Morris et al. [30].

The location and distribution of the zeros $\delta_j = \delta_j^{(q)}$ of $f$ is at present unknown, except, of course, that they are all negative. Table 1 presents the first 12 zeros for $q \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$. Fig. 1 displays the values of $\log |\delta_j^{(q)}|$. It is evident therein that the logarithms approach straight lines with slopes $-\ln q$. Thus, it makes sense to conjecture that, asymptotically for $j \gg 1$, $\delta_j^{(q)} = \mu q^{-j} + \nu + o(1)$, where $\mu$ and $\nu$ are independent of $q$. An indication that the conjecture might be true is implicit in Hahn’s proof.

3 Existence and well-posedness of the solution

The fundamental solution of the GPE (1.1) is given by the matrix equation

$$Y'(t) = AY(t) + BY(gt) + CY'(gt), \quad Y(0) = I,$$

where $Y$ is a $d \times d$ matrix of functions. Given $Y$ we can easily construct a solution to (1.1) by letting $y(t) = Y(t)y_0$. On the other hand – and identically to the standard ODE theory – the solution of (3.1) is trivially available, provided that (1.1) can be solved with an arbitrary initial input. Thus, in a sense (1.1) and (3.1) are interchangeable and we frequently find it considerably easier to work with $Y$, rather than $y$. 
Assuming that (3.1) possesses a solution, it is easy to see (e.g. by repeated differentiation) that, subject to quite general conditions on the matrix $C$, it is a $C^\infty$ function and with little more effort we can deduce that it is analytic in a nonempty open neighbourhood of the origin.\(^3\) Let

$$Y(t) = \sum_{k=0}^{\infty} \frac{1}{k!} Y_k t^k.$$  

Substitution into (3.1) yields $Y_0 = I$ and

$$ (I - q^k C) Y_{k+1} = (A + q^k B) Y_k, \quad k \in \mathbb{Z}^+. \quad (3.2) $$

Let $P$ and $Q$ be two $d \times d$ matrices. We say that the ordered pair $\{P, Q\}$ is $q$-canonical if, given that $\sigma(P) = \{\lambda_1, \lambda_2, \ldots, \lambda_d\}$ and $\sigma(Q) = \{\mu_1, \mu_2, \ldots, \mu_d\}$, it is true that $\mu_k \neq q^k \lambda_j$ for all $k, j \in \{1, 2, \ldots, d\}$ and $\ell = 1, 2, \ldots$.

**Theorem 3** The initial-value problem (3.1) is well-posed if and only if the pair $\{C, q^{-1} I\}$ is $q$-canonical.

**Proof** The nonsingularity of the matrices on the left of (3.2) for all $k \in \mathbb{Z}^+$ is equivalent to $q$-canonicity of $\{C, q^{-1} I\}$. Thus, subject to $q$-canonicity, there exists a unique solution of (3.1). On the other hand, if $q^{-k^*} \in \sigma(C)$ for some $k^* \in \mathbb{Z}^+$, then, for $k = k^*$, either (3.2) has no solution or a whole affine space of solutions.

An example of ill-posedness with an infinity of solutions is

$$Y'(t) = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} Y(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} Y(qt) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} Y'(qt).$$

it is easy to verify that

$$Y_0 = \begin{bmatrix} g \\ h \end{bmatrix}$$

is consistent with (3.2) for every $g, h \in \mathbb{C}$ and can be deployed as a 'seed' of the recurrence relation

$$Y_{k+1} = \begin{bmatrix} -1 & -1 \\ 0 & 1 + q^k \end{bmatrix} Y_k, \quad k = 1, 2, \ldots.$$  

This yields a two-parametric family of solutions. It is trivial to amend the last example, e.g. by choosing a different matrix $B$, so that no solutions consistent with $Y(0) = I$ exist.

Another example of an ill-posed equation (3.1) features in §4.

**4 Asymptotic behaviour and stability**

In the present section we examine the asymptotic behaviour of (3.1), and investigate conditions that ensure $\lim_{t \to \infty} Y(t) = O$. The main tool in our analysis are Dirichlet series.

\(^3\) It will transpire soon that this neighbourhood encompasses all of $\mathbb{C}^d$. 

Several special variants of the pantograph equation were already analysed by Dirichlet series [10, 16, 17]. The investigation of the general case is hampered by the fact that explicit values of coefficients are unknown. Fortunately it is possible to derive much useful information on their asymptotic behaviour, and this is enough for our purposes.

Assuming that \( \{C, q^{-1}I\} \) is \( q \)-canonical, we wish to represent formally the solution of (3.1) as the series

\[
Y(t) = \sum_{\ell=0}^{\infty} D_{\ell} e^{\ell t A} V, \quad t \geq 0.
\]  

(4.1)

Here \( D_{\ell}, \ell \in \mathbb{Z}^+ \), and \( V \) are \( d \times d \) matrices, which are independent of \( t \), and \( \det V \neq 0 \). Since

\[
Y'(t) - AY(t) = \sum_{\ell=0}^{\infty} \left( q^\ell D_{\ell} - AD_{\ell} \right) A e^{\ell t A} V,
\]

\[
BY(qt) + CY'(qt) = \sum_{\ell=1}^{\infty} \left( BD_{\ell-1} + Cq^{\ell-1} D_{\ell-1} A \right) e^{\ell t A} V,
\]

comparison of identical exponentials in (4.2) yields

\[
AD_0 - D_0 A = 0,
\]  

(4.2)

\[
AD_\ell - q^\ell D_\ell A = -BD_{\ell-1} - q^{\ell-1} CD_{\ell-1} A, \quad \ell = 1, 2, \ldots.
\]  

(4.3)

Clearly, (4.2) is obeyed by the choice \( D_0 = I \), which we henceforth require. The solvability of the matrix equations (4.3) follows at once from a classical theorem [18]: Let \( P, Q, R \) be three complex matrices of dimensions \( d_1 \times d_2 \), \( d_2 \times d_3 \) and \( d_1 \times d_3 \), respectively. Then the solution \( X \) of the matrix equation \( PX - XQ = R \) exists and is unique if and only if \( \sigma(P) \cap \sigma(Q) = \emptyset \). In the present case, \( P = A \) and \( Q = q^\ell A \) for \( \ell = 1, 2, \ldots \), hence (4.3) possesses a unique solution if and only if the pair \( \{A, A\} \) is \( q \)-canonical.

The existence of \( D_{\ell}, \ell \in \mathbb{Z}^+ \) falls short of a complete proof that the Dirichlet expansion (4.1) exists. We need also to demonstrate that a nonsingular matrix \( V \) exists so that \( Y(0) = I \). Let

\[
D(t) = \sum_{\ell=0}^{\infty} D_{\ell} t^\ell
\]

be the (formal) generating function of the sequence \( \{D_{\ell}\}_{\ell=0}^{\infty} \). Multiplying (4.3) by \( t^\ell \) and summing up for \( \ell = 1, 2, \ldots \) readily yields the recurrence relation

\[
(A + tB) D(t) = (I - tC) D(qt) A,
\]

consequently, as long as \( \det A \neq 0 \),

\[
\det D(t) = \frac{\det (I - tC)}{\det (I + tBA^{-1})} \det D(qt).
\]  

(4.4)

We now iterate (4.4) to obtain

\[
\det D(t) = \prod_{j=0}^{k} \frac{\det (I - tq^j C)}{\det (I + tq^j BA^{-1})} \det D(q^{k+1} t).
\]
Letting $k \to \infty$ and exploiting $\det D(0) = \det D_0 = 1$ gives

$$\det D(t) = \prod_{j=0}^{\infty} \frac{\det (I - t q^j C)}{\det (I + t q^j B A^{-1})}. \quad (4.5)$$

Since $q \in (0, 1)$, the infinite product $(4.5)$ converges as long as $I + t q^j B A^{-1}$ is nonsingular for all $j \in \mathbb{Z}^+$. Assuming the latter (an assumption that will be subsumed in a later condition) we conclude that $\det D(t) \neq 0$ as long as $t \in \sigma(C)$ for all $j \in \sigma(C)$. In particular, $q$-canonicity of $(C, q^{-1} I)$ implies that $D(1)$ is nonsingular.\(^4\) We set $V = D(1)^{-1}$, therefore

$$Y(0) = D(1) V = I,$$

the correct initial value.

The matrix $F = -B A^{-1}$ plays an important role in our analysis – recall our requirement that $I - q^j F, j \in \mathbb{Z}^+$, be nonsingular for all $j \in \mathbb{Z}^+$.

**Proposition 4** If $(A, A)$ is $q$-canonical and $\rho(F) < 1$, where $\rho(\cdot)$ is the spectral radius, then the Dirichlet series $(4.1)$ converges.

**Proof** We note first that $\rho(F) < 1$ implies nonsingularity of $I - q^j F, j \in \mathbb{Z}^+$, hence well-posedness.

Multiplying $(4.3)$ by $A^{-1}$ provides the estimate

$$\|D_\ell\| \leq \kappa q^\ell \|D_\ell\| + (\|F\| + \kappa q^{\ell + 1}) \|D_{\ell - 1}\|, \quad (4.6)$$

where $\kappa := \|A^{-1}\| \cdot \|A\|$ is the spectral condition number of $A$. The matrix norm $\|\cdot\|$ is, for the time being, arbitrary.

Since $q \in (0, 1)$, there exists $\ell^* \in \mathbb{Z}^+$ such that $\kappa q^{\ell^* + 1} < 1$. Let us assume first that $F \neq O$. Then it follows from $(4.6)$ that

$$\|D_\ell\| \leq \frac{1 + \kappa \|C\| q^{\ell - 1} \|F\| \|D_{\ell - 1}\|}{1 - \kappa q^\ell} \|F\| \|D_{\ell - 1}\|$$

for all $\ell \geq \ell^*$. Iterating the last expression, we obtain the upper bound

$$\|D_\ell\| \leq \frac{1 - \kappa \|C\| q^{\ell^* - 1} \|F\| \|D_{\ell^* - 1}\|}{(\kappa q^{\ell^* + 1} - q)^r} \|F\| \|D_{\ell^* - 1}\|, \quad \ell = \ell^*, \ell^* + 1, \ldots. \quad (4.7)$$

Here $(z; q)_m$ is the $q$-factorial symbol (the Gauss–Heine symbol),

$$\begin{cases} 1 & : m = 0, \\ \prod_{j=0}^{m-1} (1 - q^j z) & : m = 1, 2, \ldots, \infty. \end{cases}$$

The sequence $\{|(x; q)_j|\}_{j=0}^\infty$ is always uniformly bounded [19], and it follows from $(4.7)$ that

$$\|D_\ell\| \leq c_1 \|F\|, \quad \ell \in \mathbb{Z}^+, \quad (4.8)$$

\(^4\) We are assuming implicitly that the series converges. This problem will be addressed shortly.
where $c_1 > 0$ is a constant (generous enough to extend the inequality to $\ell \in \{0, 1, \ldots, \ell^* - 1\}$).

We now revert to (4.1) and use our bound to argue that
\[
\| Y(t) \| \leq c_1 \sum_{\ell=0}^{\infty} \| F \|^\ell \| e^{\ell t A} \|
\]
Since
\[
\lim_{\ell \to \infty} \| e^{\ell t A} \| = \| I \| = 1,
\]
the sequence $\{\| \exp q^t A \| \}_{t=0}^{\infty}$ is uniformly bounded for any fixed $t \geq 0$, therefore
\[
\| Y(t) \| \leq c_3(t) \sum_{\ell=0}^{\infty} \| F \|^\ell,
\]
where $c_3(t) > 0$ is a bounded function. We conclude that $\| F \| < 1$ implies convergence.

Recall that our choice of norm has been, so far, entirely arbitrary. As is well known, given any $\varepsilon > 0$, there exists a norm $\| \cdot \|_\varepsilon$ such that $\rho(F) \leq \| F \|_\varepsilon < \rho(F) + \varepsilon$. Given that $\rho(F) < 1$, we choose $\varepsilon = \frac{1}{8}(1 - \rho(F)) > 0$ and $\| \cdot \| \equiv \| \cdot \|_\varepsilon$. It follows that $\| F \| < 1$, hence convergence.

The proof of the proposition is almost complete, except for the case $F = O$ (which, of course, is the same as $B = O$). We obtain the inequality
\[
\| D_\ell \| \leq \frac{kq^{\ell-1}}{1-\kappa q} \| D_{\ell-1} \|, \quad \ell = \ell^*, \ell^* + 1, \ldots,
\]
and hence
\[
\| D_\ell \| \leq c_3 \kappa q^{(\ell-1)q}, \quad \ell \in \mathbb{Z}^+.
\]
The proof of convergence follows as before, except that the speed of convergence is considerably faster. \hfill \Box

**Theorem 5** If $\{ A, A \}$ is $q$-canonical, $\rho(F) < 1$ and $\text{Re} \sigma(A) < 0$ then $\lim_{t \to \infty} Y(t) = O$.

**Proof** We have already proved in Proposition 4 that $\rho(F) < 1$ implies that the components of the series (4.1) tend to zero geometrically (or faster) as $\ell \to \infty$ for every fixed $t$. On the other hand, since $\text{Re} \sigma(A) < 0$, it is true that $\lim_{t \to \infty} \exp t A = O$, hence each component tends to zero as $t \to \infty$ and $\ell$ is fixed. To reconcile these two processes, we split the series
\[
Y(t) = S_N(t) + T_N(t),
\]
where $N \in \mathbb{Z}^+$ is, for the time being, arbitrary, and
\[
S_N(t) = \sum_{\ell=0}^{N-1} D_\ell e^{\ell t A} V, \quad T_N(t) = \sum_{\ell=N}^{\infty} D_\ell e^{\ell t A} V.
\]

We assume that $F \neq O$ – the proof in the special case $F = B = O$ follows similarly and is, in fact, easier. We now use (4.8) (where $\| F \| = \| F \|_\varepsilon < 1$) to justify the inequality
\[
\| T_N \| \leq c_1 \sum_{\ell=N}^{\infty} \| F \|^\ell \| e^{\ell t A} \|.
\]
On the generalized pantograph equation

Since \( \Re \sigma(A) < 0 \), the sequence \( \{ \exp q^i t A \} \) can be uniformly bounded and we deduce the upper bound
\[
\| T_N(t) \| \leq c_1 \| F \|^N,
\]
(4.9)
where \( c_1 > 0 \). Note that both \( c_1 \) and the upper bound in (4.9) are valid uniformly for all \( t \geq 0 \). Thus, given any \( \delta > 0 \), we may choose sufficiently large \( N = N_\delta \) such that \( \| T_N(t) \| < \frac{1}{2} \delta \) for all \( t \geq 0 \).

We need to show that \( \| S_N(t) \| \) can be made arbitrarily small for large values of \( t \). This is easy: employing again the inequality (4.8), we obtain
\[
\| S_N(t) \| \leq c_1 \max_{\ell=0,1,\ldots,N-1} \{ \| e^{\ell t A} \| \} \sum_{\ell=0}^{N-1} \| F \|^\ell.
\]
Since \( \Re \sigma(A) < 0 \) and we are dealing with a finite range of \( \ell \)'s, we can make
\[
\max_{\ell=0,1,\ldots,N-1} \{ \| e^{\ell t A} \| \}
\]
arbitrarily small by choosing large \( t \). In other words, there exists \( t_\delta \) such that \( \| S_N(t) \| < \frac{1}{2} \delta \) for all \( t \geq t_\delta \). It follows that
\[
\| Y(t) \| \leq \| S_N(t) \| + \| T_N(t) \| \to 0
\]
and the proof is complete.

Before we present few explicit examples, we wish to state a result that, although not strictly within the ambit of this section, exploits in its proof the identity (4.4). Termination of the series (4.1) means that \( Y \) is a finite linear combination of matrix exponentials and it is interesting to derive conditions for this phenomenon.

**Theorem 6** Suppose that \( \{ A, A \} \) is \( q \)-canonical and \( A \) is nonsingular. Let \( \sigma(C) = \{ \mu_1, \mu_2, \ldots, \mu_d \} \) and \( \sigma(F) = \{ \omega_1, \omega_2, \ldots, \omega_d \} \). The series (4.1) terminates only if there exists a permutation \( [\pi_1, \pi_2, \ldots, \pi_d] \) of \( [1, 2, \ldots, d] \) such that for every \( k \in \{1, 2, \ldots, d\} \) there exists \( s_k \in \mathbb{Z}^+ \) obeying
\[
\mu_k = q^{s_k} \omega_{s_k}.
\]

**Proof** Termination, i.e. \( D = O \) for all \( \ell \geq s+1 \), say, implies that \( A(t) := \det D(t) \) is a \( d \)-degree polynomial. According to (4.4), it is true that
\[
A(t) = \frac{p_1(t)}{p_2(t)} A(qt),
\]
(4.10)
where
\[
p_1(t) = \det (I - tC) = \prod_{k=1}^{d} \left( 1 - \frac{t}{\mu_k} \right)
\]
and
\[
p_2(t) = \det (I - tF) = \prod_{k=1}^{d} \left( 1 - \frac{t}{\omega_k} \right)
\]
(the last two formulae need to be amended in a transparent manner if \( C \) and \( F \) are singular).
Iterating (4.10) and exploiting the fact that \( \lim_{m \to \infty} A(q^m t) = A(0) = 1 \), we obtain the explicit representation

\[
A(t) = \prod_{m=0}^{\infty} \frac{p_m(q^m t)}{p_m(q^m t)} = \sum_{k=1}^{a} \frac{\left( \frac{t}{\mu_k}; q \right)_\infty}{\left( \frac{t}{\omega_k}; q \right)_\infty}.
\]

(4.11)

We conclude that \( A \) can be a polynomial if and only if zeros and poles in (4.11) undergo cancellation, so as to eliminate the denominator altogether and leave only a finite number of factors in the numerator. It is now a matter of straightforward verification that, subject to the conditions of the theorem, such cancellation takes place and

\[
A(t) = \prod_{k=1}^{a} \left( \frac{t}{\mu_k}; q \right)_{s_k}
\]

(4.1)

(thus, coincidentally, \( \sum_{k=1}^{a} s_k = ds \)). Moreover, unless the conditions are satisfied, \( A \) is not a polynomial. \( \square \)

Since a determinant can vanish, clearly the condition of Theorem 6 need not be sufficient for termination. Examples, motivated by a different technique, will be presented elsewhere [23].

As already promised, we present a few examples of GPE where the Dirichlet expansion is explicitly known. Note in each case that the results of Theorem 5 (on stability) and Theorem 6 (on termination) are verifiable in a straightforward manner.

**Example 1** Let \( A \) be a single nonsingular Jordan block, \( B \) a multiple of the identity matrix, and \( C = O \). Thus,

\[
A_{k,j} = \begin{cases}
\lambda : k = j, \\
1 : k = j + 1, \\
0 : \text{otherwise};
\end{cases} 
B_{k,j} = \begin{cases}
b : k = j, \\
0 : \text{otherwise};
\end{cases} 
C_{k,j} = 0
\]

for \( k, j = 1, 2, \ldots, d \). Denoting the entries of \( D \) by \( D_{k,j}^{(l)} \), \( k, j = 1, 2, \ldots, d \), we derive from (4.3) the recurrence

\[
\lambda D_{k,j}^{(l)} + D_{k,j+1}^{(l)} - q^l (D_{k-1,j}^{(l)} + \lambda D_{k,j}^{(l)}) = -b D_{k,j}^{(l-1)}, \quad k, j = 1, 2, \ldots, d,
\]

(4.12)

where we have assumed \( D_{k,j}^{(0)} = 0 \) when one of the subscripts is outside the allowed range \( \{1, 2, \ldots, d\} \). Let \( \nu = b/\lambda \). We seek a solution of the form \( D_{k,j}^{(l)} = d_{s}^{(l)} \), \( k, j = 1, 2, \ldots, d, \ell \in \mathbb{Z}^+ \) (in other words, the \( D_{s}^{(l)} \)'s are Toeplitz matrices). Substitution in (4.12) produces the recursion

\[
d_{s}^{(l)} = \frac{\nu}{1-q^l} d_{s}^{(l-1)} - \frac{1}{\lambda} d_{s-1}^{(l)}, \quad \ell \in \mathbb{Z}^+, s \in \mathbb{Z},
\]

subject to the initial conditions \( d_{0}^{(0)} = 1, d_{s}^{(0)} = 0 \) for \( s \neq 0 \). Their explicit solution, which can be easily checked, is

\[
d_{s}^{(l)} = \begin{cases}
(-1)^s \frac{\ell!}{s!} \frac{\nu^s}{\lambda^s \ell! \ell!} & : s \geq 0, \\
0 & : s \leq -1,
\end{cases}
\]

\( \ell \in \mathbb{Z}^+ \).
Here \((z)_m\) is the familiar factorial symbol (the Pochhammer symbol) [34],
\[
(z)_m = \begin{cases} 
1 & : m = 0, \\
\prod_{j=0}^{m-1} (z+j) & : m = 1, 2, \ldots.
\end{cases}
\]

The generating function \(D\) is also Toeplitz and upper-triangular. Moreover,
\[
d_s(t) := \sum_{\ell=0}^{\infty} d_s^{(\ell)} t^{\ell} = \frac{(-1)^s}{s!\lambda^s} \sum_{\ell=0}^{\infty} \binom{\ell}{s} \frac{(\nu t)^\ell}{[q]_\ell}, \quad s \in \mathbb{Z}^+.
\]

We now exploit the standard theory of basic hypergeometric functions [19] to argue that
\[
d_0(t) = \sum_{\ell=0}^{\infty} \frac{(\nu t)^\ell}{[q]_\ell} = \frac{1}{(\nu t; q)_\infty}.
\]

Moreover, it is straightforward to prove by induction that
\[
d_s(t) = \frac{(-1)^s}{s!\lambda^s} t \frac{d^s}{dt^s} (\nu^{-1} d_0(t)).
\]

Let \(E(t) = D^{-1}(t)\). Since \(D\) is Toeplitz and upper-triangular, so is \(E\). We now exploit the theory of Toeplitz operators. Let \(\tilde{D}(t)\) be the singly-infinite Toeplitz matrix with the entries \(\tilde{d}_{k,s}(t) = d_{k-s}(t), k, s \in \mathbb{Z}^+\), and let \(\tilde{E}(t)\) be its inverse. Then also \(\tilde{E}(t)\) is Toeplitz and upper-triangular, and \(E(t)\) is its principal \(d \times d\) minor. Moreover, if \(d(z, t)\) is the symbol of \(\tilde{D}(t)\), then \(e(z, t) = 1/d(z, t)\) is the symbol of \(\tilde{E}(t)\).

We proceed to evaluate \(d(z, t)\) (and, hence, also \(e(z, t)\)).
\[
d(z, t) = \sum_{s=-\infty}^{\infty} d_s(t) z^s = \sum_{\ell=0}^{\infty} \frac{(\nu t)^\ell}{[q]_\ell} \sum_{s=0}^{\infty} \binom{\ell}{s} \left(-\frac{z}{\lambda}\right)^s
\]
\[
= \sum_{\ell=0}^{\infty} \frac{(\nu t)^\ell}{[q]_\ell} \frac{1}{\ell^\ell} \binom{\ell}{\nu t} \left(-\frac{z}{\lambda}\right)^s
\]
\[
= \sum_{\ell=0}^{\infty} \frac{1}{[q]_\ell} \left(\frac{\nu t}{1+z/\lambda}\right)^\ell = \frac{1}{(\nu t; q)_\infty}.
\]

Note that we have used the binomial theorem [34] to sum up the \(1_F^\ell\) hypergeometric series, as well as a theorem of Heine [19] to sum up \(\Phi_0\) basic hypergeometric series.

Since \(e(z, t) = 1/d(z, t)\), we obtain the explicit expression
\[
e(z, t) = \frac{\nu t}{1+z/\lambda}_\infty.
\]

Since \(E\) is Toeplitz, there exist functions \(e_s(t)\), \(s \in \mathbb{Z}\), such that \(E_{k,s}(t) = e_{k-s}(t), k, j = 1, 2, \ldots, d\). Of course, \(e_s \equiv 0\) for \(s \leq -1\).

We use again the theory of basic hypergeometric functions [19] to justify the identity
\[
(z; q)_\infty = \sum_{\ell=0}^{\infty} \frac{q^{[\ell-1]} \ell}{[q]_\ell} (-z)^\ell.
\]
This allows us to derive $e_s(t), s \in \mathbb{Z}^+$, explicitly

$$e(z, t) = \sum_{\ell=0}^{\infty} \frac{q^{h^{(\ell-1)}\ell}}{[q]_{\ell}} \left( \frac{vt}{1+z/\lambda} \right)^{\ell}$$

$$= \sum_{\ell=0}^{\infty} \frac{q^{\frac{h^{(\ell-1)}}{\ell}(-vt)^{\ell}}}{[q]_{\ell}} \Gamma_{\ell} \left[ \ell; -; \frac{z}{\lambda} \right]$$

$$= \sum_{\ell=0}^{\infty} \frac{q^{\frac{h^{(\ell-1)}(-vt)^{\ell}}{\ell}} \sum_{s=0}^{\infty} \frac{(\ell)_{s}}{s!} \left( -\frac{z}{\lambda} \right)^{s}}{[q]_{\ell}}$$

$$= \sum_{s=0}^{\infty} \frac{(-1)^s}{s! \lambda^s} \left[ \sum_{\ell=0}^{\infty} \frac{[\ell]_q}{[q]_{\ell}} q^{\frac{h^{(\ell-1)}(-vt)^{\ell}}{\ell}} \right] \left( -\frac{z}{\lambda} \right)^{s}.$$

Consequently,

$$e_s(t) = \frac{(-1)^s}{s! \lambda^s} \sum_{\ell=0}^{\infty} \frac{(\ell)_{s}}{[q]_{\ell}} q^{\frac{h^{(\ell-1)}(-vt)^{\ell}}{\ell}}, \quad \ell \in \mathbb{Z}^+.$$

It is easy to check that

$$e_0(t) = \sum_{\ell=0}^{\infty} \frac{q^{h^{(\ell-1)}\ell}}{[q]_{\ell}} (vt)^{\ell} = \left( vt, q \right)_{\infty},$$

$$e_s(t) = \frac{(-1)^s}{s! \lambda^s} \frac{d^s}{dt^s} \left( e^{r-1} e_0(t) \right), \quad s = 0, 1, \ldots.$$

Hence we have derived both the elements of $D(t)$ and of its inverse in closed form. This allows the explicit evaluation of both $D_\ell, \ell \in \mathbb{Z}^+$, and $V$, a technical exercise best left to the reader.

Example 2 Let $B = O$ and $C = I$. Note that $1 \in \sigma(C)$, hence $\{C, q^{-1}I\}$ is not $q$-canonical and, according to Theorem 3 the initial-value problem is ill-posed. Nevertheless, not all is lost and we can still attempt to recover Dirichlet series. It is easy to verify directly that

$$D_\ell = (-1)^\ell \frac{q^{h^{(\ell-1)}\ell}}{[q]_{\ell}} I, \quad \ell \in \mathbb{Z}^+,$$

is a solution of (4.3), consequently, exploiting again the theory of basic hypergeometric functions [19]

$$D(t) = \sum_{\ell=0}^{\infty} \frac{q^{h^{(\ell-1)}\ell}}{[q]_{\ell}} (-t)^{\ell} I = (t; q)_{\infty} I,$$

a scalar multiple of the identity matrix. It follows that $D(1) = O$, which we should have suspected – after all, the problem is ill-posed! Hence, the Dirichlet series

$$Y(t) = \sum_{\ell=0}^{\infty} (-1)^\ell \frac{q^{h^{(\ell-1)}\ell}}{[q]_{\ell}} e^{\ell A} V$$

obeys the GPE, albeit with the initial condition $Y(0) = O$. 
Example 3 Another example of an ill-conditioned GPE is

\[ Y'(t) = Y(t) + CY(qt), \quad (4.13) \]

where \( d = 2 \) and \( \sigma(C) = \{q^{-p}, \lambda\} \) for \( p \in \mathbb{Z}^+ \). We assume that \( \lambda \) itself is not of the form \( q^{-k} \), \( k \in \mathbb{Z}^+ \) and again find a Dirichlet series expansion, namely

\[ Y(t) = \sum_{t=0}^{\infty} \frac{(-1)^t q^{t(\lambda-1)/t}}{[q]_t} C e^{\lambda t}. \]

Let us suppose that

\[ C = W \begin{bmatrix} q^{-p} & 0 \\ 0 & \lambda \end{bmatrix} W^{-1}. \]

Thus, it follows again from the theory of basic hypergeometric functions that

\[ Y(0) = \prod_{t=0}^{\infty} (I - q^{-t} C) V = W \begin{bmatrix} (q^{-p}; q)_\infty & 0 \\ 0 & (\lambda; q)_\infty \end{bmatrix} W^{-1} V, \]

\[ = (\lambda; q)_\infty W \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} W^{-1} V, \]

since \( (q^{-p}; q)_\infty = 0 \). Letting \( \tilde{Y}(t) := W^{-1} Y(t) \), \( \tilde{V} := W^{-1} V \), we have

\[ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tilde{V} = \frac{1}{(\lambda; q)_\infty} \tilde{Y}(0); \]

consequently, the only admissible initial condition for (4.13) is

\[ Y(0) = W \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} W^{-1} V, \]

where \( W \) is the matrix of eigenvectors of \( C \), and \( V \) is an arbitrary \( 2 \times 2 \) matrix. Note that rank \( Y(0) \leq 1 \). In particular, and provided that \( W_{1,1}, W_{1,2} \neq 0 \), the matrix

\[ V = \begin{bmatrix} V_{1,1} & V_{1,2} \\ \xi V_{1,1} & \xi V_{1,2} \end{bmatrix}, \]

where

\[ \xi = \frac{W_{1,2} W_{2,1}}{W_{1,1} W_{2,2}}, \]

yields \( Y(0) = O \). Of course, this \( V \) is singular, but the Dirichlet series solution remains valid.

Theorem 5 provides sufficient conditions for \( \lim_{t \to \infty} Y(t) = O \). Our next object of study is the necessity of these conditions.\(^6\) The easier to examine is the condition \( \Re \sigma(A) < 0 \).

Lemma 7 Suppose that \( \{A, A\} \) is \( q \)-canonical, \( \rho(F) < 1 \) and \( \max \Re \sigma(A) > 0 \). Then \( \lim_{t \to \infty} \|Y(t)\| = \infty \).

\(^6\) Note that the conditions were necessary, as well as sufficient, in Examples 1–3.
Proof We follow the method of proof of Theorem 5, with the same notation. Thus, for every \( \delta > 0 \) we can find \( N \geq 1 \) such that \( e^{-\delta N t \lambda_{\text{max}}} \| T_N(t) \| < \delta \) uniformly for all \( t \geq 0 \), where \( \lambda_{\text{max}} = \max \Re \sigma(A) \). On the other hand, for sufficiently large \( t \), \( \exp t \lambda_{\text{max}} \) dominates \( S_{\rho}(t) \), hence the latter becomes unbounded in norm.\(^6\) Therefore

\[
e^{-\delta N t \lambda_{\text{max}}} \| Y(t) - S_{\rho}(t) \| = e^{-\delta N t \lambda_{\text{max}}} \| T_N(t) \| < \delta, \quad e^{-\delta N t \lambda_{\text{max}}} \| S_N(t) \| = O(1),
\]

implies that \( \| Y(t) \| \) becomes unbounded as \( t \to \infty \).

The necessity – or otherwise – of \( \rho(F) < 1 \) is more complicated, and to that end we need to examine derivatives of \( Y \). Let \( Y_m \) be the \( m \)th derivative of \( Y \). Repeated differentiation of (1.1) proves that \( Y_m \) obeys the GPE

\[
Y'_m(t) = A Y_m(t) + q^m B Y_m(q t) + q^m C Y'_m(q t), \quad Y_m(0) = \frac{d^m}{dt^m} Y(0).
\] (4.14)

There is a minor problem in trying to extend Theorems 4 and 5 to \( Y_m \), namely that \( Y_m(0) \) might be singular. This, however, can be easily absorbed into the matrix \( V \). An extreme case, namely \( Y_m(0) = O \), corresponds to polynomial termination of \( Y \), since, by (3.2), \( Y_m(0) = O \) implies that \( Y_k(0) = O \) for all \( k \geq m \). This implies that \( \| Y \| \) cannot be uniformly bounded, and there is no case to answer.

To emphasize the dependence on \( A, B \) and \( C \), we denote \( Y(t) = \mathcal{Y}(t; A, B, C) \). Thus, according to (4.14),

\[
Y_m(t) = \mathcal{Y}(t; A, q^m B, q^m C) Y_m(0).
\] (4.15)

Proposition 8 Let \( \{ A, A \} \) be \( q \)-canonical, \( \Re \sigma(A) < 0 \) and \( \rho(F) < q^{-m} \), where \( m \in \mathbb{Z}^+ \). Then

\[
\lim_{t \to \infty} Y_m(t) = O.
\]

Proof An immediate consequence of Theorem 5 and identity (4.15).

Let us denote by \( D_{\alpha, \ell} \) the coefficients in the Dirichlet expansion of \( \mathcal{Y}(t; A, \alpha B, \alpha C) \). It follows at once from (4.2) and (4.3) that

\[
D_{\alpha, \ell} = \alpha^\ell D_{1, \ell} = \alpha^\ell D_{\ell}, \quad \ell \in \mathbb{Z}^+,
\]

therefore

\[
\mathcal{Y}(t; A, \alpha B, \alpha C) = \sum_{\ell=0}^{\infty} \alpha^\ell D_{\ell} e^{q^\ell t A} D^{-1}(\alpha)
\]

(provided that \( D(\alpha) \) is nonsingular). Since \( Y_m \) is within the present framework, with \( \alpha = q^m \) and \( \det D(q^m) = 0 \) is ruled out by \( \rho(F) < q^{-m} \) and (4.5), we obtain

\[
Y_m(t) = \sum_{\ell=0}^{\infty} q^m D_{\ell} e^{q^\ell t A} D^{-1}(q^m).
\] (4.16)

Repeated integration of (4.16) yields

\[
Y_{m-k}(t) = \sum_{j=0}^{k-1} \frac{1}{j!} Y_{m-k+j}(0) t^j + \sum_{\ell=0}^{\infty} q^{(m-k)} D_{\ell} A^{-k} \left( e^{q^\ell t A} - \sum_{j=0}^{k-1} \frac{1}{j!} (q^\ell t A)^j \right) D^{-1} (q^m)
\]

\(^6\) If \( \lambda_{\text{max}} \) originates in a multiple eigenvalue, there might also be a contribution from polynomial terms, but this makes no difference whatsoever.
for \( k = 1, 2, \ldots, m \). In particular, we obtain a representation in mixed Dirichlet–Taylor series
\[
Y(t) = \sum_{j=0}^{m-1} \frac{1}{j!} Y_j(0) t^j + \sum_{\ell=0}^{\infty} D_{\ell} A^{-m} \left( e^{t A} - \sum_{j=0}^{m-1} \frac{1}{j!} (q^t A)^j \right) D^{-1}(q^m). \tag{4.17}
\]

**Theorem 9** Subject to \( q \)-canonicity of \( (A, A) \), the expansion (4.17) converges for all \( t \geq 0 \) whenever \( \rho(F) < q^{-m} \).

**Proof** For every fixed \( t \geq 0 \) and sufficiently large \( \ell \) (hence, sufficiently small \( q^t \)) it is true that
\[
A^{-m} \left( e^{t A} - \sum_{j=0}^{m-1} \frac{1}{j!} (q^t A)^j \right) \approx \frac{1}{m!} q^m t^m. \tag{4.18}
\]

As in the proof of Theorem 5, we choose a norm \( \| \cdot \| \) such that \( \| F \| < q^{-m} \). Hence, for large \( m \), (4.8) and (4.18) imply that
\[
\left\| D_{\ell} A^{-m} \left( e^{t A} - \sum_{j=0}^{m-1} \frac{1}{j!} (q^t A)^j \right) \right\| \approx c t^m (q^m \| F \|),
\]
where \( c > 0 \) is a constant. It follows that the norm of Dirichlet part of (4.17) is dominated asymptotically by a convergent geometric series, and the proof follows.

Although (4.17) is a useful generalization of (4.1), catering for the case \( 1 \leq \rho(F) < q^m \), \( m \geq 1 \), it is of lesser utility in analysing stability. Instead, we first confine our attention to \( 1 < \rho(F) < q^{-1} \). According to Proposition 8,
\[
\lim_{t \to \infty} (Y'(t) - CY'(qt)) = O,
\]
and substitution in (1.1) shows that for every \( \delta > 0 \) there exists \( t_\delta \) such that
\[
\| Y(t) - FY(qt) \| < \delta, \quad t \geq t_\delta. \tag{4.19}
\]
The norm in (4.19) is such that \( 1 < \| F \| < q^{-1} \). Likewise, we can choose \( 0 < t_0 < t_1 < \ldots < t_s \), such that
\[
\| Y(q^{-j} t) - FY(q^{-j+1} t) \| \leq q^j \delta, \quad t \geq t_j
\]
for \( j = 1, 2, \ldots, s \). Therefore,
\[
\| Y(q^{-s} t) - F^s Y(t) \| \leq \sum_{j=0}^{s-1} \| F^j (Y(q^{-s+j} t) - FY(q^{-s+j+1} t)) \|
\]
\[
\leq \sum_{j=0}^{\infty} (q \| F \|)^j \delta < \frac{\delta}{1 - q \| F \|}, \quad t \geq t_s.
\]
It follows that
\[
Y(q^{-s} t) \approx F^s Y(t), \quad t \geq 1, s \in \mathbb{Z}^+.
\tag{4.20}
\]
We conclude that \( \lim_{t \to \infty} \| Y(t) \| = \infty \), since we can always choose large \( t \) such that rank \( Y(t) = d \). Note in passing the interesting observation that a rank-deficient choice of \( Y(0) \) might well lead to a situation whereby the influence of the dominant eigenvalue(s) of \( F \) can
be discarded and the solution is stable. An example will be provided in §6. A further important observation that follows from (4.20) by letting \( t = q^{-s} \) is that, for \( t \gg 1 \),

\[
\| Y(t) \| \approx \rho(F)^{-(\ln t^{-1} \ln t) / \ln q} \| Y(t) \|.
\]

We conclude that there exists a constant \( c > 0 \) such that

\[
\| Y(t) \| \approx c t^{-\log_q \rho(F)}, \quad t \gg 1.
\]  (4.21)

Since \( g \in (0, 1) \) and \( \rho(F) > 1, \log_q \rho(F) \) is negative and we have an alternative proof that \( \| Y(t) \| \) becomes unbounded as \( t \to \infty \). This proof can be readily extended to \( 1 < \rho(F) \) (removing the restriction that \( \rho(F) < q^{-1} \)) by operating on higher derivatives. Thus, for example, if \( q^{-1} < \rho(F) < q^{-s} \) then we use (4.14) and (4.20) to deduce

\[
Y'(q^{-s} t) \approx q^{-s} F^s Y'(t), \quad t \gg 1, s \in \mathbb{Z}^+,
\]

and (4.20) follows by integration.

**Theorem 10** If \( \{ A, A \} \) is \( q \)-canonical and \( \rho(F) > 1 \) then the estimate (4.21) is valid and \( \lim_{t \to \infty} \| Y(t) \| = \infty \). \( \square \)

It is important to emphasize that the implication that every solution of the GPE (1.1) is unbounded when \( \rho(F) > 1 \) is false. As long as we choose 'good' initial conditions, rather than insisting on the general condition of (3.1), it is possible for the Dirichlet series to truncate, and in that case \( \sigma(A) \) determines solely whether the solution is stable.

**Example 4** Let

\[
y''(t) - (\alpha_1 + \alpha_2) y'(t) + \alpha_1 \alpha_2 y(t) = -\nu q^2 \alpha_1 y(qt) + \nu y'(qt),
\]

where \( \alpha_1, \alpha_2, \nu \in \mathbb{C}, \alpha_2 = q \alpha_1, q^2 \alpha_1 \). This second-order equation can be easily expressed in the format of (1.1), but it is more convenient to consider it in its present form (which will be deliberated in greater detail in §6). It is easy to verify that

\[
y(t) = e^{\alpha_1 t} + \frac{1 + q}{\alpha_2 - q \alpha_1} \nu e^{\alpha_1 t} + \frac{q}{(\alpha_2 - q \alpha_1)(\alpha_2 - q^2 \alpha_1)} \nu^2 e^{\alpha_1 t}.
\]

Thus, \( \Re \alpha_1, \Re \alpha_2 < 0 \) implies that \( \lim_{t \to \infty} y(t) = 0 \), irrespective of the value of \( \rho(F) \) (which can be made arbitrarily large by 'tuning' \( \nu \)).

5 The behaviour on the stability boundary

As long as \( \{ A, A \} \) is \( q \)-canonical and the theory of §4 is valid, there are two portions of the stability boundary, namely

1. \( \Re \sigma(A) < 0, \rho(F) = 1 \), and
2. \( \Re \sigma(A) = 0, \rho(F) \leq 1 \).

The behaviour in both cases is incompletely understood.

When \( \rho(F) = 1 \) then the estimate (4.21) yields \( \| Y(t) \| = O(1) \) as \( t \gg 1 \). Hence \( Y(t) \) lies on a bounded \( \mathbb{C}^{2 \times d} \)-valued manifold.
Example 5 We consider the equation \( Y'(t) = AY(t) - AY(qt), Y(0) = I \). It is easy to verify that, irrespective of what the matrix \( A \) is, \( Y(t) \equiv I \) is the unique (the equation is well-posed!) solution.

Example 6 Let \( d = 2 \) and

\[
A = \alpha I, \quad B = -\alpha \begin{bmatrix} e^{i\theta} & 1 \\ 0 & e^{i\theta} \end{bmatrix}, \quad C = O,
\]

where \( \text{Re} \alpha < 0 \) and \( 0 < |\theta| \leq \pi \). Therefore

\[
F = \begin{bmatrix} e^{i\theta} & 1 \\ 0 & e^{i\theta} \end{bmatrix}, \quad \rho(F) = 1.
\]

It is easy to verify that

\[
D_r = \frac{1}{|q|^r} \begin{bmatrix} e^{i\theta} & \ell e^{i(\ell-1)\theta} \\ 0 & e^{i\theta} \end{bmatrix}, \quad \ell \in \mathbb{Z}^+.
\]

obeys (4.2) and (4.3) and we obtain the formal Dirichlet series (4.1). It is easy to show, however, that the series does not converge. We derive instead the mixed Dirichlet series for \( Y' \). Its closed form is

\[
Y'(t) = I + \sum_{\ell=0}^{\infty} \frac{q^\ell}{|q|^\ell} \begin{bmatrix} e^{i\theta} & \ell e^{i(\ell-1)\theta} \\ 0 & e^{i\theta} \end{bmatrix} (e^{i\theta} t - 1) \tilde{V},
\]

where

\[
\tilde{V} = \alpha(q e^{i\theta}, q) \begin{bmatrix} 1 & -\sum_{\ell=0}^{\infty} \frac{q^\ell}{1-q^{\ell+1} e^{i\theta}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1-e^{i\theta} & -1 \\ 0 & 1-e^{i\theta} \end{bmatrix}.
\]

The functions

\[
\sum_{\ell=0}^{\infty} \frac{(qz)^\ell}{|q|^\ell} e^{i\ell t},
\]

\[
\sum_{\ell=0}^{\infty} \frac{\ell (qz)^\ell}{|q|^\ell} e^{i\ell t}
\]

are both analytic with radius of convergence \( q^{-1} \), therefore both \( Y'(t) \) and \( Y(t) \) are bounded. Moreover, it is easy to verify that, since \( \text{Re} \alpha < 0 \), the bound is uniform in \( t \). We deduce that the function \( \| Y(t) \| \) is uniformly bounded for all \( t \geq 0 \), but, of course, we have no clue as to the precise nature of the evolution of \( Y \).

It is too early even to conjecture about the nature of the trajectory of \( Y \) when \( \rho(F) = 1, \text{Re} \sigma(A) < 0 \), except that it is uniformly bounded. It can tend to a steady state (Example 5) – even to zero (Example 4), whereas computer simulation suggests that, in general, the trajectory of \( y'(t) = ay(t) + ae^{i\theta} y(qt) + cy'(qt), 0 < |\theta| < \pi, \) approaches a circle in the phase plane \( (\text{Re} y, \text{Im} y) \).

The behaviour of \( Y \) on the other stability boundary, \( \max \text{Re} \sigma(A) = 0, \rho(F) \leq 1 \), is better understood and it provided the original motivation for the study of the Dirichlet series [7, 13, 17]. Figures 2 and 3 display the solutions of \( y''(t) + 2y(t) = 5iy'(qt) \) and \( y''(t) + y(t) = \)
10y'(t/11) respectively in the 'phase plane' (Re y, Im y). This suppression of time in the plot is helpful in revealing geometric features. The first impression from Figs. 2–3 is that \( \{y(t)\}_{t \geq 0} \) is close to periodic, and it is validated by our analysis.

We recall that an \( L_{\infty}[0, \infty) \) function \( f \) is said to be almost periodic [27] if for every \( \delta > 0 \) there exists \( T_\delta > 0 \) such that

\[
|f(t + T_\delta) - f(t)| < \delta, \quad t \geq 0.
\]

\(^7\) The precise value of \( y(0) \) and \( y'(0) \) is immaterial to the matter in hand.
The extension to matrices is straightforward: the matrix function \( Y(t) \) is almost periodic if all its entries are almost-periodic functions.

**Theorem 11** Suppose that \( \{A, A\} \) and \( \{C, q^{-1} I\} \) are \( q \)-canonical, that \( A \) is diagonalizable, \( \det A \neq 0, \rho(F) < 1 \), and that max \( \Re \sigma(A) = 0 \). Then \( Y(t) \) is almost periodic.

**Proof** All the conditions of Proposition 4 are satisfied, hence the Dirichlet expansion (4.1) is valid. Moreover, it is easy, by following the method of proof of Theorem 5, to verify that \( \| Y \| \) is an \( L_\infty[0, \infty) \) function, since \( \| T_\sigma \| \) therein can be made arbitrarily small, and \( \| S_\sigma \| \) can be uniformly bounded. Therefore there exist coefficients \( \gamma_{k,j}^{(\ell, n)} \) such that

\[
(Y(t))_{k,j} = \lim_{L \to \infty} \sum_{\ell = 0}^{L} \sum_{s=1}^{d} \gamma_{k,j}^{(\ell, n)} e^{\ell \alpha_s t}, \quad k, j = 1, 2, \ldots, d,
\]

where \( \sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_d\} \). Moreover, (5.1) is valid uniformly for \( t \in [0, \infty) \). We now invoke a theorem of H. Bohr [27]: almost periodic functions are exactly all the \( L_\infty \) limits of exponential polynomials, i.e. functions of the form

\[
\sum_{\ell = 0}^{L} \sum_{s=1}^{d} \alpha_s e^{i\mu_s},
\]

where the \( \alpha_s \)'s are complex and \( \mu_s \)'s real. (To be faithful to the historical record, Bohr introduced almost periodic functions as limits of exponential polynomials.) We separate the sum in (5.1) into two components, \( P_L(t) + Q_L(t) \), where \( P_L \) includes all the \( \lambda_s \)'s with \( \Re \lambda_s = 0 \) and \( Q_L \) absorbs all the remaining elements. It follows from the Bohr theorem that, as \( L \to \infty \), \( P_L \) tends to an almost periodic function, whereas an argument that parallels the proof of Theorem 5 can be used to argue that \( Q_L \to 0 \).

Almost periodic scalar cases of the GPE have been already investigated by Frederickson [17] and by Derfel & Shevko [13].

Let \( \{i\phi_{\ell,s}\}_{\ell,s=1}^{\hat{d}} \) be all the eigenvalues of \( A \) that reside along \( i\Re, 1 \leq \hat{d} \leq d \). If \( q \) is rational, and all the \( \phi_{\ell,s} \)'s are integer multiples of the same real number \( \phi^* \), say, we can explicitly derive an 'almost period' \( T_\sigma \). Suppose that \( q = u/v \), where the integers \( (u,v) \) are relatively prime and \( 1 \leq u \leq v - 1 \). Suppose that the function \( P_L \) from the proof of Theorem 11 is of the form

\[
P_L(t) = \sum_{\ell = 0}^{L} \sum_{s=1}^{\hat{d}} p^{\ell, n}(t) e^{i\phi_s t}.
\]

Moreover, let \( \phi_s = \omega_s \phi^*, \omega_s \in \mathbb{Z}, s = 1, 2, \ldots, \hat{d} \). Let \( N \) be a suitably large integer. Then

\[
P_{\omega}(vN 2\pi/\phi^* + t) = \sum_{\ell = 0}^{L} \sum_{s=1}^{\hat{d}} p^{\ell, n}(t) \exp(iq' \phi_s t + 2\pi i u' v^{-\omega_s} w_s),
\]

consequently

\[
P_{\omega}(vN 2\pi/\phi^* + t) - P_L(t) = \sum_{\ell>N+1}^{\infty} \sum_{s=1}^{\hat{d}} p^{\ell, n}(t) e^{i\omega'_s t (e^{2\pi i u' v^{-\omega_s} w_s} - 1)}.
\]

Since the \( p^{\ell, n}(t) \)'s decay geometrically, the sum on the right is uniformly convergent and can be made arbitrarily small by choosing large \( N \).
The aforementioned argument is valid for every $k, l \in \{1, 2, \ldots, d\}$ and it produces the promised 'almost period'.

**Theorem 12** Let the conditions of Theorem 11 be satisfied and assume, in addition, that the real parts of all the eigenvalues lying on $i\mathbb{R}$ are integer multiples of the same real number $\phi^*$ and that $q$ is rational, $q = u/v$. Then for every $\delta > 0$ there exists $N_\delta \in \mathbb{Z}^+$ such that

$$\left\| Y \left( \nu^N, \frac{2\pi}{\phi^*} + t \right) - Y(t) \right\| < \delta$$

for every $t \geq 0$ and integer $N \geq N_\delta$.

Theorem 12 is illustrated by Fig. 4, where the solution of $y''(t) + y(t) = -10y'(\frac{1}{4}t)$ is depicted (again, in the $(\text{Re} y, \text{Im} y)$ plane) for $t \in [0, 2\pi 4^n]$ for $n = 2, 3, 4, 5$. 

\[ \square \]
The last theorem by no means exhausts all the richness of behaviour that is apparent in Figs 2–4. The first feature that is indicated therein is the presence of ‘symmetry’. This can be affirmed by analytic means.

Given an \( L_\infty \) function \( f \), we say that it is an almost rotationally \( m \)-symmetric, if for every \( k \in \{0, 1, \ldots, m\} \) and \( \delta > 0 \) there exists \( T_{k, \delta} > 0 \) such that

\[
|f(t + T_{k, \delta}) - e^{2\pi ik/(m+1)}f(t)| < \delta.
\]

(5.2)

Every function that is almost rotationally \( m \)-symmetric is almost periodic.

We assume again that \( q = u/v \), where \( u \) and \( w \) are relatively prime. Furthermore, we stipulate that \( v \geq u + 2 \). Let us suppose that the series

\[
f(t) = \sum_{t=0}^{\infty} p_t e^{i\vartheta t}
\]

is \( L_\infty[0, \infty) \). Note that it is almost periodic, hence it obeys (5.2) for \( k = 0 \).

**Proposition 13** For every \( k \in \{1, 2, \ldots, v-u-1\} \) and \( N \in \mathbb{Z}^+ \) there exists \( S_{k, N} \in \{1, 2, \ldots, v-u-1\} \) such that

\[
\begin{align*}
\left(\frac{u}{v}\right)^{S_{k, N}} \cdot v^N &\equiv \frac{k}{v-u} \pmod{1}, \\
\ell &\in \{0, 1, \ldots, N\}.
\end{align*}
\]

(5.3)

**Proof** Let

\[
\xi_{\ell} = \frac{u^{\ell+1} S_{k, N} - k}{v-u}, \quad \ell \in \{0, 1, \ldots, N\}.
\]

The statement (5.3) can be rephrased as \( \xi_{\ell} \in \mathbb{Z}, \ell = 0, 1, \ldots, N \). Let \( \ell \in \{1, 2, \ldots, N\} \). Then

\[
\xi_{\ell-1} \in \mathbb{Z} \Rightarrow \frac{u^{\ell-1} v^{N-(v-u+u)} S_{k, N} - k}{v-u} \in \mathbb{Z} \Rightarrow u^{\ell-1} v^{N} + \xi_{\ell} \in \mathbb{Z}
\]

\[
\Rightarrow \xi_{\ell} \in \mathbb{Z}.
\]

Thus, if \( \xi_0 \in \mathbb{Z} \) then (5.3) is true. In other words, to prove the lemma it is enough to demonstrate the existence of \( S_{k, N} \in \{1, 2, \ldots, v-u-1\} \) such that

\[
\frac{v^N S_{k, N} - k}{v-u} \in \mathbb{Z}.
\]

(5.4)

To this end, we consider the map \( \nu : \mathbb{Z}_{v-u} \to \mathbb{Z}_{v-u} \), where \( \mathbb{Z}_m \) is the cyclic group of order \( m \), defined via

\[
\nu(x) = (u^N x - k) \mod (v-u), \quad x \in \mathbb{Z}_{v-u}.
\]

Suppose that there exist \( x, y \in \mathbb{Z}_{v-u} \), \( x \neq y \), such that \( \nu(x) = \nu(y) \). It follows that \( v^N(x - y) = 0 \mod (v-u) \), and this is impossible – since \( v \) and \( u \) are relatively prime, so are \( v^N \) and \( v-u \), whereas, of course, \( |x - y| \leq v-u-1 \). It follows that \( \nu \) is bijective and \( [\nu(0), \nu(1), \ldots, \nu(v-u)] \) is a permutation of \( [0, 1, \ldots, v-u] \). Consequently, there exists \( \bar{x} \in \mathbb{Z}_{v-u} \) such that \( \nu(\bar{x}) = 0 \). It is easy to affirm that \( \bar{x} \neq 0 \). We choose \( S_{k, N} = \bar{x} \) and it follows from our construction that (5.4) is satisfied. This completes the proof. \( \square \)
Let
\[ T_{k,N} = \frac{2\pi}{v-u} S_{k,N} v^N. \]

It follows from (5.3) that
\[ e^{i\alpha T_{k,N}} = e^{\frac{2\pi i \alpha}{v-u}}. \]

Consequently,
\[ f(t + \tilde{T}_{k,N}) - e^{\frac{2\pi i \nu}{v-u}} f(t) = \sum_{n=-N+1}^{\infty} p_n e^{i\alpha' n^N} \frac{S_{k,N}}{v-u} \left( e^{2\pi i \nu n} - 1 \right). \]

Since \( f \) is in \( L_\infty[0,1] \), it is apparent that the magnitude of the series on the right can be made arbitrarily small by choosing sufficiently large \( N \), and we conclude that \( f \) is almost rotationally \((v-u-1)\)-symmetric.
Theorem 14 Suppose that

1. $q = \frac{u}{v}$, where $u \leq v - 2$, $u$ and $v$ relatively prime,
2. $\{A, A\}$ and $\{C, q^{-1}I\}$ are $q$-canonical,
3. $A$ is diagonalizable and $\det A \neq 0$,
4. $\rho(F) \neq 1$,
5. $\max \Re \sigma(A) = 0$ and the real parts of all the eigenvalues lying on $i\mathbb{R}$ are integer multiples of the same real number.

Then the solution of (1.1) is almost rotationally $(v - u - 1)$-symmetric.

Proof Identically to the proof of Theorem 12, we demonstrate that for every $k \in \{1, 2, \ldots, v - u - 1\}$ and $\delta > 0$ there exists $N_\delta$ such that for all $N \geq N_\delta$ it is true that

$$|P_L(t + T_{k,n}) - e^{\frac{2\pi i t}{v}k}P_L(t)| < \delta, \quad t \geq 0.$$ 

An illustration of the last theorem is provided in Fig. 5, where the solution of $y''(t) + y(t) = \frac{1}{2}i y'(qt)$ is displayed for $q \in \{\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}\}$.

Self-similarity is another feature that can be discerned in Figs 2-5, and it is emphasized in Fig. 6, where the solution of $y''(t) + y(t) = 5i y'(qt)$ is displayed in four different scales: $|\Re y| \leq 50, |\Re y| \leq 5, |\Re y| \leq 0.5$ and $|\Re y| \leq 0.05$. It is apparent that each ‘strand’ of $y$ in

**Figure 6.** Solutions of $y''(t) + y(t) = 5i y'(t/4)$ in the ‘windows’ $50 \times 10^{-j}$, $j = 0, 1, 3, 4$. 

the plane \((\text{Re} \, y, \text{Im} \, y)\) is, upon magnification, a bundle of (four – is it because \(q = \frac{1}{4}\?) knotted ‘strands’ and this continues \textit{ad infinitum}. The Hausdorff dimension of the underlying fractal set is, for the time being, beyond conjecture.

6 The high-order pantograph equation

Let \(\mathcal{A}\) and \(\mathcal{B}\) be two linear differential operators with constant coefficients,

\[
\mathcal{A} := \sum_{k=0}^{n} a_k \frac{d^k}{dt^k}, \quad \mathcal{B} := \sum_{k=0}^{m} b_k \frac{d^k}{dt^k},
\]

where \(a_0, a_n, b_0, b_m \neq 0\). The subject matter of the present section is the scalar, high-order equation

\[
\mathcal{A}y(t) = \mathcal{B}y(qt),
\]

given in tandem with the initial conditions

\[
y^{(k)}(0) = y_{k}, \quad k = 0, 1, \ldots, n-1.
\]

Of course, nothing is easier than converting (6.1) into the GPE (1.1) and exploiting the theory from the four previous sections. However, much additional insight can be gained by taking advantage of the special form of (6.1).

We let

\[
a(z) := \sum_{k=0}^{n} a_k z^k = a_0 \prod_{j=1}^{n} \left(1 - \frac{z}{\alpha_j}\right),
\]

\[
b(z) := \sum_{k=0}^{m} b_k z^k = b_0 \prod_{j=1}^{m} \left(1 - \frac{z}{\beta_j}\right),
\]

be the characteristic polynomials of \(\mathcal{A}\) and \(\mathcal{B}\), respectively, and set \(\rho = b_0/a_0\). Moreover, we define the exponential basic hypergeometric function \(m \Xi_n\) as

\[
m \Xi_n \left[ g_1, g_2, \ldots, g_m; h_1, h_2, \ldots, h_n; q, \kappa, z \right] = \sum_{\ell=0}^{\infty} \frac{(g_1; q)_\ell (g_2; q)_\ell \cdots (g_m; q)_\ell}{(h_1; q)_\ell (h_2; q)_\ell \cdots (h_n; q)_\ell} \kappa^\ell.
\]

The function is well-defined, as long as none of \(h_1, h_2, \ldots, h_n\) is of the form \(q^{-p}\) for some \(p \in \mathbb{Z}^+\).

To permit shortcuts in this somewhat bulky notation, we let \(g := [g_1, g_2, \ldots, g_m], (c; q)_\rho := \prod_{j=1}^{m} (c_j; q)_\rho\), adopt similar notation for \(d\) and let

\[
m \Xi_n \left[ g_1, g_2, \ldots, g_m; h_1, h_2, \ldots, h_n; q, \kappa, z \right] \equiv m \Xi_n \left[ g_1, g_2, \ldots, g_m; h_1, h_2, \ldots, h_n; q, \kappa, z \right].
\]

Lemma 15 Let \(s \in \{1, 2, \ldots, n\},\)

\[
g_s := \left[ \frac{\alpha_1}{\alpha_2}, \frac{\alpha_2}{\alpha_3}, \ldots, \frac{\alpha_n}{\beta_m} \right], \quad h_s := \left[ \frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}, \ldots, \frac{\alpha_m}{\beta_m} \right].
\]

The function

\[
y_s(t) := m \Xi_{n-1} \left[ g_s; h_s; q, \rho, \alpha_s t \right]
\]

is a formal solution of (6.1).
On the generalized pantograph equation

Proof If \( \mathcal{E} = \sum e_k \frac{d^k}{dt^k} \) is a linear differential operator with constant coefficients and \( \tilde{E}(z) = \sum e_k z^k \) then \( \mathcal{E} e^{\gamma t} = E(\gamma) e^{\gamma t} \). Thus, acting on \( y_s \), as defined by (6.3), with the operator \( \mathcal{A} \), we obtain

\[
\mathcal{A} y_s(t) = \sum_{\ell=0}^{\infty} \frac{(g_s; q)_{\ell}}{(h_s; q)_{\ell}} \rho^{\ell} A(\alpha_s \rho^\ell).
\]

However, \( A(\alpha_s) = 0 \), therefore

\[
\mathcal{A} y_s(t) = a_0 \sum_{\ell=1}^{\infty} \frac{(g_s; q)_{\ell}}{(h_s; q)_{\ell-1}} \rho^{\ell-1} \prod_{k=1}^{n} \left(1 - \frac{\alpha_s}{\alpha_k} \rho^\ell \right).
\]

Because of our definition of \( h_s \),

\[
\frac{1}{(h_s; q)_\ell [q]_{\ell-1}} \prod_{k=1}^{n} \left(1 - \frac{\alpha_s}{\alpha_k} \rho^\ell \right) = \frac{1}{(h_s; q)_\ell [q]_{\ell-1}}, \quad \ell = 1, 2, \ldots
\]

Consequently,

\[
\mathcal{A} y_s(t) = a_0 \sum_{\ell=1}^{\infty} \frac{(g_s; q)_{\ell}}{(h_s; q)_{\ell-1}} \rho^{\ell} \tag{6.4}
\]

Likewise, acting with \( \mathcal{B} \),

\[
\mathcal{B} y_s(qt) = a_0 \sum_{\ell=1}^{\infty} \frac{(g_s; q)_{\ell}}{(h_s; q)_{\ell-1}} \rho^{\ell} B(\alpha_s \rho^\ell)
\]

\[
= a_0 \sum_{\ell=1}^{\infty} \frac{(g_s; q)_{\ell-1}}{(h_s; q)_{\ell-1}} \rho^{\ell-1} \prod_{k=1}^{m} \left(1 - \frac{\alpha_s}{\beta_k} \rho^{\ell-1} \right)
\]

\[
= a_0 \sum_{\ell=1}^{\infty} \frac{(g_s; q)_{\ell}}{(h_s; q)_{\ell-1}} \rho^{\ell}
\]

and comparison with (6.4) completes the proof. \( \square \)

It has been stipulated in our discussion that \( a_0, b_0 \neq 0 \). The first requirement cannot be abandoned, but we can take the case \( b_0 = 0 \) on board with minimal effort. Since \( b_0 = (-1)^m b_m \beta_1 \beta_2 \ldots \beta_m \), we have

\[
\begin{pmatrix} \alpha_s \\ \beta_{m+1} \end{pmatrix} b_0^\ell = (-1)^m (b_m \beta_1 \beta_2 \ldots \beta_m)^{\ell-1} \prod_{j=1}^{\ell-1} (\beta_m - \alpha_s \rho^j) \rightarrow (\beta_m \alpha_s)^{\ell} \rho^{(\ell-1)\ell}
\]

In general, if \( \beta_{m-1} = \beta_{m-2} = \ldots = \beta_m = 0 \), say, and \( \beta_{m-1} \neq 0 \) then easy induction affirms that \( \prod_{k=m+1}^{n} \alpha_s \beta_k \) need be replaced with \( b_0 \alpha_s \rho^\ell \rho^{(\ell-1)\ell} \). Thus, if \( b \) has an \( r \)-fold zero at the origin, (6.3) need be replaced with

\[
y_s(t) = \sum_{\ell=0}^{\infty} \frac{(g_s; q)_{\ell}}{(h_s; q)_{\ell}} \rho^{\ell} \prod_{k=1}^{m} \left(1 - \frac{\alpha_s}{\beta_k} \rho^\ell \right) \tag{6.5}
\]

where

\[
\bar{g}_s = \begin{bmatrix} \alpha_s & \alpha_s & \ldots & \alpha_s \\ \beta_1 & \beta_2 & \ldots & \beta_{m+1} \end{bmatrix}, \quad \beta_{m+1} = \beta_{m+2} = \ldots = b_m = 0.
\]
In the case \( r = 0 \) (i.e. \( b_0 \neq 0 \)) (6.3) and (6.5) coincide.

Incidentally, the function \( y_s \) in (6.3) can be alternatively written as

\[
y_s(t) = \sum_{i=0}^{\infty} \frac{\prod_{j=0}^{i-1} b(\alpha_s q^j)}{\prod_{j=i}^{\infty} a(\alpha_s q^j)} e^{t^i \alpha_s^i}.
\]

This is true since

\[
\prod_{j=1}^{m} \left( \frac{1 - \alpha_s q^k}{\beta_s q^k} \right) = \frac{b(\alpha_s q^m)}{b_0}, \quad k \geq 0,
\]

\[
\prod_{j=1}^{n} \left( \frac{1 - \alpha_s q^k}{\alpha_s q^k} \right) = \frac{A(\alpha_s q^n)}{a_0(1 - q^n)}, \quad k \geq 1,
\]

implies

\[
(g_s; q)_\ell = b_0^{\ell - 1} \prod_{j=0}^{\ell - 1} b(\alpha_s q^j),
\]

\[
(h_s; q)_\ell = a_0^{\ell - 1} \prod_{j=0}^{\ell - 1} a(\alpha_s q^j), \quad \ell \in \mathbb{Z}^+.
\]

It is not difficult to prove that (6.6) remains valid when \( b(0) = 0 \).

Although the high-order equation (6.1) can be easily converted into the GPE vector equation (1.1), there is a crucial conceptual difference between a special solution of (6.1) and the fundamental solution of (3.1). In particular, it is perfectly possible for \( \lim_{t \to -\infty} y_s(t) = 0 \), while another choice of initial conditions might lead to unboundedness. Fortunately, the method of proof of §4 – estimates of Dirichlet series – remains valid and we can readily deduce an equivalent of Theorem 5.

**Lemma 16** Let us suppose that \( \alpha_1, \alpha_2, \ldots, \alpha_n \neq 0, \alpha_k \neq \rho^p \alpha_s \) for all \( k \in \{1, 2, \ldots, n\} \) and \( p \in \mathbb{Z} \), \( |\rho| < 1 \) and \( \Re \alpha_s < 0 \). Then \( \lim_{t \to -\infty} y_s(t) = 0 \).

Note that we do not require in the lemma that \( \Re \alpha_k < 0 \) for all \( k = 1, 2, \ldots, n \) – it suffices that \( \Re \alpha_s < 0 \). Similarly, \( \Re \alpha_s = 0, \alpha_s \equiv 0 \), and \( |\rho| < 1 \) imply almost-periodic behaviour of \( y_s \), irrespective of the remaining \( \alpha_s \).

We stipulate in the sequel of this section that for every \( p \in \mathbb{Z} \) and \( k, j \in \{1, 2, \ldots, n\} \) it is true that \( \alpha_s \neq \rho^p \alpha_s \). In particular, it follows that the numbers \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are distinct and we have \( n \) distinct functions \( y_s, y_s, \ldots, y_s \). We devote the remainder of this section to the determination of the dimension of the linear space \( \mathcal{W}_s := \text{Sp} \{ y_s(t), y_s(t), \ldots, y_s(t) \} \) and to the question whether we can fit arbitrary initial values (6.2) by linear combinations of the \( y_s \)'s. Although our exposition is consistent with the condition that \( b(0) \neq 0 \) (and hence (6.3) is valid), the results can be readily extended to cater for the general case.

Let

\[
W(t) := \det \begin{bmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\
y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\
\vdots & \vdots & \ddots & \vdots \\
y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{bmatrix}
\]

be the Wronskian of \( \{ y_1(t), y_2(t), \ldots, y_n(t) \} \). Clearly, \( W(0) \neq 0 \) implies that \( \dim \mathcal{W}_s = n \) and we can fit arbitrary initial values.
Lemma 17 Subject to the above conditions, the value of the Wronskian at the origin is

\[ \frac{1}{(\rho; q)_\infty} \text{VDM} (\alpha_1, \alpha_2, \ldots, \alpha_n) \]

when \( n > m \) and

\[ \frac{(b_n/a_n; q)_\infty}{(\rho; q)_\infty} \text{VDM} (\alpha_1, \alpha_2, \ldots, \alpha_n) \]

when \( n = m \). Here

\[ \text{VDM} (\tau_1, \tau_2, \ldots, \tau_n) = \det \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \tau_1 & \tau_2 & \ldots & \tau_n \\ \vdots & \vdots & \ddots & \vdots \\ \tau_1^{n-1} & \tau_2^{n-1} & \ldots & \tau_n^{n-1} \end{bmatrix} \]

is a Vandermonde determinant.

Proof Since

\[ \frac{d}{dt} m \Xi_n [g; h, q, \kappa, t] = m \Phi_n [g; h, q, \kappa, 0] = m \Phi_n [g; h, q, \kappa], \]

where \( m \Phi_n \) is a basic hypergeometric function [19],

\[ m \Phi_n [g; h, q, \kappa] = \sum_{r=0}^\infty \frac{(g_1; q)_r (g_2; q)_r \ldots (g_m; q)_r}{(h_1; q)_r (h_2; q)_r \ldots (h_n; q)_r} [q]_r \]

it follows by induction that

\[ \frac{d^k y_1 (0)}{dt^k} = \alpha_1^k m \Phi_{n-1} [g_2; h, q, q^k \rho], \quad k \in \mathbb{Z}^+. \quad (6.7) \]

Set \( b_{m+1} = b_{m+2} = \ldots = b_n = 0 \) if \( n > m \). We wish to explore the dependence of \( W(0) \) on \( \rho \) and \( \nu := b_n/a_n \), and for that purpose we denote \( W(0) = F(\rho, \nu) \). Letting

\[ f_\nu (\rho) := m \Phi_{n-1} [g_2; h, q, \rho], \]

it follows from (6.7) and substitution into (6.1) that

\[ \sum_{k=0}^n a_k \alpha_k^r f_\nu (q^k \rho) = \sum_{k=0}^n b_k \alpha_k^r f_\nu (q^k \rho). \quad (6.8) \]

As a notational convention, we set

\[ \alpha_k^r f(z) := [\alpha_1^k f_1(z), \alpha_2^k f_2(z), \ldots, \alpha_n^k f_n(z)], \quad k \in \mathbb{Z}^+. \]

Therefore

\[ F(\rho, \nu) = \det \begin{bmatrix} f(\rho) \\ \alpha f(q \rho) \\ \alpha^2 f(q^2 \rho) \\ \vdots \\ \alpha^{n-1} f(q^{n-1} \rho) \end{bmatrix} \]
We can replace the top row by a linear combination of the rows without affecting the value. In particular,

\[
F(\rho, \nu) = \frac{1}{a_0} \det \begin{bmatrix}
\sum_{k=0}^{n-1} a_k \alpha^k f(q^k \rho) \\
\alpha f(q\rho)
\end{bmatrix},
\]

It now follows from (6.8) that

\[
F(\rho, \nu) = \frac{1}{a_0} \det \begin{bmatrix}
\sum_{k=0}^{n-1} b_k \alpha^k f(q^k \rho) - a_n \alpha^n f(q^n \rho) \\
\alpha f(q\rho)
\end{bmatrix},
\]

\[
= \sum_{k=0}^{n-1} \frac{b_k}{a_0} \begin{bmatrix}
\alpha^k f(q^k \rho) \\
\alpha f(q\rho)
\end{bmatrix} - \frac{a_n - b_n}{a_0} \det \begin{bmatrix}
\alpha^n f(q^n \rho) \\
\alpha f(q\rho)
\end{bmatrix}.
\]

Except for \( k = 0 \), the terms in the sum vanish, since the determinants possess two identical rows, and we are left with

\[
F(\rho, \nu) = \rho F(\rho, \nu) - \frac{a_n - b_n}{a_0} \det \begin{bmatrix}
\alpha^n f(q^n \rho) \\
\alpha f(q\rho)
\end{bmatrix}.
\]

To identify the remaining determinant, we bring the top row to the bottom and extract a factor of \( \alpha_s \) from the \( s \)th column, \( s = 1, 2, \ldots, n \). Hence

\[
\det \begin{bmatrix}
\alpha^n f(q^n \rho) \\
\alpha f(q\rho)
\end{bmatrix} = (-1)^n \prod_{s=1}^{n} \alpha_s \det \begin{bmatrix}
f(q_s) \\
\alpha f(q^s \rho)
\end{bmatrix}.
\]

Since

\[
(-1)^n \frac{a_n - b_n}{a_0} \prod_{s=1}^{n} \alpha_s = 1 - \nu,
\]
we obtain the recurrence relation
\[ F(\rho, v) = \frac{1-v}{1-\rho} F(q\rho, qv). \quad (6.9) \]

The reason for \( v \) being replaced by \( qv \) on the right is that, when \( m = n \) (otherwise anyway \( v = 0 \)),
\[ v = \frac{b_n}{a_n} = \rho \prod_{s=1}^{n} \beta_s. \]

Thus, as soon as we replace \( \rho \) by \( q\rho \), we need to multiply \( v \) by \( q \).

Iterating (6.9) yields
\[ F(\rho, v) = \frac{(v; q)_\infty}{(\rho; q)_\infty} F(0, 0). \]

Since \( F(0, 0) = VDM(\alpha_1, \alpha_2, \ldots, \alpha_n) \), the proof is complete. \( \square \)

**Theorem 18** Subject to the conditions of Lemma 17, \( W(0) \neq 0 \) and the initial-value problem (6.2) can always be solved as a linear combination of \( y_1, y_2, \ldots, y_n \) if and only if there exists no \( p \in \mathbb{Z}^+ \) such that \( a_n = q^p b_n \). In particular, this is the case if \( n > m \).

**Proof** Because of the definition of the \( q \)-factorial, the only eventuality whereby \( F(\rho, v) = 0 \) is when \( 1 - q^p v = 0 \) for some \( p \in \mathbb{Z}^+ \). The proof follows from the definition of \( v \). \( \square \)

### 7 Resonance

The condition that \( (A, \Delta) \) is \( q \)-canonical is not necessary for well-posedness, and its main purpose is to ensure that the formal Dirichlet expansion (4.1) exists. The present section is devoted to the case of non-\( q \)-canonical \( (A, \Delta) \). Although no general theory is available at present, we demonstrate by several examples that the situation is akin to the familiar phenomenon of resonance in classical ODEs. The clear indication is that a hopeful approach is to consider generalized Dirichlet expansions, allowing the coefficients \( D_\gamma \) to depend polynomially on \( t \).

**Example 7** Let
\[ Y'(t) = \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix} Y(t) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} Y(qt), \quad Y(0) = I. \quad (7.1) \]

It is straightforward to verify that the solution is
\[ Y(t) = \begin{bmatrix} e^t & t e^{qt} \\ 0 & e^{qt} \end{bmatrix}, \quad t \geq 0. \]

Lest an impression is left that, upon the failure of \( q \)-canonicity, there are solution components with nontrivial polynomial terms, we consider a generalized version of (7.1), namely
\[ Y'(t) = \begin{bmatrix} 1 & 0 \\ 0 & q^p \end{bmatrix} Y(t) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} Y(qt), \quad Y(0) = I, \]
where \( \kappa \in \mathbb{R}\setminus \{0, 1\} \). The solution is

\[
Y(t) = \begin{bmatrix}
e^t & e^{q^t} - q e^{q^t} \\
q(1 - q^{-1}) & 0
\end{bmatrix} e^{q^t}
\]

and it consists of purely exponential terms — even when \( \kappa \) is an integer \( \geq 2 \) and \( \{A, A\} \) ceases to be \( q \)-canonical. Incidentally, letting \( \kappa \to 1 \) and using the l'Hôpital rule reproduces the solution of (7.1).

**Example 8** We seek a solution of

\[
y''(t) - (1 + q) y'(t) + q y(t) = b y(q t), \quad b \in \mathbb{C}\setminus \{0\},
\]

assuming that it is of the form

\[
y(t) = \sum_{r=0}^{\infty} D_r e^{q^r} + t \sum_{r=0}^{\infty} E_r e^{q^r}.
\]

Substitution into (7.2) and comparison of coefficients yield \( E_0 = 0 \) and

\[
q(1 - q^{-1}) (1 - q^r) D_r = b D_{r-1} + (1 + q - 2q^r) E_r, \quad r = 1, 2, \ldots,
\]

and

\[
(1 - q^{-1}) (1 - q^r) E_r = b E_{r-1}, \quad r = 1, 2, \ldots.
\]

We commence by observing that the solution of (7.4) is

\[
E_r = \omega \frac{b^r}{[q]_{r-1} [q]_r}, \quad r = 1, 2, \ldots,
\]

where

\[
\omega = \frac{1 - q}{b} E_1.
\]

Moreover, (7.3) with \( r = 1 \) gives \( \omega = -D_0 \) and

\[
E_r = -D_0 \frac{b^r}{[q]_{r-1} [q]_r}, \quad r = 1, 2, \ldots
\]

Wishing to find two linearly independent solutions, we first let \( D_0 = 0 \). Hence (7.5) implies that \( E_r \equiv 0 \) and (7.3) yields

\[
D_r = \frac{1 - q}{[q]_{r-1} [q]_r} \frac{(b)^{r-1}}{[q]_r} D_1, \quad r = 1, 2, \ldots
\]

We derive the solution

\[
y(t) = (1 - q) y_0 \sum_{r=0}^{\infty} \frac{1}{[q]_r [q]_{r+1}} \frac{(b)^{r-1}}{[q]_r} e^{q^{r+1} t}.
\]

Next, we let \( D_0 = -1 \) and \( D_1 = 0 \). Setting \( D_r^* := b^{-r} D_r, \ r \in \mathbb{Z}^+ \), (7.3) reduces to

\[
D_r^* = \frac{1}{[q]_r} + \frac{1}{q(1 - q^{-1}) [q]_{r-1} [q]_r} + \frac{1}{q(1 - q^{-1})(1 - q^r)} D_{r-1}^*, \quad r = 1, 2, \ldots
\]
On the generalized pantograph equation

This is satisfied by

$$D^*_\ell = \frac{1}{[q]_{\ell-1} [q]_{\ell}} \left\{ \frac{1}{1 - q^\ell} + \frac{2}{q^\ell} \sum_{k=1}^{\ell-1} \frac{q^k}{1 - q^k} - \frac{1}{1 - q^{\ell-1}} \right\}, \quad \ell = 1, 2, \ldots$$

Moreover,

$$\sum_{k=1}^{\ell-1} \frac{q^k}{1 - q^k} = \sum_{k=1}^{\ell-1} q^k \sum_{j=0}^{\infty} q^{k+j+1} = \sum_{j=0}^{\infty} \left( \sum_{k=0}^{\ell-1} q^{k+j+1} - 1 \right)$$

$$= \sum_{j=0}^{\infty} \frac{q^{j+1} - q^{\ell+j+1}}{1 - q^{j+1}} = \sum_{j=1}^{\infty} q^{j-\ell}$$

therefore

$$D^*_\ell = \left\{ \frac{2}{[q]_{\ell-1} [q]_{\ell}} \sum_{j=1}^{\infty} \frac{q^{j-\ell} - q^{\ell}}{1 - q^{j}} \right\} \left( \frac{b}{q} \right)^\ell, \quad \ell = 2, 3, \ldots,$$

and we obtain a second solution, namely

$$y(t) = -e^t \sum_{\ell=2}^{\infty} \left\{ \frac{2}{[q]_{\ell-1} [q]_{\ell}} \sum_{j=1}^{\infty} \frac{q^{j-\ell} - q^{\ell}}{1 - q^{j}} \right\} \left( \frac{b}{q} \right)^\ell e^{\ell t}$$

$$+ \sum_{\ell=1}^{\infty} \frac{1}{[q]_{\ell-1} [q]_{\ell}} b^\ell e^{\ell t}. \quad (7.7)$$

Clearly, the solutions given by (7.6) and (7.7) are linearly independent, and they span the general solution space. Note that (7.6) is a proper Dirichlet series but, of course, it is exceptional: any other choice of initial values necessarily brings in a linear factor.

Example 9 Consider the equation (6.1) with $a(\alpha) = a'(\alpha) = 0$. We seek a solution of the form

$$y(t) = \sum_{\ell=0}^{\infty} D^*_\ell e^{\ell a_1 t} + \sum_{\ell=0}^{\infty} E^*_\ell e^{\ell a_1 t}.$$

Since

$$y^{(k)}(t) = \alpha_1^k \sum_{\ell=0}^{\infty} D^*_\ell q^{\ell k} e^{\ell a_1 t} + k \alpha_1^{k-1} \sum_{\ell=0}^{\infty} E^*_\ell q^{(k-1)\ell} e^{\ell a_1 t} + \alpha_1^k \sum_{\ell=0}^{\infty} E^*_\ell q^{\ell k} e^{\ell a_1 t},$$

$a(\alpha) = a'(\alpha) = 0$ give

$$\mathcal{A} y(t) = \sum_{\ell=1}^{\infty} \left( D^*_\ell a(\alpha) q^\ell + E^*_\ell a'(\alpha) q^\ell \right) e^{\ell a_1 t} + \sum_{\ell=0}^{\infty} E^*_\ell a(\alpha) q^\ell e^{\ell a_1 t},$$

$$\mathcal{B} y(q) = \sum_{\ell=1}^{\infty} \left( D^*_\ell b(\alpha) q^{\ell-1} + E^*_\ell b'(\alpha) q^{\ell-1} \right) e^{\ell a_1 t} + \sum_{\ell=0}^{\infty} E^*_\ell b(\alpha) q^{-\ell} e^{\ell a_1 t}.$$

Substitution into (6.1) and comparison of the coefficients yield the equations

$$a(\alpha) q^\ell E^*_\ell = b(\alpha) q^{\ell-1} E^*_\ell,$$

$$a(\alpha) q^\ell D^*_\ell + a'(\alpha) q^\ell E^*_\ell = b(\alpha) q^{\ell-1} D^*_{\ell-1} + b'(\alpha) q^{\ell-1} E^*_{\ell-1}.$$
for \( \ell = 1, 2, \ldots \). \( D_0 \) and \( E_0 \) are arbitrary. Their solution is

\[
D_\ell = \frac{\prod_{j=1}^{\ell-1} b(\alpha_j, q^j)}{\prod_{j=1}^{\ell-1} a(\alpha_j, q^j)} \left( D_0 + \left( \frac{\sum_{j=0}^{\ell-1} b'(\alpha_j, q^j)}{\sum_{j=0}^{\ell-1} a'(\alpha_j, q^j)} \right) E_0 \right),
\]

\[
E_\ell = \frac{\prod_{j=1}^{\ell-1} b(\alpha_j, q^j)}{\prod_{j=1}^{\ell-1} a(\alpha_j, q^j)} E_0, \quad \ell = 1, 2, \ldots.
\]

The substitution \( D_0 = 1 \), \( E_0 = 0 \) produces the standard solution \( y_1 \), as given by (6.6), whereas \( D_0 = 0 \), \( E_0 = 1 \) yields a second, linearly independent solution, namely

\[
\tilde{y}_1(t) = \sum_{\ell=1}^{\infty} \left( \frac{\sum_{j=0}^{\ell-1} b'(\alpha_j, q^j)}{\sum_{j=0}^{\ell-1} a'(\alpha_j, q^j)} \right) e^{\sigma_1 t} + \gamma_1(t).
\]

The pattern is similar to the previous examples.

We complete this section by an analysis of the GPE

\[
Y'(t) = AY(t) + BY(qt), \quad Y(0) = I,
\]

where \( d = 2 \) and \( \sigma(A) = \{\lambda, q\lambda\}, \lambda \neq 0 \). To simplify matters, we note that, without loss of generality, we can let \( \lambda = 1 \), otherwise we need just to rescale \( t \). Moreover, having distinct eigenvalues, \( A \) is diagonalizable. Let \( U \) be the matrix of eigenvectors of \( A \). Rescaling \( Y \) by \( U^{-1} \) we may assume without loss of generality that

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix}.
\]

Of course, the matrix \( B \) is replaced by \( U^{-1} B U \) and the initial condition is \( Y(0) = U^{-1} \), but this causes no difficulties.

We seek a solution of the form

\[
Y(t) = \left\{ \sum_{\ell=0}^{\infty} D_\ell e^{\sigma_1 t} + t \sum_{\ell=0}^{\infty} E_\ell e^{\sigma_1 t} \right\} V.
\]

Substitution into (7.8) yields

\[
Y'(t) V^{-1} = \sum_{\ell=0}^{\infty} q^\ell D_\ell e^{\sigma_1 t} + t \sum_{\ell=0}^{\infty} E_\ell e^{\sigma_1 t} + \sum_{\ell=0}^{\infty} q^\ell E_\ell e^{\sigma_1 t}
\]

\[
= \begin{bmatrix} 1 & 0 \\ 0 & q \end{bmatrix} \left\{ \sum_{\ell=0}^{\infty} D_\ell e^{\sigma_1 t} + t \sum_{\ell=0}^{\infty} E_\ell e^{\sigma_1 t} \right\}
\]

\[
+ B \left\{ \sum_{\ell=1}^{\infty} e^{\sigma_1 t} + qt \sum_{\ell=1}^{\infty} E_{\ell-1} e^{\sigma_1 t} \right\}.
\]

We compare the coefficients. When \( \ell = 0 \) we have

\[
\begin{bmatrix} 0 & 0 \\ 0 & 1-q \end{bmatrix} D_0 = -E_0, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1-q \end{bmatrix} E_0 = O,
\]

consequently

\[
D_0 = \begin{bmatrix} u & v \\ 0 & 0 \end{bmatrix}, \quad E_0 = O,
\]
where \( u, v \in \mathbb{C} \) are arbitrary constants. \( \ell = 1 \) leads to
\[
\begin{bmatrix}
1 - q & 0 \\
0 & 0
\end{bmatrix} D_1 = E_1 - BD_0,
\begin{bmatrix}
1 - q & 0 \\
0 & 0
\end{bmatrix} E_1 = O,
\]
therefore
\[
D_1 = \begin{bmatrix}
-\frac{u}{1 - q} B_{1,1} & -\frac{v}{1 - q} B_{1,1} \\
\frac{w}{x} & \frac{z}{x}
\end{bmatrix}, \quad E_1 = B_{2,1} \begin{bmatrix} 0 & 0 \\ u & v \end{bmatrix},
\]
where \( w, x \in \mathbb{C} \) are, again, arbitrary. Finally, for \( \ell \geq 2 \) we obtain the nonsingular equations
\[
\begin{bmatrix}
1 - q^\ell & 0 \\
0 & q - q^\ell
\end{bmatrix} D_\ell = E_\ell - BD_{\ell - 1},
\begin{bmatrix}
1 - q^\ell & 0 \\
0 & q - q^\ell
\end{bmatrix} E_\ell = -qB E_{\ell - 1}.
\]

**Theorem 19** The solution of the GPE (7.8) can be formally expanded into the generalized Dirichlet series (7.9).

In principle, stability analysis of (7.8) can be based on the expansion (7.9), similar to the work of §4. Of course, the presence of a linear term makes stability less likely, although not impossible. No such analysis has been attempted to date.

### 8 The general pantograph equation of the second kind

In this section we explore the GPE of the second kind,
\[
Y''(t) = AY(t) + Y(qt) B, \quad Y(0) = I.
\]
(8.1)

Here \( Y, A \) and \( B \) are complex \( d_1 \times d_2, d_1 \times d_1 \) and \( d_2 \times d_2 \) matrices, respectively. Note that, in departure from (1.1), we require \( Y(0) = I - \) this, of course, entails no loss of generality.

Another remark is that the neutral term \( Y'(qt) \) is missing altogether, mainly for the sake of simplicity.\(^8\) Surprisingly enough, (8.1) is considerably simpler than (1.1), although it yields itself to similar analysis.

The initial-value problem is always well-posed. Let \( Y(t) = \sum_{k=0}^{\infty} \frac{1}{k!} Y_k t^k \). Substitution into (8.1) yields
\[
Y_{k+1} = AY_k + q^k Y_k B, \quad k \in \mathbb{Z}^+.
\]
Therefore all Taylor coefficients are well-defined and it is easy to verify that, as long as \( A \neq O \),
\[
\| Y(t) \| \leq \sum_{k=0}^{\infty} \left( \frac{\| B \|}{\| A \|} \right)^k (q \| t \|)^k, \quad t \geq 0,
\]
\(^8\) In general, there are eight different variations on the GPE (1.1), because we can act on each of the terms \( Y(t), Y(qt) \) and \( Y'(qt) \) by multiplying it with a matrix from the left or from the right, although this can be halved by transposition.
whereas $A = O$ yields

$$
\| Y(t) \| \leq \sum_{k=0}^{\infty} \frac{q^{|k-1|} t^k}{k!} \| B \|^k, \quad t \geq 0.
$$

**Theorem 20** Subject to $\det A \neq 0$, the formal Dirichlet series expansion

$$
Y(t) = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} e^{t\Lambda}}{[q]_{\ell}} A^{-\ell} V B', \quad t \geq 0,
$$

(8.2)

where $V$ is an arbitrary $d_1 \times d_2$ matrix, obeys the equation $Y'(i) = A Y(i) + Y(qi) B$. Moreover, if both $A$ and $B$ are diagonalizable, $\sigma(A) = (\lambda_1^{(A)}, \lambda_2^{(A)}, \ldots, \lambda_{d_1}^{(A)})$, $\sigma(B) = (\lambda_1^{(B)}, \lambda_2^{(B)}, \ldots, \lambda_{d_1}^{(B)})$ and

$$
\lambda_*^{(B)} := \max_{j=1, 2, \ldots, d_1} |\lambda_j^{(B)}|, \quad \lambda_*^{(A)} := \min_{j=1, 2, \ldots, d_1} |\lambda_j^{(A)}|
$$

(8.3)

then the series (8.2) converges and there exists a matrix $V$ such that $Y(0) = I$.

**Proof** Substituting (8.2) into the equation, we obtain

$$
AY(i) + Y(qi) B = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} e^{t\Lambda}}{[q]_{\ell}} A^{-\ell+1} V B' - \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell} e^{t\Lambda}}{[q]_{\ell-1}} A^{-\ell+1} V B'
$$

$$
= \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{[q]_{\ell}} e^{t\Lambda} A^{-\ell+1} V B' = Y'(qt).
$$

Subject to (8.3), there exists $c > 0$ such that

$$
\| A^{-\ell} V B' \| \leq c \left( \frac{\lambda_*^{(B)}}{\lambda_*^{(A)}} \right)^\ell, \quad \ell \in \mathbb{Z}^+,
$$

hence the series (8.2) converges. Finally, let

$$
A = U_A D_A U_A^{-1}, \quad B = U_B D_B U_B^{-1},
$$

where $D_A = \text{diag}(\lambda_1^{(A)}, \lambda_2^{(A)}, \ldots, \lambda_{d_1}^{(A)})$, $D_B = \text{diag}(\lambda_1^{(B)}, \lambda_2^{(B)}, \ldots, \lambda_{d_1}^{(B)})$, and set

$$
W := U_A^{-1} U_B, \quad \tilde{V} := U_A^{-1} V U_B.
$$

The satisfaction of $Y(0) = I$ by (8.2) is equivalent to

$$
\sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{[q]_{\ell}} D^{\ell}_A \tilde{V} D^{\ell}_B = W.
$$

Componentwise, the last identity is

$$
\sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{[q]_{\ell}} \left( \frac{\lambda_*^{(B)}}{\lambda_*^{(A)}} \right)^\ell \tilde{V}_{k,j} = W_{k,j}, \quad k = 1, 2, \ldots, d_1, j = 1, 2, \ldots, d_2.
$$

Thus, summing up the basic hypergeometric $_0\Phi_0$ series,

$$
\tilde{V}_{k,j} = \left( \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{[q]_{\ell}} \left( \frac{\lambda_j^{(B)}}{\lambda_j^{(A)}} \right)^\ell \right) W_{k,j} = \left( \frac{\lambda_j^{(B)}}{\lambda_j^{(A)}} ; q \right)_\infty W_{k,j}
$$
and we have (after trivial manipulation) an explicit expression of a matrix $V$ such that $Y(0) = I$.

Note that the condition (8.3) (and, indeed, the diagonalizability of $A$ and $B$) are not necessary if the expansion (8.2) terminates. This happens when $A^{d_i} V B^i = O$ for all $i \geq L$, say. $A$ being nonsingular, it follows that $V B^i = O$, $i \geq L$. This happens if and only if rank $B \leq d_i - 1$, and all the $d_i$ rows of $V$ are left-eigenvectors of $B$ (that need not be distinct) that correspond to zero eigenvalues. This usually rules out $Y(0) = I$.

Stability analysis presents no great difficulties and, acting on (8.2) in the spirit of the proof of Theorem 5, we can prove that, as long as all the conditions of Theorem 20 are satisfied, $\lim_{t \to \infty} Y(t) = O$ when $\text{Re} \lambda_k^{(A)} < 0$, $k = 1, 2, \ldots, d_i$.

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