Damping Mechanisms for Regularized Transformation-Acoustics Cloaking

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Dedicated to Professor G. Uhlmann on the occasion of his 60th birthday

Abstract

The regularized transformation-acoustics invisibility cloaking via the ‘blow-up a-small-ball’ construction is considered. In order to defeat the cloak-busting inclusions, a lossy layer is incorporated into the construction. Two mechanisms are respectively introduced in [15] and [21]. We generalize the two mechanisms and evaluate the corresponding cloaking performances. The exterior boundary conditions on the lossy layers are assessed and they are compared to those for the sound-soft and sound-hard layers. Moreover, the cloaking of active contents is assessed and the results show that in order to cloak a source/sink, the lossy layer must be properly chosen.

1 Introduction

This paper is concerned with the invisibility cloaking for acoustic waves via the approach of transformation optics (cf. [12,13,17,25]). There are many theoretical and experimental developments in this field and we refer to [5, 10, 11, 24, 27, 28] for the state-of-the-art surveys.

The method of transformation acoustics makes use of the transformation properties of the acoustical parameters, namely the density tensor and the acoustic modulus. The invisibility construction is based on a blow-up transformation between the virtual space and the physical space. For the ideal/perfect invisibility cloaking considered in [17, 25], it is a singular ‘blow-up-a-point’ transformation. The cloaking media achieved in this way possess singular structures. The singularity poses much challenge to both theoretical analysis and practical construction. In order to avoid the singular structure, it is natural to introduce regularization into the construction. Instead of the perfect cloak, one considers the approximate cloak or near cloak. Various regularized near-cloaking

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schemes have been proposed for the acoustic cloaking. In [7, 8, 26], the regularization relies on truncation of singularities, whereas in [15, 16, 19] the ‘blow-up-a-point’ transformation in [13, 17, 25] is regularized to be the ‘blow-up-a-small-region’ transformation. However, it is shown in [14] that the two schemes are equivalent to each other. Hence, in the present study, we shall only consider the ‘blow-up-a-small-region’ construction. Particularly, it is shown in [15] that if one intends to nearly cloak an arbitrary content, it is necessary to incorporate a certain mechanism to defeat the cloak-busting inclusions. It is proposed in [15] to incorporate a lossy layer right between the cloaked region and the cloaking region, and the lossy layer possesses a large loss parameter in the virtual space, which shall be referred to as a \textit{high-loss scheme} in the present paper. In [18, 21], the authors proposed to employ a lossy layer with a high density tensor instead, and this shall be referred to as a \textit{high-density scheme} in the present paper. The high-density construction is shown to produce significantly enhanced accuracy of approximation compared to that of the high-loss construction. Special mechanisms of different nature are also developed in [2–4, 19], where a sound-hard or sound-soft layer is employed to isolate the cloaked region from the exterior space and they produce a certain homogeneous boundary condition on the interior interface of the cloaking layer. It is heuristically pointed out in [15, 21] that the sound-soft (res. sound-hard) layer is an ideal state corresponding to the limiting case of a high-lossy (res. high-density) layer. In the present article, we shall give a more rigorous and quantitative study in this aspect. Before we proceed to discuss more about our results, we briefly introduce the mathematical formulation of the acoustic cloaking problem under our consideration.

Let \( \Sigma \) be a bounded Lipschitz domain in \( \mathbb{R}^N \), \( N = 2, 3 \). For a regular acoustic medium supported in \( \Sigma \), we denote it by \((\Sigma; g, q)\), where \( g = (g^{ij}) \) is a symmetric-matrix valued function and \( q \) is a scalar function, and they satisfy

\[
\lambda |\xi|^2 \leq g(x) \xi \cdot \xi \leq \lambda^{-1} |\xi|^2, \quad \Re q(x) \geq \lambda, \quad \Im q(x) \geq 0 \quad \text{for a.e. } x \in \Sigma, \tag{1.1}
\]

where \( 0 < \lambda < 1 \) is a constant, \( \Re \) and \( \Im \) denote taking the real and imaginary parts, respectively. The time-harmonic acoustic wave propagation in \((\Sigma; g, q)\) is governed by the following Helmholtz equation

\[
\begin{align*}
\nabla \cdot (g \nabla u) + \omega^2 qu &= f \quad \text{in } \Sigma, \\
\sum_{i,j=1}^{N} \nu_i g^{ij} \partial_j u &= \psi \in H^{-1/2}(\partial \Sigma) \quad \text{on } \partial \Sigma, \tag{1.2}
\end{align*}
\]

where \( \nu = (\nu_i)_{i=1}^{N} \) is the outward unit normal to \( \Sigma \), \( \omega \in \mathbb{R}_+ \) is called the wave number, \( f \in H^{-1}(\Sigma) \) denotes a source/sink term inside \( \Sigma \), and \( u \in H^1(\Sigma) \) denotes the wave pressure (cf. [6, 23]). Associated with (1.2), we introduce the Neumann-to-Dirichlet (NtD) operator \( \Lambda \) as follows

\[
\Lambda(\psi) = u|_{\partial \Sigma} : H^{-1/2}(\Sigma) \to H^{1/2}(\Sigma), \tag{1.3}
\]

where \( u \) is the solution to (1.2). It is always assumed that \( \omega^2 \) is not an eigenvalue so that we have a well-defined boundary operator \( \Lambda \). Here, we remark that \( \Lambda \) is a linear
operator if \( f = 0 \) in (1.2), and generally nonlinear if \( f \neq 0 \). Next, we let \( \tilde{\Sigma} \) be a bounded Lipschitz domain as well and it is assumed that there exists a bi-Lipschitz transformation \( F \) which maps \( \Sigma \) to \( \tilde{\Sigma} \). Let \( (\tilde{\Sigma}; \tilde{g}, \tilde{q}) \) be a regular medium defined to be the push-forward of \((\Sigma; g, q)\) according to the following formulae

\[
\begin{align*}
\tilde{g} &= F_* g := [(\det(DF))^{-1} (DF \cdot g \cdot DF^T)] \circ F^{-1}, \\
\tilde{q} &= F_* q := [(\det(DF))^{-1} q] \circ F^{-1}, \\
\tilde{f} &= F_* f := [(\det(DF))^{-1} f] \circ F^{-1},
\end{align*}
\] (1.4)

where \( DF \) denotes the Jacobian matrix of \( F \). In the following, we shall write \((\tilde{\Sigma}; \tilde{g}, \tilde{q}, \tilde{f}) := F_*(\Sigma; g, q, f)\) to denote the push-forward introduced above. Then, the following result is in order (see [9, 10, 15]). Let \( F : \Sigma \to \Sigma \) be a bi-Lipschitz mapping such that \( F|_{\partial \Sigma} = \text{Identity} \), and set \((\Sigma; \tilde{g}, \tilde{q}, \tilde{f}) = F_*(\Sigma; g, q, f)\). Then \( u \in H^1(\Sigma) \) satisfies (1.2) iff \( \tilde{u} \in H^1(\Sigma) \) also satisfies (1.2) but with \((g, q, f)\) replaced by \((\tilde{g}, \tilde{q}, \tilde{f})\). Moreover, one has that \( \Lambda = \tilde{\Lambda} \), where \( \tilde{\Lambda} \) is the NtD operator associated with \((\Sigma; \tilde{g}, \tilde{q}, \tilde{f})\).

Now, let \( \Omega \) and \( D \) be two simply connected Lipschitz domains in \( \mathbb{R}^N, N = 2, 3 \). It is assumed that \( D \) is convex containing the origin and \( D \supseteq \Sigma \). Let \( D_\rho := \{ \rho x; x \in D \} \) for any \( \rho \in \mathbb{R}_+ \). We assume that there exists a bi-Lipschitz mapping \( F^{(1)}_\rho \) such that

\[
F^{(1)}_\rho : \Omega \setminus D_\rho \to \Omega \setminus D, \quad F^{(1)}_\rho|_{\partial \Omega} = \text{Identity}.
\] (1.5)

Let \( F^{(2)}_\rho = x/\rho : D_\rho \to D \) and

\[
F_\rho(x) := \begin{cases} 
F^{(1)}_\rho(x), & x \in \Omega \setminus D_\rho, \\
F^{(2)}_\rho(x), & x \in D_\rho.
\end{cases}
\] (1.6)

Set

\[
(\Omega \setminus \overline{D}; g, q) = (\Omega \setminus \overline{D}; g_\alpha, q_\alpha) := (F_\rho)_* (\Omega \setminus \overline{D}_\rho; I, 1),
\] (1.7)

and

\[
(D \setminus \overline{D}_{1/2}; g, q) = (D \setminus \overline{D}_{1/2}; g_t, q_t) := (F_\rho)_* (D_\rho \setminus D_{\rho/2}; g'_t, q'_t),
\] (1.8)

with

\[
g'_t = \gamma \rho^\alpha t, \quad q'_t = \alpha + i \beta \rho^\gamma, \quad s, t \in \mathbb{R},
\] (1.9)

where \( \alpha, \beta \) and \( \gamma \) are positive constants independent of \( \rho \). Finally, we also let \( (D_{1/2}; g, q) = (D_{1/2}; g_a, q_a) \) be an arbitrary but regular medium supported inside \( D_{1/2} \), and \( f_a \in H^{-1}(D_{1/2}) \) be a source/sink term located inside \( D_{1/2} \). \((\Omega; g, q, f_a)\) describes a cloaking device with the cloaking region \( (\Omega \setminus \overline{D}; g_\alpha, q_\alpha) \), the cloaked region \((D_{1/2}; g_a, q_a, f_a)\), and the lossy layer \((D \setminus \overline{D}_{1/2}; g_t, q_t)\) right between the cloaking and cloaked regions. For the source-free case, namely \( f_a = 0 \), the above cloaking construction has been investigated in [15] with \((s, t) = (0, -2)\) in (1.9), and [18, 21] with \((s, t) = (2, 0)\) in (1.9), respectively. In order to present a quick discussion on the results obtained therein, we introduce the ‘free-space’ Helmholtz equation

\[
\begin{cases} 
\Delta u_0 + \omega^2 u_0 = 0 & \text{in } \Omega, \\
\frac{\partial u_0}{\partial \nu} = \psi & \text{in } H^{-1/2}(\partial \Omega) \quad \text{on } \partial \Omega.
\end{cases}
\] (1.10)
Clearly, in order to assess the cloaking performance, it suffices to evaluate

Moreover, and

\[ u \]

and in [18,21] with \((s,t) = (2,0)\) in (1.9) that

\[
\| \Lambda_\rho - \Lambda_0 \|_{L(H^{-1/2}(\partial D),H^{1/2}(\partial D))} \leq C \rho^N. \tag{1.12}
\]

In both (1.11) and (1.12), the estimates are shown to be optimal and the constant \(C\) is shown to be independent of \((g_a,q_a)\).

In this work, we shall consider the above two constructions with general \((s,0)\) and \((0,t)\) in (1.9). As mentioned earlier, they shall be referred to as the high-density scheme and the high-loss scheme, respectively. Also, as discussed earlier, we are particularly interested in assessing the wave field \(u\) on the exterior boundary of the lossy layer, namely \(\partial D\), and compare with those for the sound-soft and sound-hard layers. For a sound-soft layer, the wave pressure vanishes on \(\partial D\) and one would have a homogeneous Dirichlet boundary condition, and for a sound-hard layer, the wave velocity vanishes on \(\partial D\) and one would have a homogeneous Neumann boundary condition. The results obtained would clarify some heuristic beliefs in the literature that a high-lossy layer is a finite-realization of a sound-soft layer and a high-density layer is a finite-realization of a sound-hard layer. Moreover we shall consider a delta-point source, \(f_a = \eta_0 \delta_0\), located inside the cloak region, and the obtained results show that in order to cloak active contents, an appropriate lossy layer must be chosen. Furthermore, the results shall clarify some points in [14] on the interior active cloaking and in [20] on the general study of cloaking of active contents.

In the rest of the paper, we shall consider our study by taking \(\Omega = B_R\) and \(D = B_1\), where \(R > 1\) and \(B_s\) denotes a central ball of radius \(s \in \mathbb{R}_+\). Moreover, we shall let \(g_a\) be a positive constant multiple of the identity matrix and \(q_a\) be a positive constant. With no confusion in the context, we shall just take \(g_a\) as a positive constant. We set

\[
(\Omega; g', q', f'_a) = (F_{\rho}^{-1})^*(\Omega; g, q, f_a), \tag{1.13}
\]

and \(u_\rho = (F_{\rho})^* u\). Clearly, we have \((B_R \setminus B_\rho; g', q') = (B_R \setminus B_\rho; I, 1), (B_{\rho/2}; g', q') = (B_{\rho/2}; g'_a, q'_a)\) and, \((B_{\rho/2}; g', q') = (B_{\rho/2}; g'_a, q'_a)\) with \(g'_a = \rho^{-2} g_a\) and \(q'_a = \rho^{-N} q_a\). Moreover,

\[
\begin{align*}
\nabla \cdot (g' \nabla u_\rho) + \omega^2 q' u_\rho &= f'_a \quad \text{in } B_R, \\
\sum_{i,j=1}^{N} \nu_i (g')^{ij} \partial_j u_\rho &= \psi \quad \text{on } \partial B_R.
\end{align*}
\tag{1.14}
\]

Clearly, in order to assess the cloaking performance, it suffices to evaluate \(\| u_\rho - u_0 \|_{H^{1/2}(\partial B_R)}\).

For the convenience of the following discussion, we recall that it holds in \(\mathbb{R}^2\) (cf. [21])

\[
\psi(x) = \sum_{n=-\infty}^{\infty} \psi_n(R) e^{in\theta} \in H^{-1/2}(\partial B_R),
\]

\[
\psi_n(R) = \sigma_n(R) e^{n\theta}.
\]

\[
\psi_n(R) = \int_{\partial B_R} \psi_n(R) e^{-in\theta} dS.
\]

\[
\psi_n(R) = \int_{\partial B_R} \psi_n(R) e^{-in\theta} dS.
\]

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\]
we have (see, e.g. [21])
\[ \| \psi \|^2_{H^{-1/2}(\partial B_R)} = \sum_{n=-\infty}^{\infty} (1 + n^2/R^2)^{-1/2} |\psi_n \sqrt{2\pi R}|^2. \]  
(1.15)

Then the ‘free-space’ solution \( u_0(x) \in H^1(B_R) \) of (1.10) is given by
\[ u_0(x) = \sum_{n=-\infty}^{\infty} \frac{\psi_n J_n(\omega|x|)}{\omega J'_n(\omega R)} e^{in\theta}, \quad x \in B_R. \]  
(1.16)

2 The high-density scheme

We shall first consider the high-density construction, namely \((B_R; g, q, f_a)\) with \((s, 0)\) in (1.9), \(s \in \mathbb{R}\). In the sequel, \((B_R; g', q', f'_a)\) shall be referred to as the virtual image of \((B_R; g, q, f_a)\) and (1.14) shall be referred to as the Helmholtz equation in the virtual space.

2.1 The case with \( g'_l = \gamma \rho^{-2}\delta, \quad 0 \leq \delta < 2, \quad q'_l = (\alpha + i\beta) \) and \( f_a = 0 \)

We first consider the 2D case with \( \delta = 0 \). Let \( \omega_a = \omega \sqrt{g'_l/g'_s} = \omega \sqrt{g_a/g_a \rho^{-1}} \) and \( \omega_l = \omega \sqrt{q'_l/q'_s} = \omega \sqrt{(\alpha + i\beta)/\gamma \rho^{-1}} \), where we shall choose the branch of \( \sqrt{(\alpha + i\beta)/\gamma} \) with \( 3\sqrt{(\alpha + i\beta)/\gamma} > 0 \) throughout the rest of the paper. We see that \( \omega \rho, \omega_a \rho \) are constants independent of \( \rho \).

Proposition 2.1. In \( \mathbb{R}^2 \), when \( g'_l = \gamma \rho^{-2}\delta, \quad q'_l = (\alpha + i\beta) \) and \( f_a = 0 \), we have
\[ \begin{align*}
& \left\| u_\rho - u_0 \right\|_{H^{1/2}(\partial B_R)} \leq C \rho^2 \left\| \psi \right\|_{H^{-1/2}(\partial B_R)} , \\
& \left\| \frac{\partial u_R^+(\rho)}{\partial \nu} \right\|_{H^{-1/2}(\partial B_1)} \leq C \rho \left\| \psi \right\|_{H^{-1/2}(\partial B_R)} .
\end{align*} \]  
(2.1)

Remark 2.2. This means the near-cloak in this case is within \( \rho^2 \)-approximation of the ideal cloak. Moreover, we see that the lossy layer with \( (s, t) = (2, 0) \) is an approximation to a sound-hard layer.

Proof. The proposition can be proved in a similar manner to that given in [21]. However, for our subsequent study, we shall present the proof in some detail. Assume the solution to the virtual problem (1.14) is given by
\[ u_\rho(x) = \begin{cases}
\sum_{n=-\infty}^{\infty} e_n J_n(\omega_n |x|) e^{in\theta}, & x \in B_{\rho/2}, \\
\sum_{n=-\infty}^{\infty} e_n J_n(\omega_n |x|) e^{in\theta} + \sum_{n=-\infty}^{\infty} d_n H^{(1)}_n(\omega_n |x|) e^{in\theta}, & x \in B_{\rho} \setminus \overline{B}_{\rho/2}, \\
\sum_{n=-\infty}^{\infty} a_n J_n(\omega_n |x|) e^{in\theta} + \sum_{n=-\infty}^{\infty} b_n H^{(1)}_n(\omega_n |x|) e^{in\theta}, & x \in B_R \setminus \overline{B}_{\rho} ,
\end{cases} \]  
(2.2)
Set \( u_a = u_\rho |_{B_{2/3}} \), \( u_l = u_\rho |_{B_{2/3}} \), and \( u_R = u_\rho |_{B_R} \). By the standard transmission conditions on \( \partial B_{2/3}, \partial B_2 \) and the boundary condition on \( \partial B_R \), we have
\[
\begin{align*}
\begin{cases}
  u_a(x) = u_l(x), & g_a \frac{\partial u_a(x)}{\partial \nu(x)} = g_l \frac{\partial u_l(x)}{\partial \nu(x)}, & x \in \partial B_{2/3}, \\
u_l(x) = u_R(x), & g_l \frac{\partial u_l(x)}{\partial \nu(x)} = \frac{\partial u_R(x)}{\partial \nu(x)}, & x \in \partial B_2, \\
u_R(x) = \psi(x), & x \in \partial B_R.
\end{cases}
\end{align*}
\] (2.3)

Plugging the series representations (2.2) into (2.3), we have the following linear system of equations for the coefficients,
\[
\begin{align*}
&\begin{cases}
  c_n J_n(\omega a \rho/2) = c_n J_n(\omega l \rho/2) + d_n H_n^{(1)}(\omega l \rho/2), \\
\sqrt{g_a g_l} J_n'(\omega a \rho/2) = \sqrt{g_a g_l} J_n'(\omega l \rho/2) + d_n H_n^{(1)'}(\omega l \rho/2), \\
  c_n J_n(\omega a \rho) + d_n H_n^{(1)}(\omega a \rho) = a_n J_n(\omega l \rho) + b_n H_n^{(1)}(\omega l \rho), \\
\sqrt{g_a g_l} [c_n J_n'(\omega a \rho) + d_n H_n^{(1)'}(\omega a \rho)] = a_n J_n'(\omega l \rho) + b_n H_n^{(1)'}(\omega l \rho), \\
  a_n \omega J_n'(\omega a \rho) + b_n \omega H_n^{(1)'}(\omega a \rho) = \psi_n.
\end{cases}
\end{align*}
\] (2.4)

Let \( A = \sqrt{\frac{g_a g_l}{g_a g_l}} = \sqrt{\frac{g_a g_l}{\gamma (\omega + \omega l)}} \rho^{-2} \) and notice that \( \omega a \rho \) and \( \omega l \rho \) are all constants. From the first two equations of (2.4) we have
\[
\begin{align*}
&\begin{cases}
  d_n = -\frac{J_n(\omega a \rho/2)}{H_n^{(1)'}(\omega a \rho/2)} c_n \quad \text{if} \quad J_n(\omega a \rho/2) = 0, \\
d_n = -\frac{2J_n(\omega a \rho/2) - AJ_n(\omega l \rho/2) J_n^{(1)'}(\omega a \rho/2)}{J_n^{(1)'}(\omega l \rho/2) - AH_n^{(1)'}(\omega l \rho/2)} c_n \quad \text{if} \quad J_n(\omega a \rho/2) \neq 0.
\end{cases}
\end{align*}
\] (2.5)

If \( \Im q_n' \neq 0 \), we have \( J_n(\omega a \rho/2) \neq 0 \). Denoting the expressions before \( c_n \) in (2.5) by \( \Upsilon_n \), namely \( d_n := \Upsilon_n c_n \). So while \( \rho \to +0 \), we have
\[
\begin{align*}
&\begin{cases}
  \Upsilon_n = -\frac{J_n(\omega a \rho/2)}{H_n^{(1)'}(\omega a \rho/2)} \quad \text{if} \quad J_n(\omega a \rho/2) = 0, \\
\Upsilon_n \sim -\frac{\Upsilon_n(\omega a \rho/2)}{H_n^{(1)'}(\omega a \rho/2)} \quad \text{if} \quad J_n(\omega a \rho/2) \neq 0.
\end{cases}
\end{align*}
\] (2.6)

Substituting \( d_n \) into the third and fourth equations of (2.4), we have by straightforward calculations
\[
\begin{align*}
b_n = -\frac{\sqrt{g_a g_l} J_n'(\omega a \rho) + \Upsilon_n H_n^{(1)'}(\omega a \rho)}{J_n(\omega a \rho) + \Upsilon_n H_n^{(1)}(\omega a \rho)} a_n, \\
b_n = -\frac{\sqrt{g_a g_l} J_n'(\omega l \rho) + \Upsilon_n H_n^{(1)'}(\omega l \rho)}{J_n(\omega l \rho) + \Upsilon_n H_n^{(1)}(\omega l \rho)} a_n.
\end{align*}
\] (2.7)

Let \( \Gamma_n \) denote the expression before \( a_n \) in (2.7), namely \( b_n := \Gamma_n a_n \). Then we have
\[
\begin{align*}
\mathcal{H}_n(h) &= \frac{J_n'(\omega a \rho) + \Upsilon_n H_n^{(1)'}(\omega a \rho)}{J_n(\omega a \rho) + \Upsilon_n H_n^{(1)}(\omega a \rho)}, \\
\mathcal{H}_n(h) &= \frac{J_n'(\omega l \rho) + \Upsilon_n H_n^{(1)'}(\omega l \rho)}{J_n(\omega l \rho) + \Upsilon_n H_n^{(1)}(\omega l \rho)}.
\end{align*}
\] (2.8)
As $\rho \to +0$,
\[
\mathcal{H}_n(\rho) \to \mathcal{H}_n^0 = \frac{J_n(\omega \rho) + \Gamma_n H_n^{(1)'}(\omega \rho)}{J_n(\omega \rho) + \Gamma_n H_n^{(1)'}(\omega \rho)},
\]
where $\Gamma_n = -\frac{J_n(\omega \rho) / \omega}{H_n^{(1)'}(\omega \rho)}$. Plugging (2.7) into the last equation in (2.4), we have
\[
u_R(x) = \sum_{n=-\infty}^{\infty} \frac{\psi_n J_n(\omega R) + \Gamma_n H_n^{(1)}(\omega R)}{\omega J_n(\omega R)} e^{i\theta}, \quad x \in \partial B_R.
\]
Hence,
\[
[u_R(x) - u_0(x)]|_{\partial B_R} = \sum_{n=-\infty}^{\infty} \frac{\psi_n J_n(\omega R)}{\omega J_n(\omega R)} \left[ \Gamma_n \frac{H_n^{(1)}(\omega R) - H_n^{(1)'}(\omega R)}{J_n(\omega R)} \right] e^{i\theta}. \tag{2.10}
\]
Therefore we have
\[
\|u_R - u_0\|_{L^{1/2}(\partial B_R)}^2 = \sum_{n=-\infty}^{\infty} \left( 1 + \frac{n^2}{R^2} \right)^{1/2} \left| \frac{\psi_n J_n(\omega R)}{\omega J_n(\omega R)} \right|^2 \|\tilde{h}_n\|^2_{H^{-1/2}(\partial B_R)} \tag{2.11}
\]
where
\[
\tilde{h}_n := \left| \frac{\Gamma_n \left[ H_n^{(1)}(\omega R) - H_n^{(1)'}(\omega R) \right]}{1 + \Gamma_n H_n^{(1)'}(\omega R)} \right|.
\]
Since $H_n^{(1)'}(\omega r) = (-1)^n H_n^{(1)}(\omega r)$, we only need consider $n \geq 0$ in estimating the series in the last inequality of (2.11). By the fact that $\mathcal{T}_n, \mathcal{H}_n$ are all constants which together with the asymptotic behaviors of $J_n(\omega \rho), H_n^{(1)}(\omega \rho), J_n'(\omega \rho), H_n^{(1)'}(\omega \rho)$ as $\rho \to +0$ (cf. [1, 22]), one can further show
\[
\begin{align*}
\Gamma_0 &\sim -i(\sqrt{\gamma(\alpha + i\beta)} H_n^0 + \omega^2 / 2)^{\frac{n}{2}}, \quad n = 0, \\
\Gamma_n &\sim \pi i (\frac{\omega \rho}{2\pi n})^{2n}, \quad n \geq 1.
\end{align*} \tag{2.12}
\]
Then using the estimates in (2.12), together with the use of the asymptotic developments of the Bessel and Hankel functions for large $n$ (cf. [1]), one can verify that there exists
a sufficiently large integer $N_1$ such that

$$
\begin{cases}
\tilde{h}_0 \sim -i(\sqrt{\gamma(\alpha + i\beta)}H_0^0 + \omega/2)^{\pi \omega^2 n} \frac{H_n^{(1)}(\omega R)}{J_n(\omega R)}, \quad n = 0, \\
\tilde{h}_n \sim \pi i(\omega^2 n) \frac{H_n^{(1)}(\omega R)}{J_n(\omega R)}, \quad 1 \leq n \leq N_1, \\
\tilde{h}_n \sim 2(\frac{\rho}{R})^{2n}, \quad n > N_1.
\end{cases}
$$

(2.13)

Hence from (2.13), we readily see that there exists a constant $C_1$ independent of $\rho$ for $\rho$ sufficiently small such that

$$|	ilde{h}_n| \leq C_1 \rho^2, \quad n \leq N_1,$$

(2.14)

and for $n > N_1$

$$|	ilde{h}_n| \leq 2 \left(\frac{\rho}{R}\right)^{2n}.$$

(2.15)

Here it is emphasized that due to the asymptotic developments of $\tilde{h}_0$ and $\tilde{h}_1$, (2.14) is the best estimate one could achieve, namely $C_1 \rho^2$ could not be improved. Now, using (2.14), we see that

$$\sum_{n=0}^{N_1} \left(1 + \frac{n^2}{R^2}\right) \left|\frac{J_n(\omega R)}{\omega J_n^{(1)}(\omega R)} \tilde{h}_n\right|^2 \leq C_2 \rho^4. $$

(2.16)

Let $N_1$ be sufficiently large such that $|\frac{J_n(\omega R)}{\omega J_n^{(1)}(\omega R)}| < 1$ for $n > N_1$, then for $\rho < \min\{R/4, 1\}$

$$\sum_{n>N_1} \left(1 + \frac{n^2}{R^2}\right) \left|\frac{J_n(\omega R)}{\omega J_n^{(1)}(\omega R)} \tilde{h}_n\right|^2 \leq \frac{\rho^4}{R^4} \sum_{n>N_1} \left(1 + \frac{n^2}{R^2}\right) \left|2 \left(\frac{\rho}{R}\right)^{2(n-1)}\right|^2 < C_3 \rho^4. $$

(2.17)

Combining (2.11), (2.16) and (2.17), we have

$$\|u_\rho - u_0\|_{H^{1/2}(\partial B_R)} \leq C \rho^2 \|\psi\|_{H^{-1/2}(\partial B_R)}.$$  

(2.18)

Moreover, from the optimality of the estimate (2.14), we readily see the sharpness of (2.18). For the normal derivatives on $\partial B_\rho$, we have

$$\frac{\partial u_+}{\partial v} \bigg|_{\partial B_\rho} = \sum_{n=-\infty}^{\infty} \omega l_n e^{i n \theta}$$

(2.19)

where

$$l_n := a_n J_n(\omega \rho) + b_n H_n^{(1)'}(\omega \rho).$$

By (2.4), (2.7) and (2.8), we have

$$l_n = \frac{\psi_n \sqrt{\frac{\pi}{R^2} H_n(\rho)} \left[ J_n(\omega \rho) H_n^{(1)}(\omega \rho) - J_n(\omega \rho) H_n^{(1)'}(\omega \rho) \right]}{\sqrt{\frac{\pi}{R^2} H_n(\rho) H_n^{(1)}(\omega \rho) - H_n^{(1)'}(\omega \rho)}},$$

(2.20)
By the Wronskian $J_n(t)Y'_n(t) - J'_n(t)Y_n(t) = \frac{2}{\pi t}$, we have
\[ I_n = \frac{\psi_n \sqrt{g'_n q'_n} \mathcal{H}_n(\rho)}{\omega [J'_n(\omega R) + \Gamma_n H_n^{(1)}(\omega R)] \sqrt{g'_n q'_n} \mathcal{H}_n(\rho) H_n^{(1)}(\omega \rho) - H_n^{(1)}(\omega \rho)} \cdot \frac{-2i}{\pi \omega \rho}. \]

By (2.12) and $\sqrt{q'_n} = \rho \sqrt{\gamma(\alpha + i\beta)}$, we have
\[ \| \frac{\partial u^+_R(\rho)}{\partial \nu} \|_{H^{-1/2}(\partial B_1)} \leq C\rho \| \psi \|_{H^{-1/2}(\partial B_R)}, \] (2.21) where $C$ is a generic constant that does not depend on $\rho$.

For the 3D case, similar to Proposition 2.1, one has

**Remark 2.3.** In $\mathbb{R}^3$, for $g'_l = \gamma \rho^{2-\delta}$, $q'_l = \alpha + i\beta$ and $f_a = 0$, there also exists a positive constant $C$ independent of $\rho$ and $\psi \in H^{-1/2}(\partial B_R)$, such that
\[ \| u - u_0 \|_{H^{1/2}(\partial B_R)} \leq C\rho \| \psi \|_{H^{-1/2}(\partial B_R)} \] (2.22)
where $C$ is independent of $\rho$ and $\psi$.

Next we consider the case with $0 < \delta < 2$ and we have

**Proposition 2.4.** In $\mathbb{R}^2$, when $g'_l = \gamma \rho^{2-\delta}$, $q'_l = \alpha + i\beta$, and $f_a = 0$, we have that for any $\psi \in H^{-1/2}(\partial B_R)$,
\[ \| u - u_0 \|_{H^{1/2}(\partial B_R)} \leq C\rho^{1-\delta} \| \psi \|_{H^{-1/2}(\partial B_R)} \] (2.23)
where $C$ is a positive constant independent of $\rho$ and $\psi$.

**Remark 2.5.** This means the near-cloak in this case is within $\rho^{2-\delta}$-approximation of the ideal cloak. Moreover, when $0 < \delta < 1$, the lossy layer tends to be a sound hard layer and when $1 \leq \delta < 2$, the layer does not tend to be a sound hard layer.

**Proof.** The proof follows from a completely similar argument to that in Proposition 2.1. The only difference one needs special care is that in this case, $\omega l \rho = \omega \sqrt{\alpha + i\beta/g \rho^{2}}$, $A = \sqrt{\frac{2g'}{q'_n}} = \sqrt{\frac{2\omega \rho}{\pi(\alpha + i\beta)}} \rho^{-2+\frac{\delta}{2}}$. It can then be verified that
\[ \gamma_n \to - \frac{J_n(\omega l \rho/2)}{H_n^{(1)}(\omega l \rho/2)}, \quad \text{as} \quad \rho \to 0^+. \] (2.24)

The asymptotic properties of $\mathcal{H}_n(\rho)$ as $\rho \to 0^+$ are as follows
\[
\begin{align*}
\mathcal{H}_0 & \sim \frac{\sqrt{\gamma \rho^{-2}}}{\omega \sqrt{\alpha + i\beta} \ln 2}, \\
\mathcal{H}_n & \sim \frac{n \sqrt{\gamma \rho^{-2}}}{\omega \sqrt{\alpha + i\beta}}.
\end{align*}
\]
Then one can show that $\Gamma_n$ has the following asymptotic properties as $\rho \to 0^+$,

\[
\begin{align*}
\Gamma_0 &\sim -i\gamma \rho^{2-\delta}, \\
\Gamma_n &\sim \frac{2 \ln 2}{(2\pi n)^2} - in\pi (\omega \rho)^{2n},
\end{align*}
\tag{2.25}
\]

Hence the proof is complete. □

**Remark 2.6.** Similar to the 2D case, in $\mathbb{R}^3$, when $g'_0 = \gamma \rho^{2-\delta}$, $q'_0 = \alpha + i\beta$, $0 < \delta < 2$, and $f_\alpha = 0$, one has

\[
\|u_\rho - u_0\|_{H^{1/2}(\partial B_R)} \sim \rho^{3-\delta}\|\psi\|_{H^{-1/2}(\partial B_R)} , \quad \|\frac{\partial u_\rho}{\partial \nu}\|_{H^{1/2}(\partial B_R)} \sim C \rho^{1-\delta}\|\psi\|_{H^{-1/2}(\partial B_R)}.
\]

### 2.2 The case with $g'_0 = \gamma \rho^{2-\delta}$, $0 \leq \delta < 2$, $q'_0 = \alpha + i\beta$ and $f_\alpha = \eta_0 \delta_0$

**Proposition 2.7.** In $\mathbb{R}^2$, let $g'_0 = \gamma \rho^2$, $q'_0 = \alpha + i\beta$, and $f_\alpha = \eta_0 \delta_0$, where $\eta_0$ is a constant which represents the strength of the source. In this case, we have

\[
\|u_\rho - u_0\|_{H^{1/2}(\partial B_R)} = \mathcal{O}(1).
\tag{2.26}
\]

This means that in this case the construction produces no near-cloaking effect.

**Proof.** We shall make use of the same notations as that in Section 2.1. In the virtual space, one has

\[
u_\rho(x) = \begin{cases}
\sum_{n=-\infty}^{\infty} c_n J_n(\omega |x|) e^{in\theta} + \eta_0 \frac{\eta_0}{\rho^2} H_0^{(1)}(\omega |x|), & x \in B_{\rho/2}, \\
\sum_{n=-\infty}^{\infty} c_n J_n(\omega |x|) e^{in\theta} + \sum_{n=-\infty}^{\infty} d_n H_n^{(1)}(\omega |x|) e^{in\theta}, & x \in B_\rho \setminus \overline{B}_{\rho/2}, \\
\sum_{n=-\infty}^{\infty} a_n J_n(\omega |x|) e^{in\theta} + \sum_{n=-\infty}^{\infty} b_n H_n^{(1)}(\omega |x|) e^{in\theta}, & x \in B_R \setminus \overline{B}_\rho.
\end{cases}
\tag{2.27}
\]

By the transmission boundary condition on $\partial B_{\rho/2}$ and $\partial B_\rho$, we have

\[
\begin{align*}
&c_0 J_0(\omega_0 \rho/2) + \frac{\eta_0}{\rho^2} H_0^{(1)}(\omega_0 \rho/2) = c_0 J_0(\omega_0 \rho/2) + d_0 H_0^{(1)}(\omega_0 \rho/2), \\
&\sqrt{g_0^2 q_0^2} c_0 J_0(\omega_0 \rho/2) + \frac{\eta_0}{\rho^2} H_0^{(1)}(\omega_0 \rho/2) = \sqrt{g_0^2 q_0^2} c_0 J_0(\omega_0 \rho/2) + d_0 H_0^{(1)}(\omega_0 \rho/2), \\
&c_0 J_0(\omega_0 \rho) + d_0 H_0^{(1)}(\omega_0 \rho) = a_0 J_0(\omega_0 \rho) + b_0 H_0^{(1)}(\omega_0 \rho), \\
&\sqrt{g_0^2 q_0^2} c_0 J_0(\omega_0 \rho) + d_0 H_0^{(1)}(\omega_0 \rho) = \sqrt{g_0^2 q_0^2} c_0 J_0(\omega_0 \rho) + b_0 H_0^{(1)}(\omega_0 \rho), \\
&a_0 J_0(\omega R) + b_0 H_0^{(1)}(\omega R) = \psi_0/\omega.
\end{align*}
\tag{2.28}
\]
\[
\begin{aligned}
\text{For } n \neq 0, \text{ one can show}
\begin{cases}
\epsilon_n J_n(\omega_0 \rho/2) = c_n J_n(\omega_0 \rho/2) + d_n H_n^{(1)}(\omega_0 \rho/2), \\
\sqrt{\eta_0} e_n J'_n(\omega_0 \rho/2) = \sqrt{\eta_0' e_n} [c_n J'_n(\omega_0 \rho/2) + d_n H_n^{(1)}(\omega_0 \rho/2)], \\
c_n J_n(\omega_0 \rho) + d_n H_n^{(1)}(\omega_0 \rho) = a_n J_n(\omega_0 \rho) + b_n H_n^{(1)}(\omega_0 \rho), \\
\sqrt{\eta_0' e_n} [c_n J'_n(\omega_0 \rho) + d_n H_n^{(1)}(\omega_0 \rho)] = a_n J'_n(\omega_0 \rho) + b_n H_n^{(1)}(\omega_0 \rho), \\
a_n J'_n(\omega_0 R) + b_n H_n^{(1)}(\omega_0 R) = \psi_n/\omega.
\end{cases}
\end{aligned}
\]

By the third and fourth equations of (2.28), we have
\[
\begin{aligned}
d_0 &= \frac{\left[J_0(\omega_0 \rho) J'_0(\omega_0 \rho) - \frac{1}{\sqrt{\eta_0' e_n}} J'_0(\omega_0 \rho) J_0(\omega_0 \rho)\right] a_0}{H_0^{(1)}(\omega_0 \rho) J_0(\omega_0 \rho) - H_0^{(1)}(\omega_0 \rho) J_0(\omega_0 \rho)}, \\
c_0 &= \frac{\left[J_0(\omega_0 \rho) H_0^{(1)}(\omega_0 \rho) - \frac{1}{\sqrt{\eta_0' e_n}} J'_0(\omega_0 \rho) H_0^{(1)}(\omega_0 \rho)\right] a_0}{J_0(\omega_0 \rho) H_0^{(1)}(\omega_0 \rho) - J'_0(\omega_0 \rho) H_0^{(1)}(\omega_0 \rho)}, \\
&\end{aligned}
\]

Let \( A = \sqrt{\frac{\eta_0' e_n}{\eta_0}} \). Using the Wronskian
\[
J_n(t) Y_n'(t) - J'_n(t) Y_n(t) = \frac{2}{\pi t},
\]
and by virtue of the first and second equation of (2.28), we further have
\[
\begin{aligned}
\left\{\begin{aligned}
&\left[J_0(\omega_0 \rho/2) J'_0(\omega_0 \rho/2) - \frac{1}{A} J'_0(\omega_0 \rho/2) J_0(\omega_0 \rho)\right] [J_0(\omega_0 \rho) H_0^{(1)}(\omega_0 \rho) - \frac{1}{\sqrt{\eta_0' e_n}} J'_0(\omega_0 \rho) H_0^{(1)}(\omega_0 \rho)] \\
- [H_0^{(1)}(\omega_0 \rho/2) J'_0(\omega_0 \rho/2) - \frac{1}{A} J'_0(\omega_0 \rho/2) H_0^{(1)}(\omega_0 \rho)] [J_0(\omega_0 \rho) J'_0(\omega_0 \rho) - \frac{1}{\sqrt{\eta_0' e_n}} J'_0(\omega_0 \rho) J_0(\omega_0 \rho)]
\end{aligned}\right\} a_0 \\
+ \left\{\begin{aligned}
&\left[J_0(\omega_0 \rho/2) J'_0(\omega_0 \rho/2) - \frac{1}{A} J'_0(\omega_0 \rho/2) J_0(\omega_0 \rho)\right] [H_0^{(1)}(\omega_0 \rho) J'_0(\omega_0 \rho) - \frac{1}{\sqrt{\eta_0' e_n}} H'_0(\omega_0 \rho) J_0(\omega_0 \rho)] \\
- [H_0^{(1)}(\omega_0 \rho/2) J'_0(\omega_0 \rho/2) - \frac{1}{A} J'_0(\omega_0 \rho/2) H_0^{(1)}(\omega_0 \rho)] [J_0(\omega_0 \rho) J'_0(\omega_0 \rho) - \frac{1}{\sqrt{\eta_0' e_n}} H'_0(\omega_0 \rho) J_0(\omega_0 \rho)]
\end{aligned}\right\} b_0 \\
&= \frac{2i}{\pi \omega_0 \rho \rho^2} \left[-H_0^{(1)}(\omega_0 \rho/2) + \frac{J_0(\omega_0 \rho/2)}{J_0(\omega_0 \rho/2)} H_0^{(1)}(\omega_0 \rho/2)\right].
\end{aligned}
\]
In the case under discussion, one has $1/A = \sqrt{\frac{2\pi}{\gamma(\alpha + i\beta)}} \rho^2$. We define the following shorthand notations for brevity:

$$A_1 = \frac{J_0(\omega_a \rho/2)}{J_0(\omega_a \rho)}, \quad A_2 = \frac{H^{(1)}_0(\omega_a \rho/2)}{J_0(\omega_a \rho/2)} \frac{J_0(\omega_a \rho/2)}{J_0(\omega_a \rho)},$$

$$A_3 = -H^{(1)}_0(\omega_a \rho) + \frac{J_0(\omega_a \rho/2)}{J_0(\omega_a \rho/2)} H^{(1)}_0(\omega_a \rho/2),$$

$$c_0 = \sqrt{\frac{\gamma(\alpha + i\beta)}{g_0 q_0}}, \quad d_0 = 1/\sqrt{\gamma(\alpha + i\beta)}.$$

Since $\omega_a \rho = \omega \sqrt{\frac{\alpha + i\beta}{\gamma}}$ and $\omega_a \rho$ are constants, it can be shown that $A_1, A_2, A_3$ are all constants as $\rho \to +0$. Then one has that as $\rho \to 0^+$,

$$\left\{ \begin{array}{l}
[A_1 - \rho^2 c_0 J_0(\omega_a \rho)] [H^{(1)}_0(\omega_1 \rho) + d_0 \omega^2 / J_0(\omega_1 \rho)]

- [A_2 - \rho^2 c_0 H^{(1)}_0(\omega_a \rho)] [J_0(\omega_1 \rho) + d_0 \omega^2 / J_0(\omega_1 \rho)] \bigg\} a_0

+ \left\{ \begin{array}{l}
[A_1 - \rho^2 c_0 J_0(\omega_a \rho)] \frac{2i}{\pi} \ln \frac{\omega}{\omega_a} [H^{(1)}_0(\omega_1 \rho) - d_0 \omega^2 / J_0(\omega_1 \rho)]

- [A_2 - \rho^2 c_0 H^{(1)}_0(\omega_a \rho)] \frac{2i}{\pi} \ln \frac{\omega}{\omega_a} [J_0(\omega_1 \rho) - d_0 \omega^2 / J_0(\omega_1 \rho)] \bigg\} b_0

\sim \sqrt{\frac{-2i}{\pi \omega}} \frac{\eta_0}{\pi \omega^2} A_3.
\end{array} \right.$$  

Next, letting $D_0 = \sqrt{\frac{-2i}{\pi \omega}} \frac{\eta_0}{\pi \omega^2} A_3$ and dropping the lower order term, we have

$$\left\{ \begin{array}{l}
A_1 [H^{(1)}_0(\omega R) + d_0 \omega / J_0(\omega R)] - A_2 [J_0(\omega R) + d_0 \omega / J_0(\omega R)]

\frac{1}{\rho^2} - A_1 \frac{d_0 \omega^2}{\pi \omega} H^{(1)}_0(\omega R) + A_2 \frac{d_0 \omega^2}{\pi \omega} J_0(\omega R) \sim \frac{D_0}{\rho^2},
\end{array} \right.$$  

(2.34) yields that

$$\begin{cases}
a_0 \sim \frac{\psi_0/\omega - D_0 J_0^{(1)}(\omega R) / E_0}{J_0(\omega R) - C_0 \omega^2 J_0^{(1)}(\omega R) / E_0}, \\
b_0 \sim \frac{\rho^2 \psi_0/\omega - D_0 J_0^{(1)}(\omega R)}{\rho^2 J_0^{(1)}(\omega R) - E_0 J_0^{(1)}(\omega R) / C_0}.
\end{cases}$$  

(2.36)

by letting

$$C_0 := \{ A_1 [H^{(1)}_0(\omega R) + d_0 \omega / J_0(\omega R)] - A_2 [J_0(\omega R) + d_0 \omega / J_0(\omega R)] \},$$

$$E_0 := -A_1 \frac{d_0 \omega^2}{\pi \omega} H^{(1)}_0(\omega R) + A_2 \frac{d_0 \omega^2}{\pi \omega} J_0(\omega R).$$
Then by the Wronskians (2.31) and (2.35) we have further
\[\left| a_0 J_0(\omega R) + b_0 H_0^{(1)}(\omega R) - \frac{\psi_0 J_0(\omega R)}{\omega J_0(\omega R)}\right| = -\frac{2ib_0}{\pi \omega R J_0'(\omega R)}.\] (2.37)

Next, by (2.36), \(b_0 \sim O(1)\) and
\[\left| a_n J_n(\omega R) + b_n H_n^{(1)}(\omega R) - \frac{\psi_n J_n(\omega R)}{\omega J_n(\omega R)}\right| = O(\rho^{2n}), \quad n \geq 1.
\]

Using the estimates derived above, together with a similar analysis to that for the proof of Proposition 2.1, one can show that
\[\|u_\rho - u_0\|_{H^{1/2}(\partial B_R)} = O(1).\] (2.38)

**Remark 2.8.** It can be verified by using (2.36) that in \(\mathbb{R}^2\), when \(g_1' = \gamma \rho^2, \quad q_1' = \alpha + i\beta, \quad \text{and} \quad f_a = \eta_0 \delta_0,\) one has \(\|u_\rho - u_0\|_{H^{-1/2}(\partial B_1)} \sim \rho^{-1}.\) That is, the normal velocity of the wave field on \(\partial B_1\) would blow up as \(\rho \to 0^+.\)

**Proposition 2.9.** In \(\mathbb{R}^2\), when \(g_1' = \gamma \rho^{2-\delta}, \quad q_1' = \alpha + i\beta, \quad 0 < \delta < 2, \quad \text{and} \quad f_a = \eta_0 \delta_0,\) one has \(\|u_\rho - u_0\|_{H^{1/2}(\partial B_R)} = O(\rho^{-\delta}).\) That is, the construction would not yield a near-cloaking device.

**Proof.** The proof follows from a completely similar argument to that for Proposition 2.7. We only need notice the following significant differences. For this case, equations (2.27)-(2.32) and (2.35) are of the same formulas as the \(g_1' = \rho^2\) case, but with different parameters. \(1/A = \sqrt{\frac{g_1'}{q_1'}} = \sqrt{\frac{\rho^2}{\alpha^2 + \beta^2}}, \quad \omega_0 \rho = \omega \sqrt{\frac{\alpha^2 + \beta^2}{\gamma}}, \quad \sqrt{g_1/q_1} = \sqrt{\frac{\alpha + i\beta}{\gamma}}.\)

By the asymptotic behaviors of the Bessel and Hankel functions (cf. [1] and [22]), the equation (2.32) has the following asymptotics,
\[\frac{J_0'(\omega_0 \rho)}{J_0(\omega_0 \rho)} \sim \frac{2i}{\pi \omega_0 \rho^{\frac{1}{2}}} \left[H_0^{(1)}(\omega_0 \rho/2) + \frac{4 \ln 2 \rho^{\frac{\delta}{2}}}{\pi^2 \omega \sqrt{(\alpha + i\beta)\gamma}} b_0\right] \quad (2.39)\]

and this in combination with (2.35) gives
\[b_0 \sim \rho^{-\delta};\] (2.40)

which readily implies the statement of the proposition.

\[\square\]
Remark 2.10. In the same setting as Proposition 2.9, one can show that \( \| \partial u_n^i(\rho) \|_{H^{-1/2}(\partial B_R)} \sim \rho^{-1-\delta} \). That is, the normal velocity of the wave field would blow up as \( \rho \to 0^+ \).

For the three dimensional case, we have

Proposition 2.11. In \( \mathbb{R}^3 \), when \( f_a = \eta_0 \delta_0, \quad g'_1 = \gamma \rho^{2-\delta} \) and \( q_1 = \alpha + i \beta, \quad 0 \leq \delta \leq 1 \), one has

\[
\| u_\rho - u_0 \|_{H^{1/2}(\partial B_R)} \leq C_1 \rho^{1-\delta} \eta_0 + C_2 \rho^2 \| \psi \|_{H^{-1/2}(\partial B_R)},
\]

where \( C_1 \) and \( C_2 \) are independent of \( \rho, \eta_0 \) and \( \psi \); whereas when \( 1 < \delta < 2 \), we have

\[
\| u_\rho - u_0 \|_{H^{1/2}(\partial B_R)} \sim O(\rho^{1-\delta}).
\]

Proof. The 3D case can be proved in a similar manner to that for the 2D case. Let

\[
\psi(x) \big|_{\partial B_R} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \psi_n^m Y_n^m(\hat{x}) \in H^{-1/2}(\partial B_R),
\]

with

\[
\| \psi(x) \|_{H^{-1/2}(\partial B_R)}^2 = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (1 + n(n + 1)/R^2)^{-1/2} |\psi_n^m|^2 < +\infty.
\]

The free space solution \( u_0 \) to (1.14) in \( \mathbb{R}^3 \) is given by

\[
u_0 = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{\psi_n^m}{\omega n(\omega R)} j_n(\omega |x|) Y_n^m(\hat{x}).
\]

Noting \( g'_a = g_a/\rho, \quad g'_0 = g'_0/\rho^2 \) in \( B_{\rho/2} \), similar to (2.2) for the 2D case, the wave fields in the separated domains could be represented as follows

\[
\begin{align*}
u_a(x) &= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} e_n^m j_n(\omega_a |x|) Y_n^m(\hat{x}) - \frac{i\eta_0}{4\pi \rho^2} h_n^{(1)}(\omega_a |x|), \\
u_l(x) &= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} m_n^m j_n(\omega |x|) Y_n^m(\hat{x}) + \sum_{n=0}^{\infty} \sum_{m=-n}^{n} d_n^m h_n^{(1)}(\omega |x|) Y_n^m(\hat{x}), \\
u_R(x) &= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m j_n(\omega |x|) Y_n^m(\hat{x}) + \sum_{n=0}^{\infty} \sum_{m=-n}^{n} b_n^m h_n^{(1)}(\omega |x|) Y_n^m(\hat{x}).
\end{align*}

By using the standard transmission conditions and the boundary condition, one could derive the following linear systems of equations for the coefficients, when \( n = 0 \)

\[
\begin{align*}
(\alpha_0^{(0)} \delta_0(\omega_a \rho/2) - \frac{i \eta_0}{4\pi \rho^2} h_0^{(1)}(\omega_a |x|)) &= e_0^{(0)}(\omega_a \rho/2) + d_0^{(1)}(\omega_a \rho/2), \\
\sqrt{\theta'_a q'_a e_0^{(0)}(\omega_a \rho/2) - \frac{i \eta_0}{4\pi \rho^2} h_0^{(1)}(\omega_a |x|)} &= \sqrt{\theta'_a q'_a} e_0^{(0)}(\omega_a \rho/2) + d_0^{(1)}(\omega_a \rho/2), \\
(\alpha_0^{(0)} \delta_0(\omega \rho/2) + d_0^{(1)}(\omega \rho/2)) &= e_0^{(0)}(\omega \rho) + b_0^{(1)}(\omega \rho), \\
\sqrt{\theta'_a q'_a e_0^{(0)}(\omega \rho) + d_0^{(1)}(\omega \rho)} &= \sqrt{\theta'_a q'_a} e_0^{(0)}(\omega \rho) + b_0^{(1)}(\omega \rho), \\
(\omega a_0^{(0)} \delta_0(\omega R) + \omega b_0^{(1)}(\omega R)) &= \psi_0^{(0)},
\end{align*}
\]

(2.45)
and when \( n \geq 1, \)

\[
\begin{aligned}
e_m^m j_n(\omega \rho/2) &= e_m^m j_n(\omega \rho/2) + d_m^m h_n^{(1)}(\omega \rho/2), \\
\sqrt{g_d \hat{q} e_m^m j'_n(\omega \rho/2)} &= \sqrt{g_d \hat{q} e_m^m j'_n(\omega \rho/2) + d_m^m h_n^{(1)}(\omega \rho/2)}, \\
c_n^m j_n(\omega \rho) + d_m^m h_n^{(1)}(\omega \rho) &= a_m^m j_n(\omega \rho) + b_m^m h_n^{(1)}(\omega \rho), \\
\sqrt{g_d \hat{q} c_n^m j'_n(\omega \rho) + d_m^m h_n^{(1)}(\omega \rho)} &= a_m^m j'_n(\omega \rho) + b_m^m h_n^{(1)}(\omega \rho), \\
\omega_m^m j'_n(\omega R) + \omega b_m^m h_n^{(1)}(\omega R) &= \psi_m^m.
\end{aligned}
\]  

(2.46)

Letting \( \tilde{A} = \sqrt{\frac{d \hat{q}}{g_d \hat{q}'}} \) and solving (2.46), one has

\[
b_m^m = -\frac{\sqrt{\hat{q} g_d \hat{q} j'_n(\omega \rho) + \hat{Y}_n h_n^{(1)}(\omega \rho)} j_n(\omega \rho) - j'_n(\omega \rho)}{\sqrt{\hat{q} g_d \hat{q} j'_n(\omega \rho) + \hat{Y}_n h_n^{(1)}(\omega \rho)} h_n^{(1)}(\omega \rho) - h'_n(\omega \rho)} a_m^m,
\]

where

\[
\hat{Y}_n := \begin{cases} 
-j_n(\omega \rho/2) \quad \text{if} \quad j_n(\omega \rho/2) = 0, \\
-h_n^{(1)}(\omega \rho/2) - A j_n(\omega \rho/2) j'_n(\omega \rho/2) \quad \text{if} \quad j_n(\omega \rho/2) \neq 0.
\end{cases}
\]

(2.48)

Let \( \tilde{Y}_n \) denote the expression before \( a_m^m \) in (2.47). Let \( u^0_\rho, u^0_0 \) represent the zeroth order Fourier term with \( Y_n^m \) on \( \partial B_R \). Then

\[
\left( [u_\rho(x) - u_0(x)] - [u^0_\rho(x) - u^0_0(x)] \right)_{\partial B_R} = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \psi_n^m j_n(\omega R) \frac{\tilde{Y}_n h_n^{(1)}(\omega R) - h_n^{(1)}(\omega R)}{j'_{n}(\omega R)} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \tilde{Y}_n^m (\hat{x}).
\]

(2.49)

Set

\[
\tilde{g}_n = \tilde{Y}_n \left[ \frac{h_n^{(1)}(\omega R)}{j_{n}(\omega R)} - \frac{h_n^{(1)'(\omega R)}}{j'_{n}(\omega R)} \right] / \left[ 1 + \frac{\tilde{Y}_n h_n^{(1)'(\omega R)}}{j'_{n}(\omega R)} \right], \quad n \geq 1,
\]

then

\[
\left\| [u_\rho(x) - u_0(x)] - [u^0_\rho(x) - u^0_0(x)] \right\|_{H^{1/2}(\partial B_R)}^2 = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \sqrt{1 + \frac{n(n+1)}{R^2}} \left| \psi_n^m j_n(\omega R) \tilde{g}_n R \right|^2.
\]

(2.50)

By a similar asymptotic argument to the 2D case, one can show

\[
\tilde{g}_n \sim O(\omega \rho)^{2n+1},
\]

(2.51)
which is then applied to the estimate of (2.50), one can further show that
\[
\|u_\rho(x) - u_0(x) - [u_\rho^0 - u_0^0]_H^{1/2(\partial B_R)} \leq C \rho^3 \|\psi - \psi_0^0\|_{H^{-1/2}(\partial B_R)},
\]
(2.52)
Next for \( n = 0 \), by using the Wronskian
\[
j_n(t)y_n(t) - j_n'(t)y_n(t) = \frac{1}{t^2},
\]
(2.53)
and the first and second equations of (2.45), we have
\[
\left\{ \begin{array}{l}
\left[ \frac{j_0(\omega \rho/2)}{j_0(\omega \rho/2)}j_0'(\omega \rho/2) - \frac{1}{A}j_0'(\omega \rho/2)\right] [j_0(\omega \rho)h_0^{(1)}'(\omega \rho) - \frac{1}{\sqrt{g_0q_0}}j_0'(\omega \rho) h_0^{(1)}(\omega \rho)] \\
- \left[ \frac{h_0^{(1)}(\omega \rho/2)}{j_0(\omega \rho/2)}j_0'(\omega \rho/2) - \frac{1}{A}j_0'(\omega \rho/2)\right] [j_0(\omega \rho)j_0'(\omega \rho) - \frac{1}{\sqrt{g_0q_0}}j_0'(\omega \rho) j_0(\omega \rho)] a_0 \\
+ \left\{ \begin{array}{l}
\left[ \frac{j_0(\omega \rho/2)}{j_0(\omega \rho/2)}j_0'(\omega \rho/2) - \frac{1}{A}j_0'(\omega \rho/2)\right] [h_0^{(1)}(\omega \rho) h_0^{(1)}'(\omega \rho) - \frac{1}{\sqrt{g_0q_0}}h_0^{(1)}'(\omega \rho) h_0^{(1)}(\omega \rho)] \\
- \left[ \frac{h_0^{(1)}(\omega \rho/2)}{j_0(\omega \rho/2)}j_0'(\omega \rho/2) - \frac{1}{A}j_0'(\omega \rho/2)\right] [h_0^{(1)}(\omega \rho) j_0'(\omega \rho) - \frac{1}{\sqrt{g_0q_0}}h_0^{(1)}'(\omega \rho) j_0(\omega \rho)] b_0 \\
= \frac{1}{4\pi(\omega \rho)^2 \rho^2} \left[ -h_0^{(1)}(\omega_0 \rho/2) + \frac{j_0'(\omega_0 \rho/2)}{j_0(\omega_0 \rho/2)} h_0^{(1)}(\omega_0 \rho/2) \right].
\end{array} \right.
\]
(2.54)
For the first term of \( u_\rho - u_0 \), by the Wronskian (2.53) and the last equation of (2.45), we have
\[
\left| a_0^0 j_0(\omega \rho) Y_0^0 + b_0^0 h_0^{(1)}(\omega \rho) Y_0^0 - \frac{\psi_0^0 j_0(\omega \rho)}{\omega j_0'(\omega \rho)} Y_0^0 \right| = \left| \frac{b_0^0 Y_0^0}{a_0^0 j_0'(\omega \rho)} \right|.
\]
(2.55)
Clearly, \( b_0^0 \) is the dominant factor for the first term of \( u_\rho - u_0 \). Since \( \omega \rho = \omega \sqrt{\frac{\alpha + i \beta}{\gamma}} \rho^{1 - \delta} \), \( \sqrt{g_0q_0} = \sqrt{\frac{(\alpha + i \beta)\gamma}{\gamma}} \rho^{1 - \delta} \), and \( A = \sqrt{\frac{g_0q_0}{(\alpha + i \beta)\gamma}} \rho^{\delta - 3} \), we further have the following from (2.54),
\[
\frac{j_0'(\omega_0 \rho/2)}{j_0'(\omega_0 \rho/2)} \left[ a_0^0 j_0(\omega_0 \rho) - h_0^{(1)}(\omega_0 \rho/2) \right] \sim \frac{1}{4\pi(\omega_0 \rho)^2 \rho^2} \left[ -h_0^{(1)}(\omega_0 \rho/2) + \frac{j_0'(\omega_0 \rho/2)}{j_0(\omega_0 \rho/2)} h_0^{(1)}(\omega_0 \rho/2) \right],
\]
which in combination with the last equation in (2.45), implies that there exist constants \( C_1, C_2 \) independent of \( \rho \) such that
\[
b_0^0 \sim C_1 \eta_0 \rho^{1 - \delta} + C_2 \psi_0^0 \rho^{3 - \delta}.
\]
(2.56)
Thus the proof is completed.
2.3 The case with \( g'_l = \gamma \rho^{2+2\delta}, \delta > 0 \) and \( g'_a = \alpha + i\beta \)

Proposition 2.12. In \( \mathbb{R}^2 \), when \( g'_l = \gamma \rho^{2+2\delta}, \delta > 0 \) and \( g'_a = \alpha + i\beta \), and \( f_a = \eta_0 \delta_0 \), we have that

\[
\| u_\rho - u_0 \|_{H^{1/2}(\partial B_R)} \leq C_1 \rho^2 \| \psi \|_{H^{-1/2}(\partial B_R)} + C_2 \rho^\delta e^{-\frac{\eta_0 - \delta}{2}},
\]

where \( l_1 = \Im(\omega^2(\alpha + i\beta)/\gamma) \) (the positive branch) and \( C_1 \) and \( C_2 \) are positive constants independent of \( \psi \) and \( \eta_0 \). That is, the source/sink term can be exponentially cloaked in this case.

Proof. We shall make use of a similar argument to that for Proposition 2.7. For the present proof, we have the same equations as those in (2.28), (2.29), but with different \( g'_l, g'_a \). Here \( \omega_1 = \omega \sqrt{g'_l/g'_a} = \omega \rho^{-\frac{1}{2}} \sqrt{(\alpha + i\beta)/\gamma}, \omega_1 \rho = \omega \rho^{-\frac{1}{2}} \sqrt{(\alpha + i\beta)/\gamma}. \) Let \( \Re \omega_1 \rho = \rho^{-\delta} l_1, \Im \omega_1 \rho = \rho^{-\delta} l_2 \). Let \( a_n, b_n \) be the same as above. For \( n \neq 0 \), the equations are the same as those in [21] and \( a_0, b_0 \) are also the same. We only need to consider the case \( n = 0 \). Here in equation (2.32), \( A = \sqrt{\frac{g'_l}{g'_a} g'_l} = \sqrt{\frac{g'_l}{g'_a} g'_a} \rho^{2-2\delta}. \) Let \( B_1 = \frac{J'_0(\omega_1 \rho/2)}{J_0(\omega_1 \rho/2)}. \)

By the asymptotic behavior of the Bessel functions Hankel functions with large variables and positive imaginary part (cf. [1])

\[
\begin{align*}
J_n(z) & \sim \sqrt{\frac{1}{2\pi}} e^{i|\Re(z)|} e^{i(\Re(z) + \frac{n+1}{2})}, \quad |\arg z| < \pi, \\
H_n^{(1)}(z) & \sim \sqrt{\frac{1}{2\pi}} e^{-i|\Re(z)|} e^{i(\Re(z) - \frac{n+1}{2})}, \quad -\pi < \arg z < 2\pi, \\
J'_n(z) & \sim -\sqrt{\frac{1}{2\pi}} e^{-i|\Re(z)|} e^{i(\Re(z) + \frac{n+1}{2})}, \quad |\arg z| < \pi, \\
H_n^{(1)'}(z) & \sim -\sqrt{\frac{1}{2\pi}} e^{-i|\Re(z)|} e^{i(\Re(z) - \frac{n+1}{2})}, \quad -\pi < \arg z < 2\pi.
\end{align*}
\]

With the same notations \( d_0, c_0 \) as those in (2.33), we have the following asymptotic expansion as \( \rho \to 0^+ \),

\[
e^{-\frac{i\eta_0}{2}} \frac{\sqrt{\gamma}}{\pi \omega_1 \rho} [B_1 + \rho^{2+\delta} c_0 e^{i\pi/2}] [-e^{i\pi/2} + \rho^{-\delta} d_0 \frac{\omega}{2} e^{-\frac{\eta_0 - \delta}{2}}]
\]

\[
- e^{-\frac{i\eta_0}{2}} \frac{\sqrt{\gamma}}{\pi \omega_1 \rho} [B_1 + \rho^{2+\delta} c_0 e^{-i\pi/2}] [-e^{-i\pi/2} + \rho^{-\delta} d_0 \frac{\omega}{2} e^{-\frac{\eta_0 - \delta}{2}}]
\]

\[
- e^{-\frac{i\eta_0}{2}} \frac{\sqrt{\gamma}}{\pi \omega_1 \rho} [B_1 + \rho^{2+\delta} c_0 e^{i\pi/2}] \left[ \frac{2}{\pi} \ln \frac{\omega}{2} + d_0 i \frac{2}{\pi \omega \rho^{2+\delta}} e^{-\frac{i\eta_0 - \delta}{2}} \right] b_0
\]

\[
+ e^{-\frac{i\eta_0}{2}} \frac{\sqrt{\gamma}}{\pi \omega_1 \rho} [B_1 + \rho^{2+\delta} c_0 e^{-i\pi/2}] \left[ -\frac{2}{\pi} \ln \frac{\omega}{2} + d_0 i \frac{2}{\pi \omega \rho^{2+\delta}} e^{-\frac{i\eta_0 - \delta}{2}} \right] b_0
\]

\[
\sim \frac{2i \eta_0}{\pi \omega_1 \rho^2} [-H_0^{(1)'}(\omega_1 \rho/2) + J_0'(\omega_1 \rho/2) H_0^{(1)}(\omega_1 \rho/2)].
\]
Next, by dropping the exponentially decaying terms in (2.59), we further have
\[
- e^{-\frac{1}{2}a_0} \frac{\sqrt{2}}{\rho} \left[ B_1 + \rho \right] e^{-i\pi/2} e^{-i\pi/2} \left[ -e^{-i\pi/2} + \rho^{-\delta} \right] e^{-\frac{i}{2}a_0} + e^{-\frac{i}{2}a_0} \frac{\sqrt{2}}{\rho} \left[ B_1 + \rho \right] e^{-i\pi/2} \left[ -\frac{2}{\pi} \ln \rho \right] e^{-\frac{i}{2}a_0} b_0
\]
\[
\sim \frac{2i}{\pi \omega \rho^2} \left[ -H_0^1 + \frac{J_0^1}{\omega} \right] \left[ (\omega \rho/2) + \frac{J_0^1(\omega \rho/2)}{\omega} \right] .
\]
Let \( g_0 = \sqrt{2} B_1 + \rho c_0 e^{-i\pi/2} \left[ -e^{-i\pi/2} + \rho^{-\delta} d_0 \right] \) and \( y = \left[ -H_0^1 + \frac{J_0^1(\omega \rho/2)}{\omega} \right] \). One can show from (2.60) that
\[
- a_0 + \frac{\rho b_0}{d_0 \omega^2} \sim \frac{1}{\rho} e^{-\frac{1}{2}a_0} e^{-i\pi/2} e^{-\frac{i}{2}a_0} y.
\]
Let \( k_0 = \frac{\rho b_0}{d_0 \omega^2} \), \( n_0 = \frac{\rho b_0}{d_0 \omega^2} \) and \( v(\rho) = \frac{1}{\rho^2} e^{-\frac{1}{2}a_0} e^{-i\pi/2} \). Combining (2.61) and (2.35), we have
\[
\left\{ \begin{array}{l}
a_0 \sim \frac{v_0 \rho}{\omega} \frac{-n_0 e^{-i\pi/2} H_0^1(\omega \rho)}{\rho_0} \\
b_0 \sim \frac{v_0 \rho}{\omega} \frac{\rho_0^2 H_0^1(\omega \rho)}{\rho_0^2}.
\end{array} \right.
\]
Hence there exists constants \( C_1, C_2 \), such that
\[
\left| a_0 J_0(\omega R) + b_0 H_0^1(\omega R) - \frac{\psi_0 J_0(\omega R)}{\omega R} \right| \leq C_1 \rho^2 \psi_0 + C_2 \rho \delta e^{-\frac{i}{2}a_0} \eta_0,
\]
and by which the proposition is readily proved.

Remark 2.13. It can be easily seen from the proof of Proposition 2.12 that
\[
\left\| \frac{\partial u^+}{\partial \rho} \right\|_{H^{-1/2}(\partial B_1)} \leq C \rho,
\]
where \( C \) depends on \( \psi \) and \( \eta_0 \) but does not depend on \( \rho \).

3 The high-loss scheme

In this section, we shall consider the high-loss scheme, namely \( g_0^t = I \) and \( g_0^t = 1 + i \rho, \ t \in \mathbb{R} \). The cloaking performance in various settings can be assessed by following completely similar arguments to those for the high-density scheme, and we would present those results in the sequel but only sketch the proofs.
Proposition 3.1. In $\mathbb{R}^2$, when $g'_1 = I, q'_1 = 1 + i\rho^{-2+\delta}, 0 < \delta < 2$, and $f_a = 0$, we have
\[
\|u_\rho - u_0\|_{H^{1/2}(\partial\Omega)} \leq C \frac{1}{\ln \rho} \|\psi\|_{H^{-1/2}(\partial B_{\rho})}, \quad \|u_R^+(\rho)\|_{H^{1/2}(\partial B_{\rho})} \leq C \frac{1}{\ln \rho} \|\psi\|_{H^{-1/2}(\partial B_{\rho})},
\] (3.1)
where $C$ is independent of $\rho$ and $\psi$.

Proof. The proof follows from a completely similar argument to that in Proposition 2.1. The following are the major ingredients of the present proof. In this case, $\omega l\rho = \omega \rho^{1/2} \sqrt{\rho^2 - \delta + i}$, $A = \sqrt{\frac{g_2}{g'_2} g''_2} = \sqrt{\frac{g_2 g'_2}{\rho^2 - \delta + i}} \rho^{-\frac{\delta}{2}}$, $\alpha = \sqrt{\frac{\omega l\rho}{\rho}} = \rho^{-\frac{1}{2}} \sqrt{\rho^2 - \delta + i}$. As $\rho \to 0$, one has
\[
\Gamma_n \sim -\frac{J_n(\omega l\rho/2)}{H_n^{(1)}(\omega l\rho/2)}, \quad \mathcal{H}_n(\rho) = O(\rho^{-\delta/2}),
\]
and
\[
\left\{ \begin{array}{l}
\Gamma_0 \sim \frac{2}{T_{n}\pi}, \\
\Gamma_n \sim \rho^{2n}.
\end{array} \right.
\] (3.2)
Finally, by supposing $u_R^+(x)|_{\partial B_{\rho}} = \sum_{n=-\infty}^{\infty} k_n e^{i\theta}$ and using (2.4) and (2.7), one has
\[
k_n = \frac{\psi_n}{\omega[J_n'(\omega R) + \Gamma_n H_n^{(1)}(\omega R)]} \cdot \frac{J_n'(\omega R) H_n^{(1)}(\omega R) - J_n(\omega l\rho) H_n^{(1)}(\omega l\rho)}{\sqrt{g'_2 g''_2} \mathcal{H}_n(\rho) H_n^{(1)}(\omega l\rho) - H_n^{(1)}(\omega l\rho)},
\] (3.3)

Remark 3.2. In $\mathbb{R}^3$, when $g'_1 = I, q'_1 = 1 + i\rho^{-2+\delta}, 0 < \delta < 2$, and $f_a = 0$, we have
\[
\|u_\rho - u_0\|_{H^{1/2}(\partial\Omega)} \leq C \rho \|\psi\|_{H^{-1/2}(\partial B_{\rho})}, \quad \|u_R^+(\rho)\|_{H^{1/2}(\partial B_{\rho})} = O(1).
\] (3.4)

Next, we consider the near-cloak by including a delta-point source into the cloaked region.

Proposition 3.3. In $\mathbb{R}^2$, when $g'_1 = I, q'_1 = 1 + i\rho^{-2+\delta}, 0 \leq \delta < 2$, and $f_a = \eta_0 \delta_0$, we have
\[
\|u_\rho - u_0\|_{H^{1/2}(\partial\Omega)} = O\left(\frac{1}{\rho^2 \ln \rho}\right).
\] (3.5)

Proof. We only sketch the proof for the case with $\delta = 0$ and the other case with $0 < \delta < 2$ can be proved in a completely similar manner. In this case, we have $g''_1 = g''_0$ and $q''_1 = \frac{g''_0}{\rho^2}$; and $\omega_a = \omega \rho^{-1} \sqrt{g''_0}$ with $\omega_a \rho$ being a constant. $\omega' = \omega g''_1 = \sqrt{1 + i\rho^{-2}}$ and $\omega_l \rho \to \omega \sqrt{i} = (1 + i)/\sqrt{2}$ as $\rho \to +0$. Moreover, $A = \sqrt{\frac{\omega'}{\rho^{\delta+1}}} = \sqrt{\frac{\omega g''_0}{\rho^{\delta+1}} + \frac{\omega g''_0}{\rho^{\delta+1}}}$ and $\sqrt{\frac{1}{\rho^{\delta+1}}} = \frac{\sqrt{\rho^2}}{\rho^{\delta+1}} \to \frac{1}{\delta+1}$ as $\rho \to +0$. 

\[19\]
It only suffices to consider the case with \( n = 0 \). Let
\[
A'_1 = \frac{J_0(\omega(1+i)/\sqrt{2})}{J_0(\omega_a^2/2)} J'_0(\omega_a^2/2), \quad A'_2 = \frac{H_0^{(1)}(\omega(1+i)/\sqrt{2})}{J_0(\omega_a^2/2)} J'_0(\omega_a^2/2), \quad c'_0 = \frac{1+\frac{1}{\sqrt{2\ln a}}}{2}, \quad z_0 = \omega(1+i)/\sqrt{2}.
\]
The asymptotic counterpart to (2.32) for the present proof is
\[
\left\{ [A'_1 - c'_0 J'_0(z_0/2)] [H_0^{(1)'}(z_0)] + \rho^2 \frac{\sqrt{2}}{1 + i \frac{\omega}{2} H_0^{(1)}(z_0)} \right\} a_0
\]
\[
+ \left\{ [A'_1 - c'_0 J'_0(z_0/2)] [H_0^{(1)'}(z_0)] + \rho^2 \frac{\sqrt{2}}{1 + i \frac{\omega}{2} J'_0(z_0)] \right\} b_0
\]
\[
\sim \frac{2i}{\pi z_0} \frac{\eta_0}{\rho^2} A_3.
\]
By dropping the lower order terms in (3.6), we have
\[
\left\{ [A'_1 - c'_0 J'_0(z_0/2)] [H_0^{(1)'}(z_0)] - [A'_2 - c'_0 H_0^{(1)'}(z_0/2)] J'_0(z_0) \right\} a_0
\]
\[
+ \left\{ [A'_1 - c'_0 J'_0(z_0/2)] \frac{2i}{\pi} H_0^{(1)'}(z_0) - [A'_2 - c'_0 H_0^{(1)'}(z_0/2)] \frac{2i}{\pi} J'_0(z_0) \right\} \ln \frac{\omega \rho}{2} b_0
\]
\[
\sim \frac{2i}{\pi z_0} \frac{\eta_0}{\rho^2} A_3.
\]
Let
\[
g_0 = [A'_1 - c'_0 J'_0(z_0/2)] \frac{2i}{\pi} H_0^{(1)'}(z_0) - [A'_2 - c'_0 H_0^{(1)'}(z_0/2)] \frac{2i}{\pi} J'_0(z_0),
\]
\[
p_0 = [A'_1 - c'_0 J'_0(z_0/2)] H_0^{(1)'}(z_0) - [A'_2 - c'_0 H_0^{(1)'}(z_0/2)] J'_0(z_0).
\]
We have
\[
a_0 + g_0 \ln \frac{\omega \rho}{2} b_0 \sim \frac{2i}{\pi z_0 p_0} \frac{\eta_0}{\rho^2} A_3.
\]
Let \( m_0 = \frac{2i}{\pi z_0 p_0} \eta_0 A_3 \). By combining the above estimates with (2.35), we have
\[
\left\{ \begin{array}{l}
a_0 \sim \frac{\gamma_0}{\sqrt{2\ln m_0 \frac{H_0^{(1)}(\omega R)}{J'_0(\omega R)}}}, \\
J'_0(\omega R) \frac{H_0^{(1)}(\omega R)}{\ln m_0 \frac{J'_0(\omega R)}},
\end{array} \right.
\]
\[
\left\{ \begin{array}{l}
b_0 \sim \frac{\gamma_0 - m_0}{\sqrt{2\ln m_0 \frac{J'_0(\omega R)}}}, \\
H_0^{(1)}(\omega R) \frac{\ln m_0 \frac{J'_0(\omega R)}}{\gamma_0 - m_0}.
\end{array} \right.
\]
Finally, using (3.9), we have
\[
\left| a_0 J_0(\omega R) + b_0 H_0^{(1)}(\omega R) - \frac{\psi_0 J_0(\omega R)}{\omega J_0'(\omega R)} \right| = \mathcal{O}\left(\frac{1}{\rho^2 \ln \frac{\rho}{2}}\right).
\]

\[\square\]

**Remark 3.4.** In \(\mathbb{R}^3\), when \(g'_1 = I, q'_1 = 1 + i\rho^{-2+\delta}, 0 \leq \delta < 2\), and \(f_a = \eta_0\delta_0\), we have
\[
\|u_\rho - u_0\|_{H^{1/2}(\partial B_R)} \sim \rho^{-1}.
\]

(3.10)

### 3.1 The case with \(g'_1 = I, q'_1 = 1 + i\rho^{-2-\delta}, \delta > 0\)

We first consider the case without the delta-point source presented inside the cloaked region.

**Proposition 3.5.** In \(\mathbb{R}^2\), when \(g'_1 = I, q'_1 = 1 + i\rho^{-2-\delta}, \delta > 0\), and \(f_a = 0\), we have
\[
\|u_\rho - u_0\|_{H^{1/2}(\partial B_R)} \leq C \ln \rho \|\psi\|_{H^{-1/2}(\partial B_R)}, \quad \|u_R^+(\rho)\|_{H^{1/2}(\partial B_R)} \leq C \frac{\rho^{\delta/2}}{\ln \rho} \|\psi\|_{H^{-1/2}(\partial B_R)}.
\]

**Proposition 3.6.** In \(\mathbb{R}^3\), when \(g'_1 = I, q'_1 = 1 + i\rho^{-2-\delta}, \delta > 0\), and \(f_a = 0\), we have
\[
\|u_\rho - u_0\|_{H^{1/2}(\partial B_R)} \leq C \rho \|\psi\|_{H^{-1/2}(\partial B_R)}, \quad \|u_R^+(\rho)\|_{H^{1/2}(\partial B_R)} \leq C \rho^{\delta/2} \|\psi\|_{H^{-1/2}(\partial B_R)}.
\]

Next, we consider the case with a delta-point source presented inside the cloaked region.

**Proposition 3.7.** In \(\mathbb{R}^2\), when \(g'_1 = I, q'_1 = 1 + i\rho^{-2-\delta}, \delta > 0\), and \(f_a = \eta_0\delta_0\), we have
\[
\|u_\rho - u_0\|_{H^{1/2}(\partial B)} = \mathcal{O}(\ln \rho^{-1}).
\]

(3.11)

**Proof.** In this case, \(g'_1, q'_1, \omega_a\) are the same to those in Proposition 3.3. \(\omega_1 \rho = \omega \rho^{-\frac{\delta}{2}} \sqrt{\rho^{2+\delta} + i}
\]
and \(A = \sqrt{\frac{2\omega_1 \rho}{\delta}} = \rho^{-\frac{\delta}{2}} \sqrt{\frac{2\omega_0 \rho}{\delta} + i} \frac{1}{\sqrt{\delta}} = \rho^{\frac{\delta}{2}} \sqrt{\frac{2\omega_0 \rho}{\delta} + i}.\) Set \(l'_0 = l'_1 = \frac{\delta}{2}, c'_0 = \frac{1+i}{\sqrt{2\omega_1 \rho}}.\) By using (2.58), one can obtain the asymptotic counterpart to (2.32) for the present proof, and from which one can further show that
\[
\left| a_0 J_0(\omega R) + b_0 H_0^{(1)}(\omega R) - \frac{\psi_0 J_0(\omega R)}{\omega J_0'(\omega R)} \right| = \mathcal{O}\left(\frac{1}{\ln \frac{\rho}{2}}\right) + \mathcal{O}\left(\frac{\eta_0}{\ln \frac{\rho}{2}} e^{-\frac{l'_0 - \delta}{2} \rho}\right), \quad n \neq 0.
\]
and
\[
\left| a_n J_n(\omega R) + b_n H_n^{(1)}(\omega R) - \frac{\psi_n J_n(\omega R)}{\omega J_n'(\omega R)} \right| = \mathcal{O}(\rho^{2|n|}).
\]

Using the above results, the proof follows from a completely similar argument to that for Proposition 3.3.

\[\square\]

For the three-dimensional case, we have

**Proposition 3.8.** In \(\mathbb{R}^3\), when \(g'_1 = I, q'_1 = 1 + i\rho^{-2-\delta}, \delta > 0\) and \(f_a = \eta_0\delta_0\), we have
\[
\|u_\rho - u_0\|_{H^{1/2}(\partial B_R)} = \mathcal{O}(\rho).
\]

(3.12)
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