

# 4. Life Insurance

## 4.1 Survival Distribution And Life Tables

### Introduction

- $X$ , *Age-at-death*
- $T(x)$ , *time-until-death*
- *Life Table*
  - Engineers use life tables to study the reliability of complex mechanical and electronic systems.
  - Biostatistician use life tables to compare the effectiveness of alternative treatments of serious disease.
  - Demographers use life tables as tools in population projections.
  - **In this text, life tables are used to build models for insurance systems designed to assist individuals facing uncertainty about the time of their death.**

# Probability for the Age-at-Death

## 1 The Survival Function

$$s(x) = 1 - F_X(x) = 1 - \Pr(X \leq x) = \Pr(X > x), \quad x \geq 0.$$

$s(x)$  has traditionally been used as a starting point for further development in Actuarial science and demography. In statistics and probability, the d.f. usually plays this role.

$$\Pr(x < X \leq z) = F_X(z) - F_X(x) = s(x) - s(z).$$

## Time-until-Death for a Person Age $x$

The conditional probability that a newborn will die between the ages  $x$  and  $z$ , given survival to age  $x$ , is

$$\Pr(x < X \leq z | X > x) = \frac{F_X(z) - F_X(x)}{1 - F_X(x)} = \frac{s(x) - s(z)}{s(x)}$$

- The symbol  $(x)$  is used to denote a *life-age- $x$*
- The future lifetime of  $(x)$ ,  $X - x$  is denote by  $T(x)$ .

The symbols of Actuarial science

- ${}_tq_x = \Pr[T(x) \leq t], t \geq 0, \quad {}_tp_x = 1 - {}_tq_x = \Pr[T(x) > t], t \geq 0.$
- ${}_xp_0 = s(x), x \geq 0.$
- $q_x = \Pr[(x) \text{ will die within 1 year}], \quad p_x = \Pr[(x) \text{ will attain age } x + 1]$

- $(x)$  will die between ages  $x + t$  and  $x + t + u$ .

$${}_{t|u}q_x = \Pr[t < T(x) \leq t + u] = {}_{t+u}q_x - {}_tq_x = {}_t p_x - {}_{t+u} p_x.$$

If  $u = 1$ , the prefix is delete in  ${}_{t|u}q_x$  and denotes as  ${}_tq_x$ .

- 

$${}_t p_x = \frac{{}_{x+t} p_0}{{}_x p_0} = \frac{s(x+t)}{s(x)}$$

$${}_tq_x = 1 - \frac{s(x+t)}{s(x)}$$

- 

$$\begin{aligned} {}_{t|u}q_x &= \frac{s(x+t) - s(x+t+u)}{s(x)} \\ &= \left[ \frac{s(x+t)}{s(x)} \right] \left[ \frac{s(x+t) - s(x+t+u)}{s(x+t)} \right] \\ &= {}_t p_x \cdot {}_uq_{x+t} \end{aligned}$$

***Curtate-future-life time  $K(x)$***  A discrete random variable associated with the future lifetime is the number of future years completed by  $(x)$  prior to death.

$$\begin{aligned}\Pr[K(x) = k] &= \Pr[k \leq K(x) < k + 1] = \Pr[k < T(x) \leq k + 1] \\ &= {}_k p_x - {}_{k+1} p_x = {}_k p_x q_{x+k} \\ &= {}_k | q_x\end{aligned}$$

$$F_{K(x)}(y) = \sum_{h=0}^k h | q_x = {}_{k+1} q_x,$$

$y \geq 0$  and  $k$  is the greatest integer in  $y$

## Force of Mortality

$$\Pr(x < X \leq x + \Delta x | X > x) = \frac{F_X(x + \Delta x) - F_X(x)}{1 - F_X(x)} = \frac{f_X(x)\Delta x}{1 - F_X(x)}$$

### Definition of force of mortality

$$\mu(x) \hat{=} \frac{f_X(x)}{1 - F_X(x)} = \frac{-s'(x)}{s(x)}$$

In reliability theory,  $\mu(x)$  is called *failure rate* or *hazard rate* or, *hazard rate function*.

Force of mortality can be used to specify the distribution of  $X$ .

$${}_n p_x = \exp\left[-\int_0^n \mu(x+s) ds\right].$$

$${}_n p_0 = \exp\left[-\int_0^n \mu(s) ds\right].$$

$$F_X(x) = 1 - s(x) = 1 - \exp\left[-\int_0^x u(s) ds\right]$$

$$F'_X(x) = f_X(x) = \exp\left[-\int_0^x \mu(s) ds\right] \mu(x) = {}_x p_0 \mu(x).$$

$F_{T(x)}(t)$  and  $f_{T(x)}(t)$  denote, respectively, the d.f. and p.d.f. of  $T(x)$ , the future lifetime of  $(x)$ . Then we have

$$\begin{aligned} f_{T(x)}(t) &= \frac{d}{dt} {}_t q_x = \frac{d}{dt} \left[ 1 - \frac{s(x+t)}{s(x)} \right] \\ &= \frac{s(x+t)}{s(x)} \left[ -\frac{s'(x+t)}{s(x+t)} \right] \\ &= {}_t p_x \mu(x+t), \quad t \geq 0 \end{aligned}$$

## Probability Theory Functions for Age-at-Death, $X$

	d.f. $F_X(x)$	Survival Function $s(x)$		p.d.f. $f_X(x)$	Force of Mortality $\mu(x)$
For	Requirements		For	Requirements	
$x < 0$	$F_X(x) = 0$	$s(x) = 1$	$x < 0$	$f_X(x) = 0$	$\mu(x) = 0$
$x = 0$	$F_X(0) = 0$	$s(0) = 1$	$x = 0$	undefined	undefined
$x \geq 0$	nondecreasing	nonincreasing	$x > 0$	$f_X(x) \geq 0$	$\mu(x) \geq 0$
$\lim_{x \rightarrow \infty}$	$F_X(\infty) = 1$	$s(\infty) = 0$	$\lim_{x \rightarrow \infty}$	$\int_0^\infty f_X(x) dt = 1$	$\int_0^\infty \mu(x) dx = \infty$
Functions in Terms of	Relationships				
$F_X(x)$	$F_X(x)$	$1 - F_X(x)$	$F'_X(x)$	$F'_X(x) / [1 - F_X(x)]$	
$s(x)$	$1 - s(x)$	$s(x)$	$-s'(x)$	$-s'(x) / s(x)$	
$f_X(x)$	$\int_0^x f_X(u) du$	$\int_x^\infty f_X(u) du$	$f_X(x)$	$f_X(x) / \int_x^\infty f_X(u) du$	
$\mu(x)$	$1 - \exp[-\int_0^x \mu(t) dt]$	$\exp[-\int_0^x \mu(t) dt]$	$\mu(x) \exp[-\int_0^x \mu(t) dt]$	$\mu(x)$	



**Example**  $\bar{A}$  refer to the complement of the event  $A$  within the sample space and  $\Pr(\bar{A}) \neq 0$ , the following expression and identity in probability theory:

$$\Pr(A \cup B) = \Pr(A) + \Pr(\bar{A}) \Pr(B|\bar{A}).$$

Rewrite this identity in actuarial notation for the events  $A = [T(x) \leq t]$  and  $B = [t < T(x) \leq 1]$ ,  $0 < t < 1$ .

**Solution:**  $\Pr(A \cup B)$  becomes  $\Pr[T(x) \leq 1] = q_x$ ,  $\Pr(A)$  is  ${}_tq_x$ , and  $\Pr[B|\bar{A}]$  is  ${}_{1-t}q_{x+t}$ , hence

$$q_x = {}_tq_x + {}_tp_x {}_{1-t}q_{x+t}.$$

# Life Tables

## Ration of Life Table Functions to the Survival Function

$${}_tq_x = 1 - \frac{s(x+t)}{s(x)} \Rightarrow q_x = 1 - \frac{s(x+1)}{s(x)}$$

A group of  $l_0 = 100,000$  newborns. Let  $\mathcal{L}(x)$  denote the group's number of survivors to age  $x$ . We index these lives by  $j = 1, 2, \dots, l_0$  and observe that

$$\mathcal{L}(x) = \sum_{j=1}^{l_0} I_j$$

where  $I_j$  is an indicator for the survival of life  $j$ . Hence

$$l_x = \mathbb{E}[\mathcal{L}(x)] = \sum_{j=1}^{l_0} \mathbb{E}[I_j] = l_0 s(x).$$

${}_n\mathcal{D}_x$  denotes the number of deaths between ages  $x$  and  $x + n$  from among the initial  $l_0$  lives.

$${}_nd_x = \mathbb{E}[{}_n\mathcal{D}_x] = l_0[s(x) - s(x + n)] = l_x - l_{x+n}$$

From above equations, we also have

$$-\frac{1}{l_x} \frac{dl_x}{dx} = -\frac{1}{s(x)} \frac{ds(x)}{dx} = \mu(x) \quad \text{and} \quad -dl_x = l_x \mu(x) dx.$$

Since

$$l_x \mu(x) = l_{0x} p_0 \mu(x) = l_0 f_X(x)$$

We note further that

$$l_x = l_0 \exp\left[-\int_0^x \mu(y) dy\right],$$

$$l_{x+n} = l_x \exp\left[-\int_x^{x+n} \mu(y) dy\right],$$

$$l_x - l_{x+n} = \int_x^{x+n} l_y \mu(y) dy$$

## Life Table Example

### *Life Table for the Total Population: United States, 1979-81*

- The 1979-81 U.S. Life Table was not construct by observing 100,000 newborns until the last survival died.
- Instead, it was base on estimates of probabilities of death, given survival to various ages, derived from the experience of entire U.S. population in the years around 1980 census.
- In using the random survivorship group concept with this table, we must make the assumption that the probabilities derived from the table will be appropriate for the lifetimes of those who belong to the survivorship group.
- Most studies of human mortality for insurance purpose use the representation  $s(x) = l_x/l_0$ . Since  $100,000s(x)$  is displayed only for integer value of  $x$ , there is a need to interpolate in evaluating  $s(x)$  for noninteger values.

**Example** On the basis of The 1979-81 U.S. Life Table, evaluate the probability that (20) will

- a. Live to 100
- b. Die before 70
- c. Die in the tenth decade of Life.

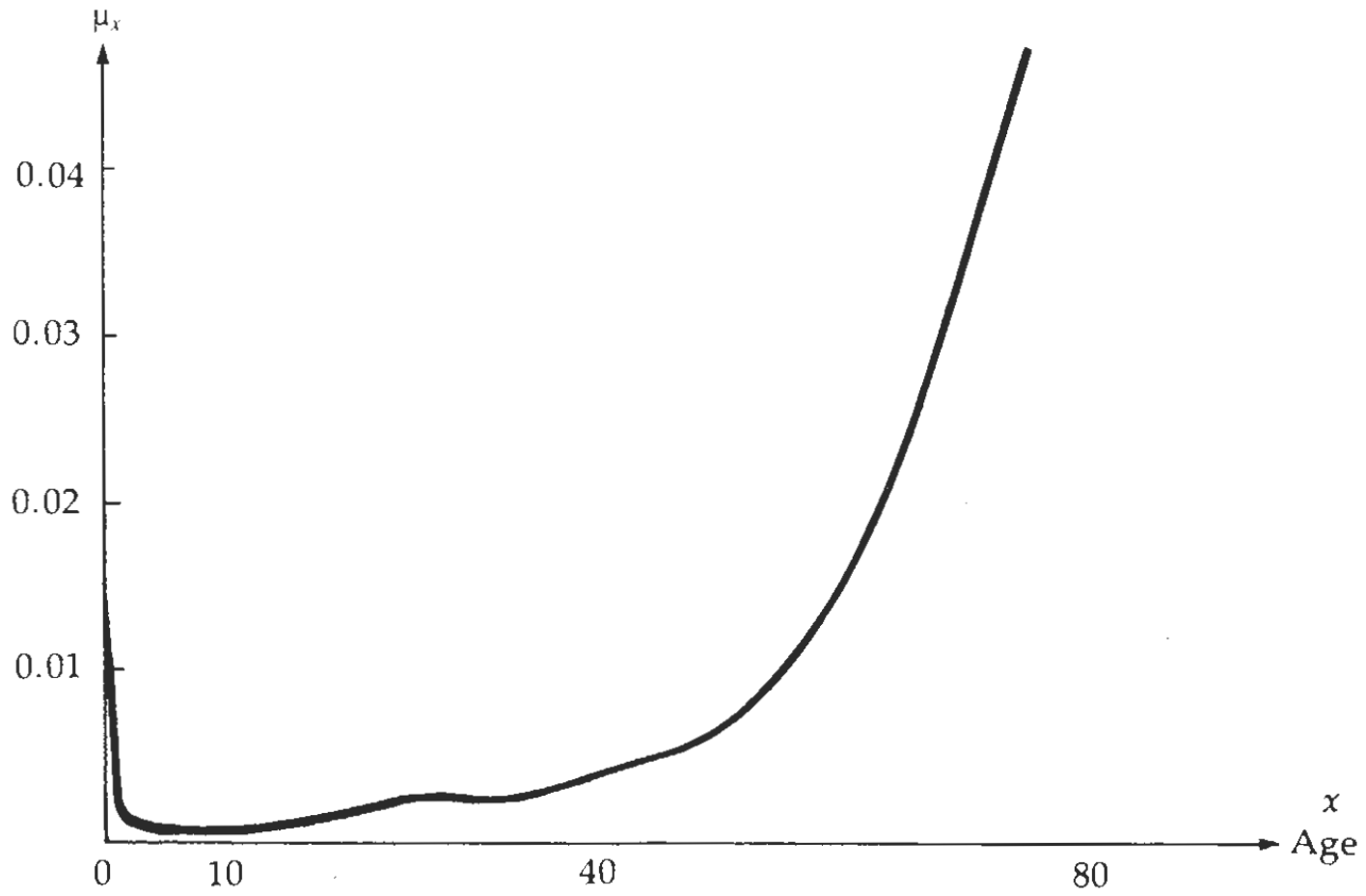
**Soltion:**

$$\text{a. } \frac{s(100)}{s(20)} = \frac{l_{100}}{l_{20}} = \frac{1,150}{97,741} = 0.0118.$$

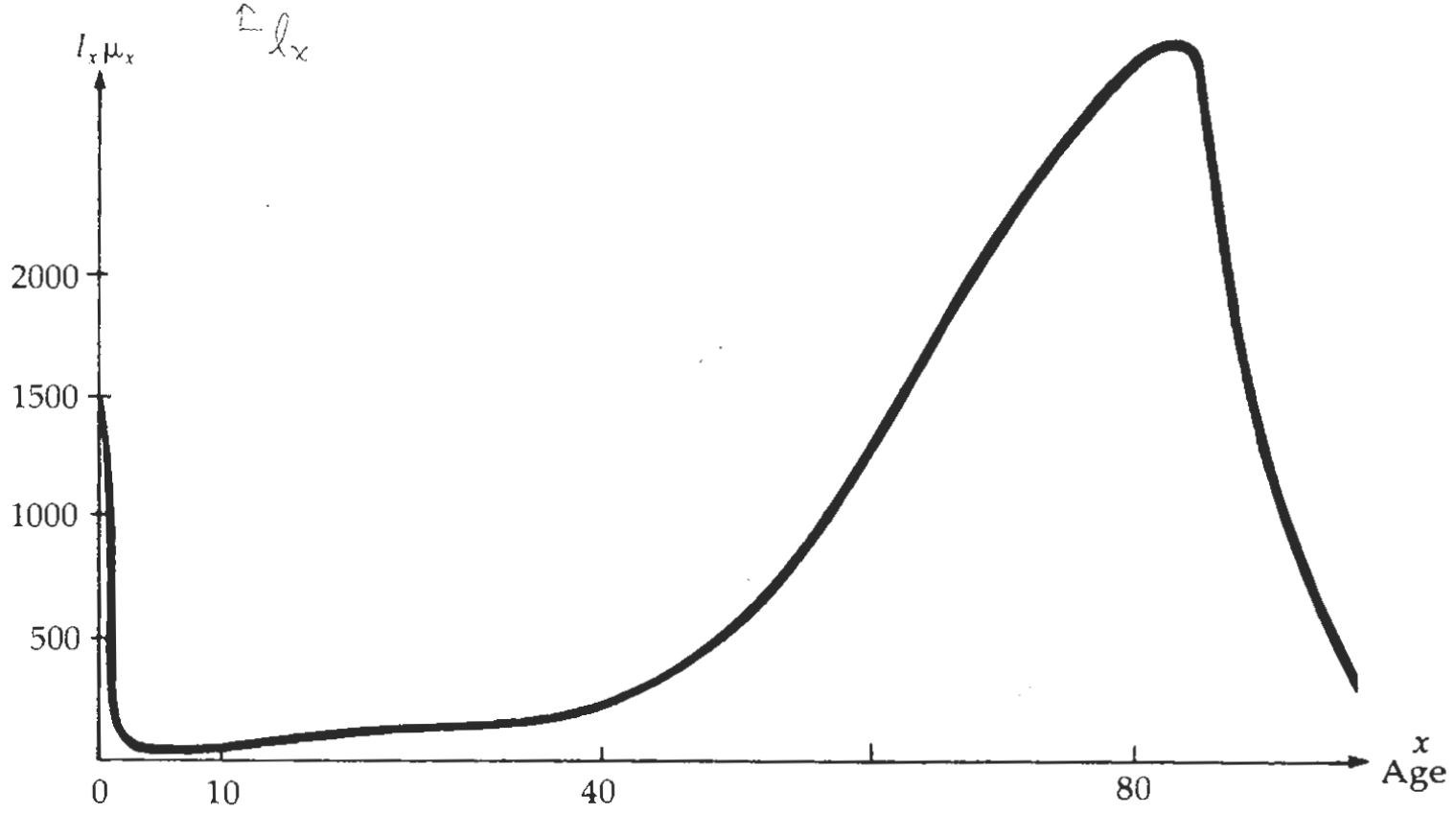
$$\text{b. } \frac{s(70) - s(20)}{s(20)} = 1 - \frac{l_{70}}{l_{20}} = 1 - \frac{68,248}{97,741} = 0.3017$$

$$\text{c. } \frac{s(90) - s(100)}{s(20)} = \frac{l_{90} - l_{100}}{l_{20}} = \frac{14,154 - 1,150}{97,741} = 0.1330.$$

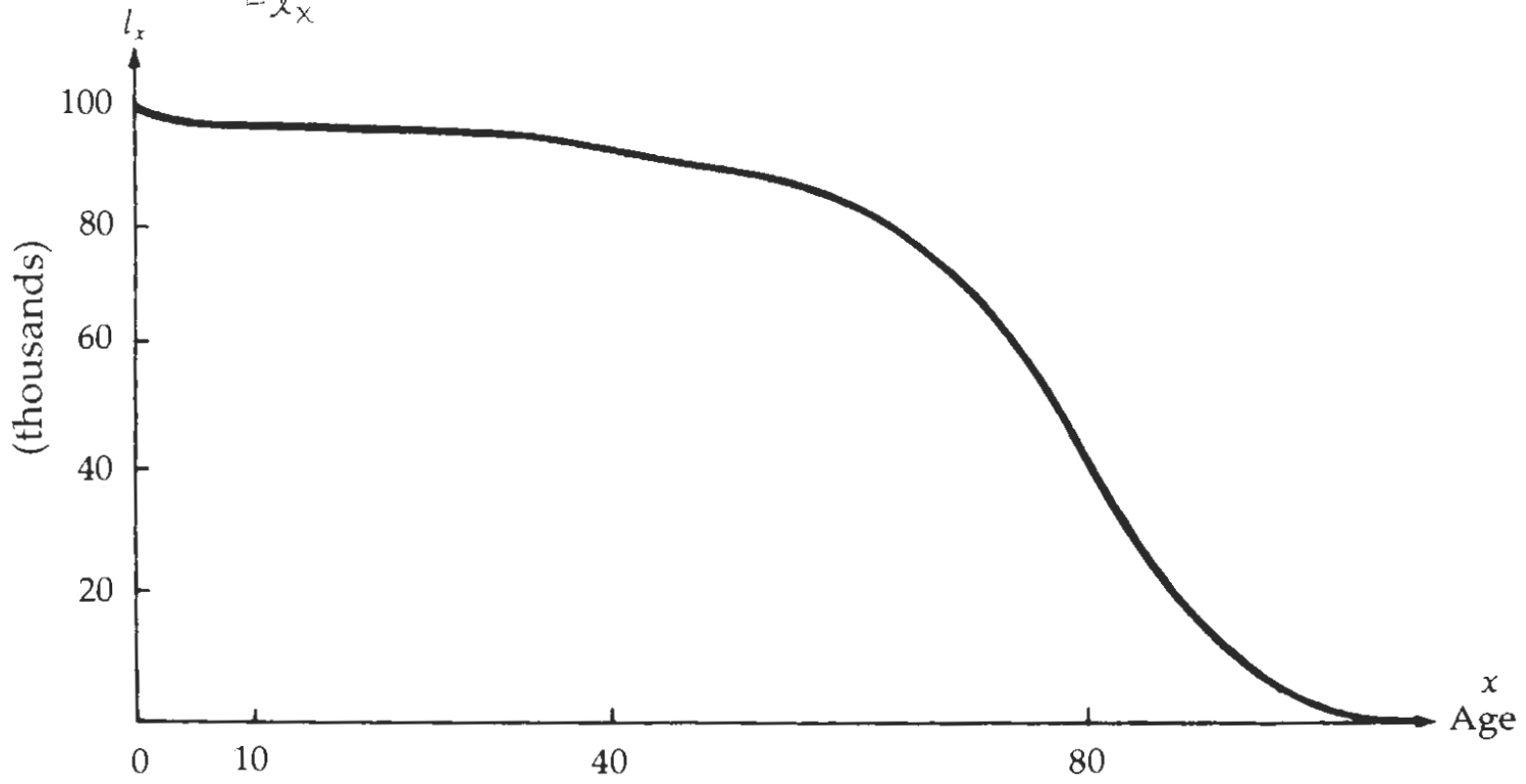
## Graph of $\mu(x)$



# Graph of $l_x \mu(x)$



Graph of  $l_x$   
 $\uparrow l_x$





- The function  $l_x\mu(x)$  is proportional to the p.d.f of the age-at-death of a newborn. Since  $l_x\mu(x)$  is the expected density of death at age  $x$ , under the random survivorship group idea, the graph of  $l_x\mu(x)$  is called the curve of deaths.
- There is a local minimum of  $l_x\mu(x)$  at about age 10. The mode of the distribution of deaths — the age at which the maximum of the curve of death occurs — is around age 80.
- The function  $l_x$  is proportional to the survival function  $s(x)$ . It can also be interpreted as the expected number living at age  $x$  out of an initial group of size  $l_0$ .
- Local extreme points of  $l_x\mu(x)$  correspond to points inflection of  $l_x$  since

$$\frac{d}{dx}l_x\mu(x) = \frac{d}{dx} \left( -\frac{d}{dx}l_x \right) = -\frac{d^2}{dx^2}l_x.$$

## 2. Life Insurance

**Introduction** In this section we develop models for life insurances designed to reduce the financial impact of the random event of untimely death.

Due to the long-term nature of the insurance, the amount of investment earnings, up to the time of payment, provides a significant element of uncertainty.

- The unknown rate of earning over.
- The unknown length of, the investment period.

In this chapter a deterministic model is used for the unknown investment earnings. Our model will be built in terms of function  $T$ , the insured's future-lifetime random variable.

## Insurances Payable at the Moment of Death

- $b_t$ , a benefit function
- $v_t$ , a discount function
- $z_t = b_tv_t$ , the present-value function.
- $T = T(x)$ , The elapsed time from policy issue to the death of the issued is the insured's future-lifetime random variable.
- $Z = b_Tv_T$ , the present value, at policy issue, of the benefit payment which is a random variable.

The first step in our analysis of a life insurance will be to define  $b_t$  and  $v_t$ . The next step is to determine some characteristics of the probability distribution of  $Z$  that are consequences of an assumed distribution for  $T$ , and we work through these steps for several conventional insurances.

## Level Benefit Insurance

**n-year term life insurance** A payment only if the insured dies within the n-year term of an insurance commencing at issue.

If a unit is payable at the moment of the death ( $x$ ), then

$$b_t = \begin{cases} 1, & t \leq n \\ 0, & t > n, \end{cases} \quad v_t = v^t, t \geq 0, \quad Z = \begin{cases} v^T, & T \leq n \\ 0, & T > n, \end{cases}$$

Where the force of interest is assumed to be constant.

**Actuarial present value** The expectation of the present-value random variable  $Z$  in insurance. ( The expected loss was called the pure premium. This word is commonly used in property-liability insurance.)

## Expected value of $Z$

$$\bar{A}_{x:\bar{n}|}^1 = \mathbb{E}[Z] = \mathbb{E}[z_T] = \int_0^\infty z_t f_T(t) dt = \int_0^n v^t {}_t p_x \mu_x(t) dt.$$

## Rule of Moments

$$\mathbb{E}[Z^j] = \int_0^n (v^t)^j {}_t p_x \mu_x(t) dt = \int_0^n e^{-(\delta j)t} {}_t p_x \mu_x(t) dt$$

Hence

$$\mathbb{E}[Z^j] @ \delta_t = \mathbb{E}[Z] @ j \delta_t$$

and

$$\text{Var}(Z) = {}^2\bar{A}_{x:\bar{n}|}^1 - (\bar{A}_{x:\bar{n}|}^1)^2$$

where  ${}^2\bar{A}_{x:\bar{n}|}^1$  is the actuarial present value for an n-year term insurance for a unit amount calculated at force of interest  $2\delta$ .

**Whole life insurance** provides for a payment following the death of the insured at any time in the future. If the payment is to be a unit amount at the moment of death ( $x$ ), then

$$b_t = 1, \quad v_t = v^t, \quad Z = v^T, \quad t \geq 0, T \geq 0.$$

The actuarial present value is

$$\bar{A}_x = E[Z] = \int_0^\infty v^t {}_t p_x \mu_x(t) dt.$$

For a life selected at  $x$  and now age  $x + h$ , the expression would be

$$\bar{A}_{[x]+h} = \int_0^\infty v^t {}_t p_{[x]+h} \mu_x(h + t) dt.$$

Whole life insurance is the limiting case of  $n$ -year term insurance as  $n \rightarrow \infty$ .

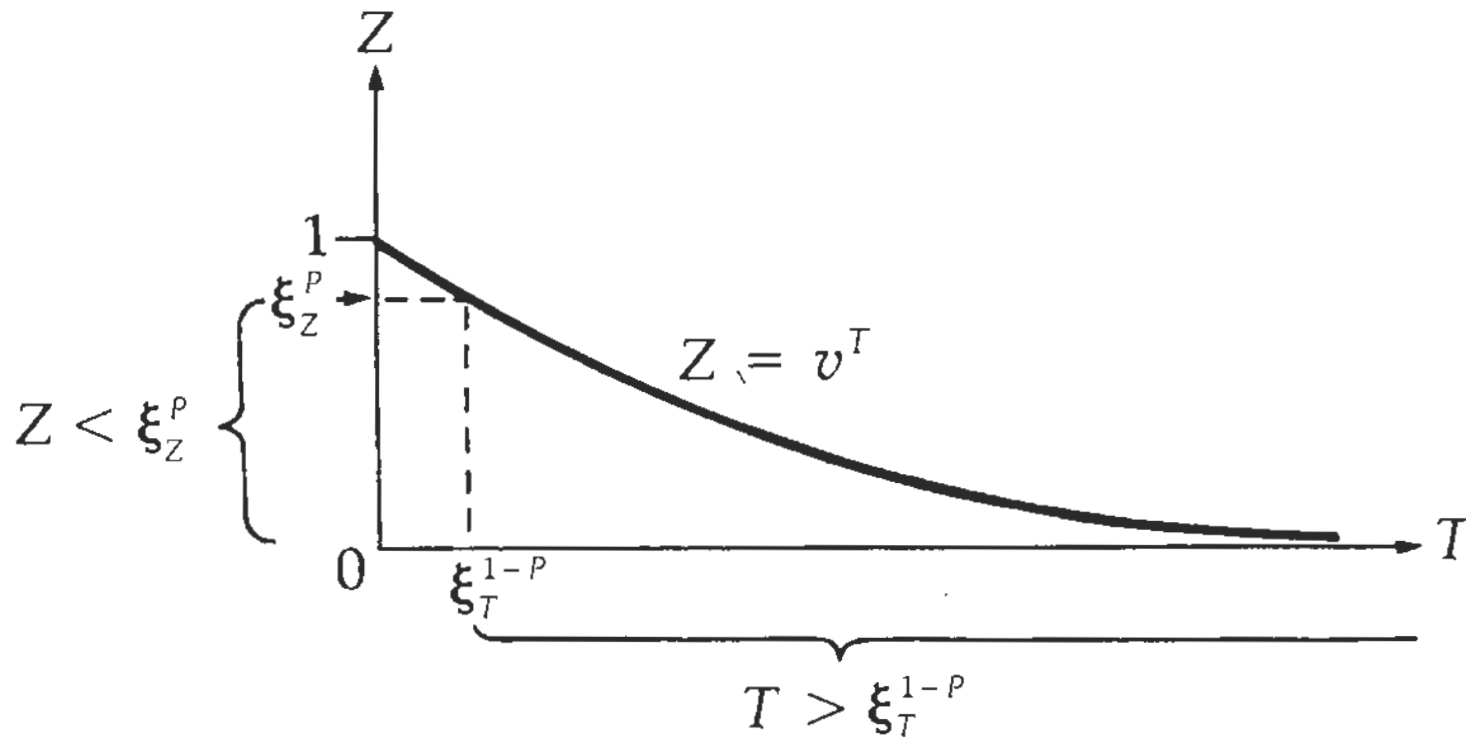
**Example 1.** The p.d.f of the future lifetime  $T$ , for  $(x)$  is assumed to be

$$f_T(t) = \begin{cases} 1/80, & 0 \leq t \leq 80 \\ 0, & \text{elsewhere.} \end{cases}$$

At a force of interest  $\delta$ , calculate  $Z$ , the present value random variable for a whole life insurance of unit amount issued to  $(x)$ :

- a. The actuarial present value
- b. The variance
- c. The 90th percentile ,  $\xi_Z^{0.9}$ .

## Relationship of $Z$ to $T$ for Whole Life Insurance





**Example 2.** For the assumption in Example 1, determine

a.  $Z$ 's d.f.

b.  $Z$ 's p.d.f

**Example 3.** Assume that each 100 independent lives

- Is age  $x$
- Is subject to a constant force of mortality,  $\mu = 0.04$ , and
- Is insured for a death benefit amount of 10 units, payable at the moment of death.

The benefit payments are to be withdrawn from an investment fund earning  $\delta = 0.06$ . Calculate the minimum amount at  $t = 0$  so that the probability is approximately 0.95 the sufficient funds will be on hand to withdraw the benefit payment at the death of each individual.

**Solution:**

$$b_t = 10, v_t = v^t, Z = 10v^T, t \geq 0, T \geq 0$$

$$S = \sum_{j=1}^{100} Z_j$$

where  $Z_j$  is the present value at  $t = 0$  for the payment to be made at the death of the life numbered  $j$ .

$$\mathbb{E}[Z] = 10\bar{A}_x = 10 \int_0^{\infty} e^{-\delta t} e^{-\mu t} \mu dt = \frac{10\mu}{\mu + \delta} = 10 \frac{0.04}{0.1} = 4,$$

$$\mathbb{E}[Z^2] = 10^2 {}^2\bar{A}_x = 100 \frac{0.04}{0.04 + 2(0.06)} = 25.$$

and  $\text{Var}(Z) = 9$ . Hence

$$\mathbb{E}[S] = 100(4) = 400, \quad \text{Var}(S) = 100(9) = 900.$$

Analytically, the required minimum amount is a number,  $h$ , such that

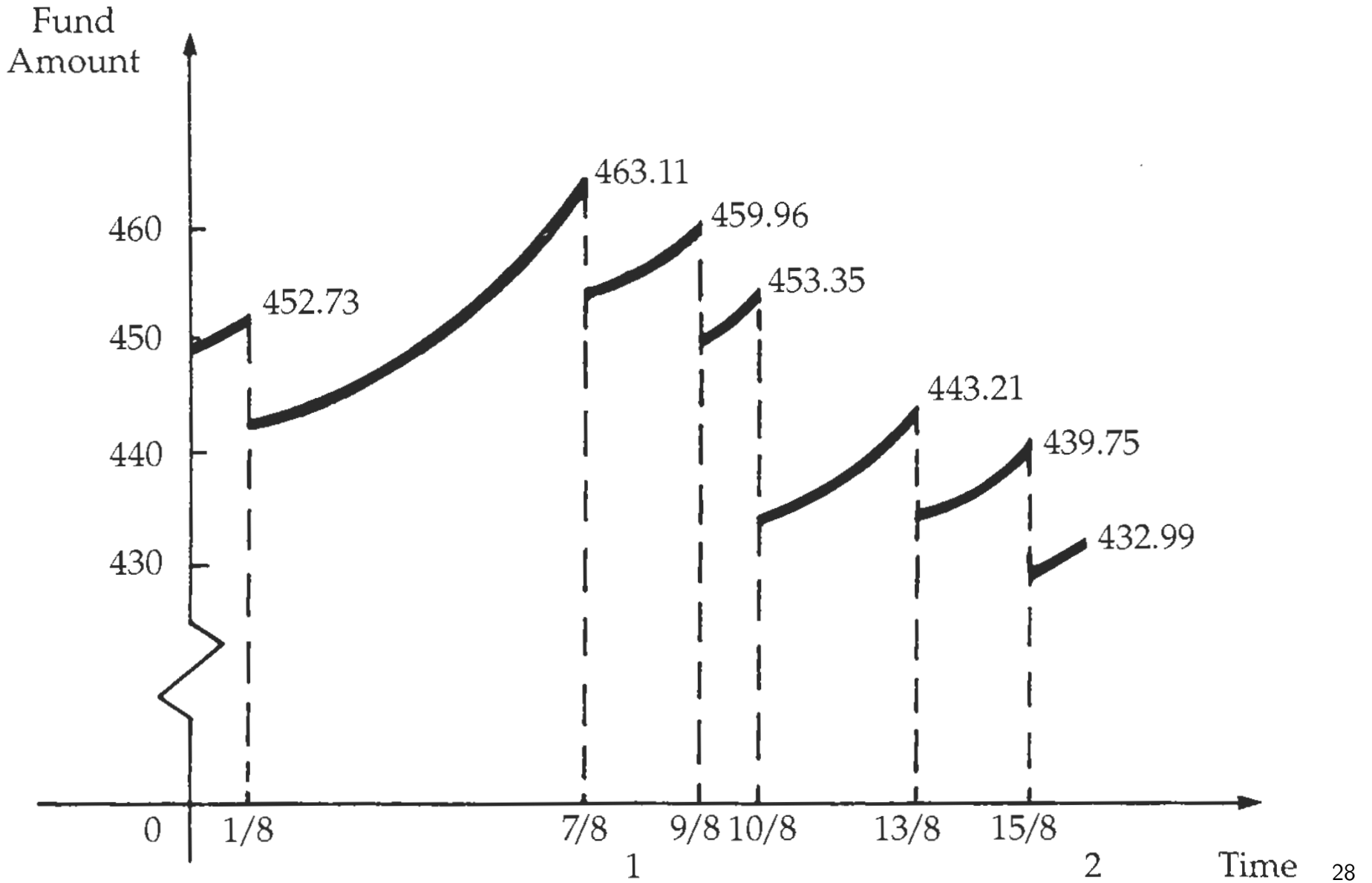
$$\Pr(S \leq h) = \Pr \left[ \frac{S - \mathbb{E}[S]}{\sqrt{\text{Var}(S)}} \leq \frac{h - 400}{30} \right] = 0.95$$

Using normal approximation, we obtain  $\frac{h-400}{30} = 1.645 \Rightarrow h = 449.35$ .

## Observations:

- The 49.35 difference between this initial fund of 449.35 and the expectation of the present value of all payments, 400, is the risk loading.
- This example used the individual risk model and a normal approximation to the probability distribution  $S$ .
- A graph of the amount in the fund during the first 2 years for a payout pattern when one death occurs at each time  $1/8, 7/8, 9/8, 13/8$  and  $15/8$ , and two deaths occur at time  $10/8$  is shown in Figure 1. Between the benefit payments, represented by the discontinuities, are exponential arcs representing the growth of the fund at  $\delta = 0.06$ .
- There are infinitely many payout patterns, each with its own graph. Both the number of claims and times of those claims affect the fund. For example, had the seven claims all occurred within the first instant, instead of the payout pattern of the Figure, the fund would have dropped immediately to 379.35 and grown to 427.72 by the end of the second year.

# Graph of an Outcome for the Fund



## Endowment Insurance

**An  $n$ -year pure endowment** provides for a payment at the end of the  $n$  years if and only if the insured survives at least  $n$  years from the time of policy issue.

$$b_t = \begin{cases} 0, & t \leq n \\ 1, & t > n, \end{cases} \quad v_t = v^n, t \geq 0, \quad Z = \begin{cases} 0, & T \leq n \\ v^n, & T > n, \end{cases}$$

$$A_{x:\overline{n}|}^1 = E[Z] = v^n E[Y] = v^n {}_n p_x$$

and

$$\text{Var}(Z) = v^{2n} \text{Var}(Y) = v^{2n} {}_n p_x {}_n q_x = {}^2 A_{x:\overline{n}|}^1 - (A_{x:\overline{n}|}^1)^2.$$

where  $Z = v^n Y$ ,  $Y$  is the indicator of the event of survival to age  $x + n$ .  $Y$  has the value 1 if the insured survives to age  $x + n$  and has the value 0 otherwise.

**N-year endowment insurance** provides for an amount to be payable either following the death of the insured or upon the survival of the insured to the end of the n-year term, whichever occurs first. If the insurance is for a unit amount and the death benefit is payable at the moment of death, then

$$b_t = 1, t \geq 0, \quad v_t = \begin{cases} v^t, & t \leq n \\ v^n, & t > n, \end{cases} \quad Z = \begin{cases} v^T, & T \leq n \\ v^n, & T > n, \end{cases}$$

The actuarial present value is denoted by  $\bar{A}_{x:\bar{n}|}$  and

$$\text{Var}(Z) = {}^2\bar{A}_{x:\bar{n}|} - (\bar{A}_{x:\bar{n}|})^2.$$

and

$$\bar{A}_{x:\bar{n}|} = \bar{A}_{x:\bar{n}|}^1 + A_{x:\bar{n}|}^1$$

$$\text{Var}(Z) = {}^2\bar{A}_{x:\bar{n}|}^1 - (\bar{A}_{x:\bar{n}|}^1)^2 + {}^2A_{x:\bar{n}|}^1 - (A_{x:\bar{n}|}^1)^2 - 2\bar{A}_{x:\bar{n}|}^1 A_{x:\bar{n}|}^1.$$

## Deferred Insurance

**M-year deferred insurance** provides for a benefit following the death of the insured only if the insured dies at least  $m$  years following policy issue.

$$b_t = \begin{cases} 1, & t > m \\ 0, & t \leq m, \end{cases} \quad v_t = v^t, t \geq 0, \quad Z = \begin{cases} v^T, & T > m \\ 0, & T \leq m, \end{cases}$$

The actuarial present value is

$${}_m|\bar{A}_x = \int_m^{\infty} v^t {}_t p_x \mu_x(t) dt.$$

**Example 4.** Consider a 5-year deferred whole life insurance payable at the moment of the death of  $(x)$ . The individual is subject to a constant force of mortality  $\mu = 0.04$ . For the distribution of the present value of the benefit payment, at  $\delta = 0.10$ :

- Calculate the expectation
- Calculate the variance
- Display the distribution function
- Calculate the median  $\xi_Z^{0.5}$ .

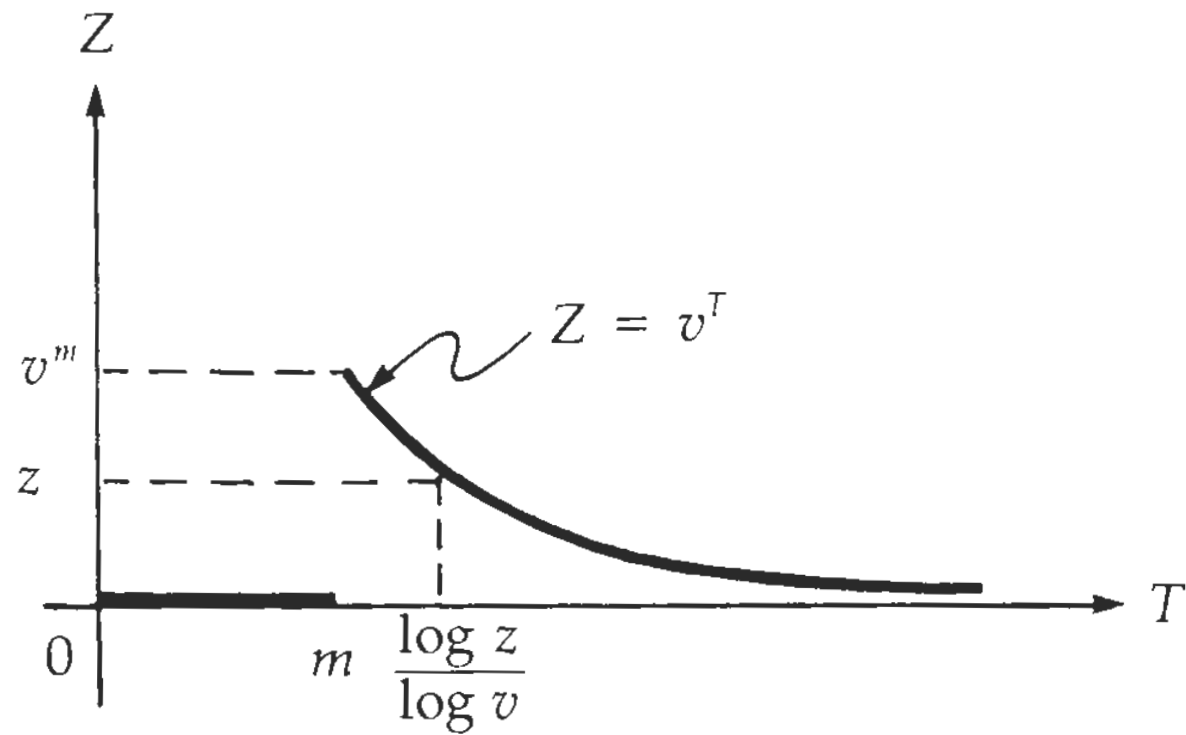
### Observations:

1. The largest value of  $Z$  with nonzero probability density in this example is  $e^{-0.1(5)} = 0.6065$ , corresponding to  $T = 5$ .
2. The distribution of  $Z$  in this example is highly skewed to the right. While its total mass is in the interval  $[0, 0.6065]$  and its mean is 0.1419, its median is only 0.0573. This skewness in the direction of large positive values is characteristic of many claim distributions in all field of insurance.



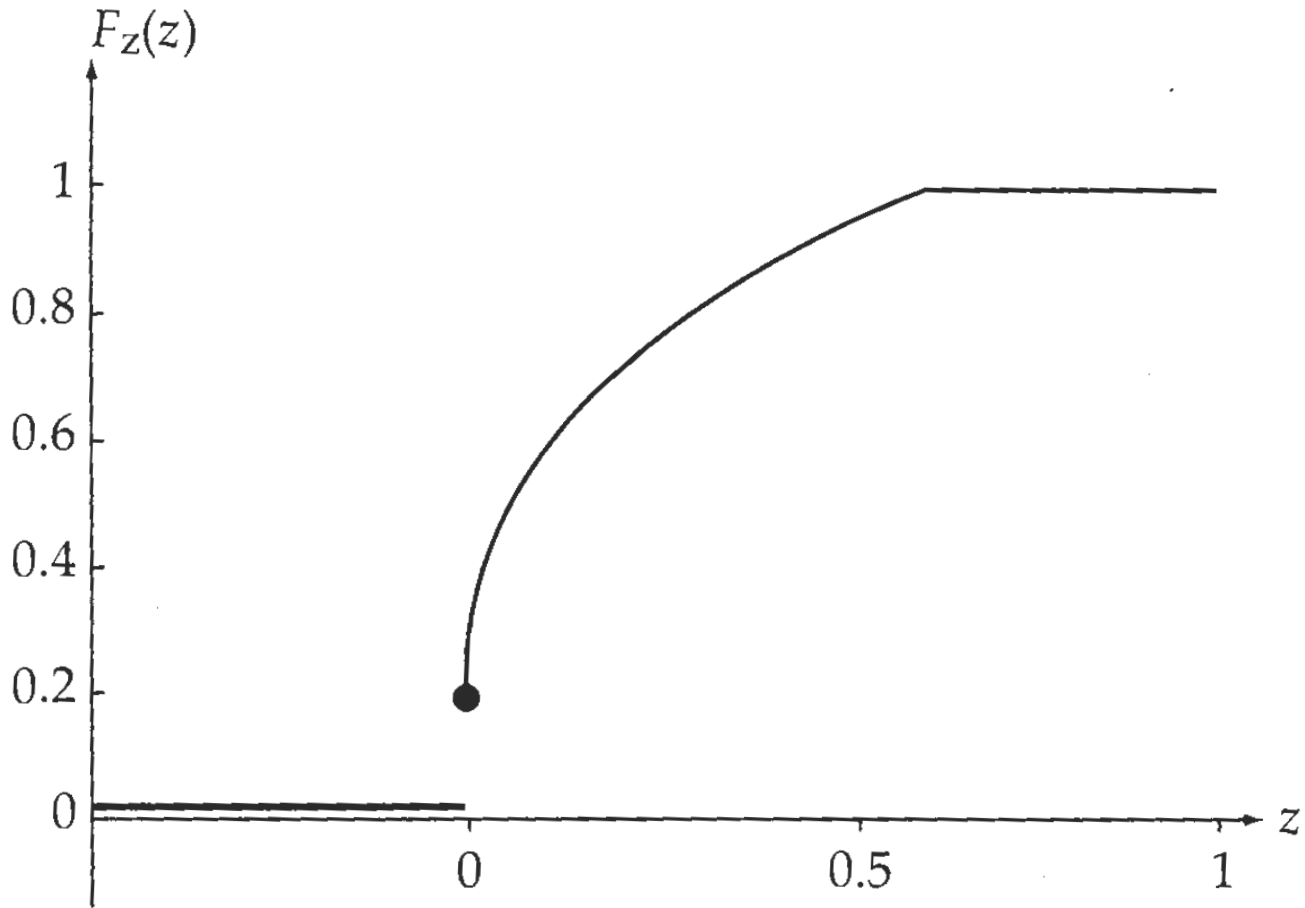
Figure 12.12

## Relationship of $Z$ to $T$ for Deferred Whole Life Insurance



QUESTION

## Distribution Function of Z



# 3. Life Annuities

## Introduction

**Life Annuity:** a series of payments made continuously or at equal interval (such as months, quarters, years) while a given life survives.

- It may be temporary, that is, limited to a given term of years. Or it may be payable for the whole of life.
- The payment intervals may commence immediately or, alternatively, the annuity may be deferred.
- Payment may be due at the beginning of the payment intervals(**annuities-due**) or at the end of such interval (**annuities-immediate**).

## **Life annuities play a major role in life insurance operations**

- Life insurances are usually purchased by a life annuity of premiums rather than by a single premium.
- The amount payable at the time of claim may be converted through a settlement option into some form of life annuity for the beneficiary.
- Some type of life insurance carry this concept even further and instead of featuring a lump sum payable on death, provide stated forms of income benefits. Thus for example, there may be monthly income payable to a surviving spouse or to a retired insured.

## **Annuities are even more central in pension systems**

In fact, a retirement plan can be regards as a system for purchasing deferred life annuities (payable during retirement) by some form of temporary annuity of contributions during active service.

## Life annuities also have a role in disability and worker's compensation insurances

- In the case of disability insurance, termination of the annuity benefit by reason of recovery of the disabled insured may need to be considered.
  - For surviving spouse benefits under workers' compensation, remarriage may terminate the annuity.
- In this chapter, we express the present value of benefits to be received by the annuitant as a function of  $T$ , the annuitant's future lifetime random variable.
- As in the preceding chapter on life insurance, unless otherwise stated we assume a constant effective annual rate of interest  $i$

## Continuous Life Annuities

- Annuities payable continuously at the rate of 1 per year, this is as a practical matter closely approximates annuities payable on a monthly basis.
- **Whole life annuity** provides for payments until death.  
The present value of payment is  $Y = \bar{a}_{\overline{T}|}$  for all  $T \geq 0$  where  $T$  is the future lifetime of  $(x)$ .

$$\begin{aligned}
 F_Y(y) &= \Pr(Y \leq y) = \Pr(\bar{a}_{\overline{T}|} \leq y) = \Pr(1 - v^T \leq \delta y) \\
 &= \Pr(v^T \geq 1 - \delta y) = \Pr\left[T \leq \frac{-\log(1 - \delta y)}{\delta}\right] \\
 &= F_T\left(\frac{-\log(1 - \delta y)}{\delta}\right) \quad \text{for } 0 < y < \frac{1}{\delta}.
 \end{aligned}$$

$$f_Y(y) = \frac{d}{dy} F_T\left(\frac{-\log(1 - \delta y)}{\delta}\right) = \frac{f_T([-\log(1 - \delta y)]/\delta)}{1 - \delta y} \quad \text{for } 0 < y < \frac{1}{\delta}$$

## The actuarial present value for a continuous whole life annuity

$$\bar{a}_x = E[Y] = \int_0^{\infty} \bar{a}_{\bar{t}|} p_x \mu(x+t) dt = \int_0^{\infty} v^t {}_t p_x dt = \int_0^{\infty} {}_t E_x dt.$$

The current payment technique for determining an actuarial present value for an annuity gives

$$APV = \int_0^{\infty} v^t \Pr[\text{payments are being made at time } t] \times [\text{Payment rate time } t] dt.$$

## Backward recursion formula

$$\bar{a}_x = \int_0^1 v^t {}_t p_x dt + \int_1^{\infty} v^t {}_t p_x dt = \bar{a}_{x:\bar{1}|} + v p_x \int_0^{\infty} v^s {}_s p_{x+1} ds = \bar{a}_{x:\bar{1}|} + v p_x \bar{a}_{x+1}$$

## A relationship, familiar from compound interest theory

$$1 = \delta \bar{a}_{\bar{t}|} + v^t \Rightarrow 1 = \delta \bar{a}_{\bar{T}|} + v^T \Rightarrow 1 = \delta \bar{a}_x + \bar{A}_x.$$

### Variance of $\bar{a}_{\bar{T}|}$

$$\text{Var}(\bar{a}_{\bar{T}|}) = \text{Var}\left(\frac{1 - v^T}{\delta}\right) = \frac{\text{Var}(v^T)}{\delta^2} = \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{\delta^2}.$$

**Example 1.** Under the assumptions of a constant force of mortality,  $\mu$ , and of a constant force of interest,  $\delta$ , evaluate

- $\bar{a}_x = \text{E}[\bar{a}_{\bar{T}|}]$
- $\text{Var}(\bar{a}_{\bar{T}|})$
- The probability that  $\bar{a}_{\bar{T}|}$  will exceed  $\bar{a}_x$ .



**N-year temporary life annuity** of 1 per year, payable continuously while ( $x$ ) survives during the next  $n$  year, is

$$Y = \begin{cases} \bar{a}_{\overline{T}|}, & 0 \leq T < n \\ \bar{a}_{\overline{n}|}, & T \geq n \end{cases}$$

**The actuarial present value of an n-year temporary life annuity**

$$\bar{a}_{x:\overline{n}|} = \mathbb{E}[Y] = \int_0^n \bar{a}_{\overline{t}|} {}_t p_x \mu(x+t) dt + \bar{a}_{\overline{n}|} {}_n p_x = \int_0^n v^t {}_t p_x dt.$$

$$Y = \begin{cases} \bar{a}_{\overline{T}|} = \frac{1-v^T}{\delta}, & 0 \leq T < n \\ \bar{a}_{\overline{n}|} = \frac{1-v^n}{\delta}, & T \geq n \end{cases} \implies \mathbb{E}[Y] = \bar{a}_{x:\overline{n}|} = \mathbb{E} \left[ \frac{1-Z}{\delta} \right] = \frac{1 - \bar{A}_{x:\overline{n}|}}{\delta}$$

$$\text{Var}(Y) = \frac{\text{Var}Z}{\delta^2} = \frac{{}^2\bar{A}_{x:\overline{n}|} - (\bar{A}_{x:\overline{n}|})^2}{\delta^2} = \frac{2}{\delta} (\bar{a}_{x:\overline{n}|} - {}^2\bar{a}_{x:\overline{n}|}) - (\bar{a}_{x:\overline{n}|})^2.$$

## N- year deferred whole life annuity

$$Y = \begin{cases} 0 = \bar{a}_{\overline{T}|} - \bar{a}_{\overline{T}|}, & 0 \leq T < n \\ v^n \bar{a}_{\overline{T-n}|} = \bar{a}_{\overline{T}|} - \bar{a}_{\overline{n}|}, & T \geq n \end{cases}$$

$$\begin{aligned} {}_n|\bar{a}_x &= \mathbf{E}[Y] = \int_n^\infty v^n \bar{a}_{\overline{t-n}|} {}_t p_x \mu(x+t) dt \\ &= v^n {}_n p_x \int_0^\infty \bar{a}_{\overline{s}|} {}_s p_{x+n} \mu(x+n+s) ds = {}_n \mathbf{E}_x \bar{a}_{x+n}. \end{aligned}$$

$$\begin{aligned} \text{Var}(Y) &= \int_n^\infty v^{2n} (\bar{a}_{\overline{t-n}|})^2 {}_t p_x \mu(x+t) dt - ({}_n|\bar{a}_x)^2 \\ &= \frac{2}{\delta} v^{2n} {}_n p_x (\bar{a}_{x+n} - {}^2\bar{a}_{x+n}) - ({}_n|\bar{a}_x)^2 \end{aligned}$$

**N-year certain and life annuity** This is a whole life annuity with a guarantee of payments for the first  $n$  years.

$$Y = \begin{cases} \bar{a}_{\overline{n}|}, & 0 \leq T < n \\ \bar{a}_{\overline{T}|}, & T \geq n \end{cases}$$

$$\begin{aligned} \bar{a}_{\overline{x:\overline{n}|}} &= \mathbf{E}[Y] = \int_0^n \bar{a}_{\overline{n}|t} p_x \mu(x+t) dt + \int_n^\infty \bar{a}_{\overline{T}|t} p_x \mu(x+t) dt \\ &= {}_nq_x \bar{a}_{\overline{n}|} + \int_n^\infty \bar{a}_{\overline{T}|t} p_x \mu(x+t) dt = \bar{a}_{\overline{n}|} + \int_n^\infty v^t {}_t p_x dt. \end{aligned}$$

$Y$  is the sum of a constant  $\bar{a}_{\overline{n}|}$  and the random variable for the  $n$ -year deferred annuity. Hence

$$\bar{a}_{\overline{x:\overline{n}|}} = \bar{a}_{\overline{n}|} + {}_n|\bar{a}_x = \bar{a}_{\overline{n}|} + {}_n\mathbf{E}_x \bar{a}_{x+n} = \bar{a}_{\overline{n}|} + (\bar{a}_x - \bar{a}_{x:\overline{n}|})$$

**Further more**, since  $\text{Var}(Y - \bar{a}_{\overline{n}|}) = \text{Var}(Y)$ , the variance for the  $n$ -year certain and life annuity is the same as that of the  $n$ -year deferred annuity given above.

## An Expression of $\frac{d}{dx}\bar{a}_x$

$$\begin{aligned}\frac{d}{dx}\bar{a} &= \int_0^\infty v^t \left( \frac{\partial}{\partial x} {}_t p_x \right) dt = \int_0^\infty v^t {}_t p_x [\mu(x) - \mu(x+t)] dt \\ &= \mu(x)\bar{a}_x - \bar{A}_x = \mu(x)\bar{a}_x - (1 - \delta\bar{a}_x) \\ &= [\mu(x) + \delta]\bar{a}_x - 1.\end{aligned}$$

**Example 2.** Assuming that probabilities come from an aggregate table, obtain formulas for

$$a. \quad \frac{\partial}{\partial x} \bar{a}_{x:\bar{n}|} \qquad b. \quad \frac{\partial}{\partial n} {}_n | \bar{a}_x$$

**Solution:**

$$a. \quad \frac{\partial}{\partial x} \bar{a}_{x:\bar{n}|} = [\mu(x) + \delta]\bar{a}_{x:\bar{n}|} - (1 - {}_n E_x).$$

$$b. \quad \frac{\partial}{\partial n} {}_n | \bar{a}_x = -v^n {}_n p_x$$

## Summary of Continuous Life Annuities (Annuity of 1 per Annum Payable Continuously)

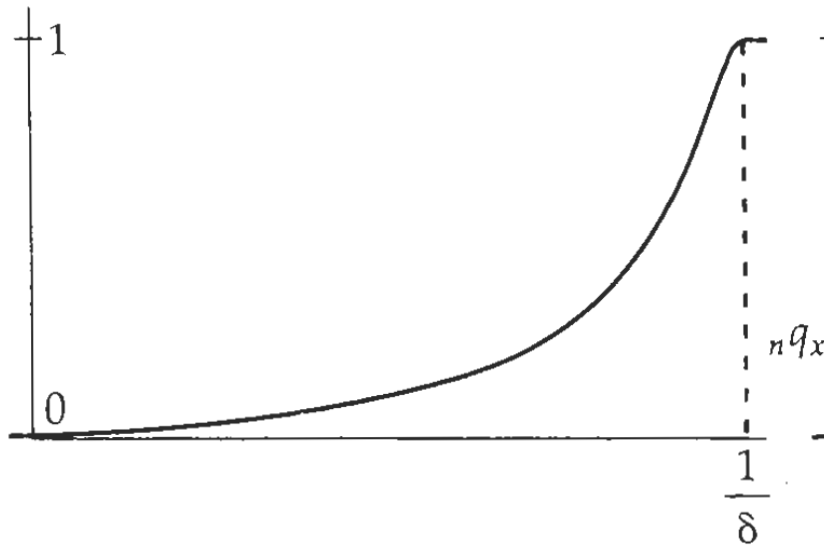
Annuity Name	Present-Value Random Variable $Y$	Actuarial Present Value $E[Y]$ Equal to
Whole Life Annuity	$\bar{a}_{\overline{T} }$ $T \geq 0$	$\bar{a}_x = \int_0^{\infty} v^t {}_t p_x dt$
$n$ -Year Temporary Life Annuity	$\begin{cases} \bar{a}_{\overline{T} } & 0 \leq T < n \\ \bar{a}_{\overline{n} } & T \geq n \end{cases}$	$\bar{a}_{x:\overline{n} } = \int_0^n v^t {}_t p_x dt$
$n$ -Year Deferred Whole Life Annuity	$\begin{cases} 0 & 0 \leq T < n \\ \bar{a}_{\overline{T} } - \bar{a}_{\overline{n} } & T \geq n \end{cases}$	${}_n \bar{a}_x = \int_n^{\infty} v^t {}_t p_x dt$
$n$ -Year Certain and Life Annuity	$\begin{cases} \bar{a}_{\overline{n} } & 0 \leq T < n \\ \bar{a}_{\overline{T} } & T \geq n \end{cases}$	$\bar{a}_{x:\overline{n} } = \bar{a}_{\overline{n} } + \int_n^{\infty} v^t {}_t p_x dt$

Additional relations are

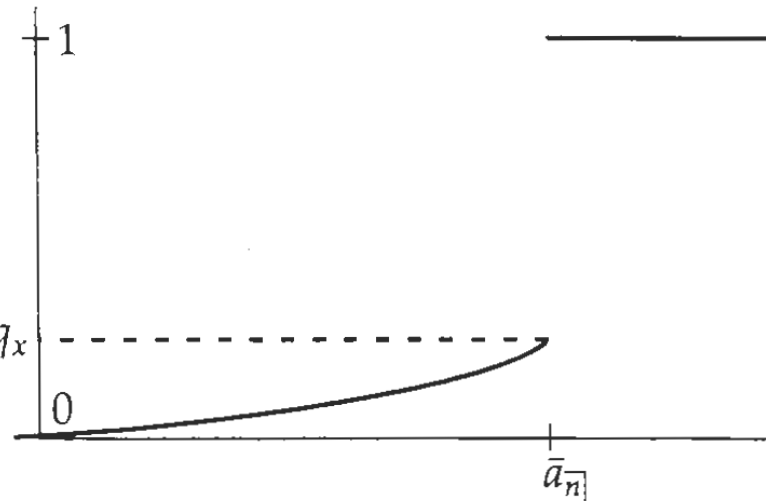
- $1 = \delta \bar{a}_x + \bar{A}_x$
- $1 = \delta \bar{a}_{x:\overline{n}|} + \bar{A}_{x:\overline{n}|}$
- ${}_n|\bar{a}_x = \bar{a}_x - \bar{a}_{x:\overline{n}|}$
- $\bar{s}_{x:\overline{n}|} = \frac{\bar{a}_{x:\overline{n}|}}{{}_n E_x} = \int_0^n (1+i)^{n-t} \frac{l_{x+t}}{l_{x+n}} dt.$

# Typical Distribution Functions for the Present-Value Random Variables, $Y$

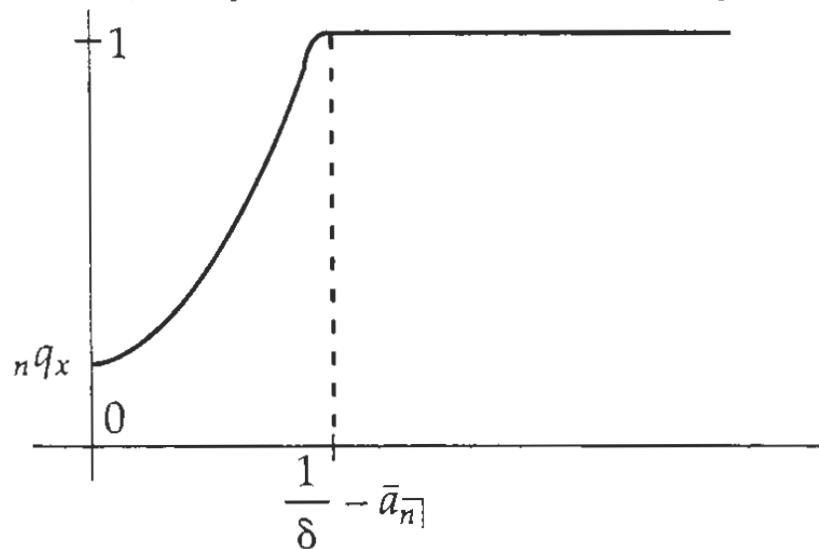
(a) Whole life annuity



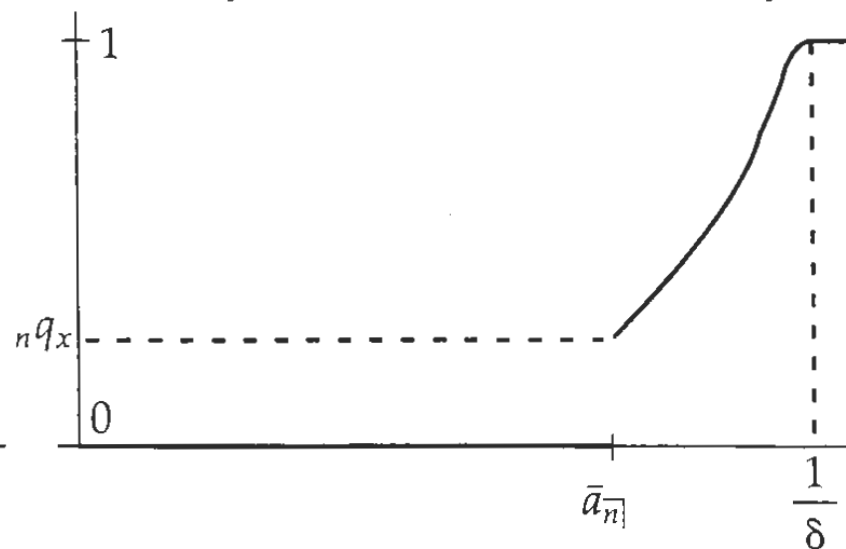
(b)  $n$ -year temporary life annuity



(c)  $n$ -year deferred life annuity



(d)  $n$ -year certain and life annuity



## Discrete Life Annuities

**Whole life annuity-due:** This annuity pays a unit amount at the beginning of each year that the annuitant ( $x$ ) survives. The present value random variable is denoted by

$$Y = \ddot{a}_{\overline{K+1}|}$$

where the random variable  $K$  is the curtate-future lifetime of ( $x$ ).

**The actuarial present value of the annuity:**

$$\begin{aligned}\ddot{a}_x &= E[Y] = E[\ddot{a}_{\overline{K+1}|}] = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} {}_k p_x q_{x+k} \\ &= 1 + \sum_{k=0}^{\infty} v^{k+1} {}_k p_x = \sum_{k=0}^{\infty} v^k {}_k p_x.\end{aligned}$$

The last term of the equation above is the current payment form of the actuarial present value for a whole life-annuity-due where the  ${}_k p_x$  term is the probability of a payment of size 1 being made at time  $k$ .

We can also obtain in succession

$$\ddot{a}_x = \mathbb{E} \left[ \frac{1 - v^{K+1}}{d} \right] = \frac{1 - A_x}{d}$$

and

$$\ddot{a}_x = \ddot{a}_{\infty|} - \ddot{a}_{\infty|} A_x, \quad 1 = d\ddot{a}_x + A_x.$$

and **the variance formula** is

$$\text{Var}(\ddot{a}_{\overline{K+1}|}) = \text{Var} \left( \frac{1 - v^{K+1}}{d} \right) = \frac{\text{Var}(v^{K+1})}{d^2} = \frac{{}^2A_x - (A_x)^2}{d^2}.$$



**N-year temporary life annuity-due** of 1 per year is

$$Y = \begin{cases} \ddot{a}_{\overline{K+1}|}, & 0 \leq K < n \\ \ddot{a}_{\overline{n}|}, & K \geq n \end{cases}$$

Its actuarial present value is

$$\ddot{a}_{x:\overline{n}|} = \mathbf{E}[Y] = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|} {}_k p_x q_{x+k} + \ddot{a}_{\overline{n}|} n p_x = \sum_{k=0}^{n-1} v^k {}_k p_x.$$

Since  $Y = (1 - Z)/d$ , where

$$Z = \begin{cases} v^{K+1}, & 0 \leq K < n \\ v^n, & K \geq n \end{cases}$$

is the present -value random variable for a unit of endowment insurance, payable at the end of the year or at maturity, we have

$$\ddot{a}_{x:\overline{n}|} = \frac{1 - \mathbf{E}[Z]}{d} = \frac{1 - A_{x:\overline{n}|}}{d} \implies 1 = d\ddot{a}_{x:\overline{n}|} + A_{x:\overline{n}|}.$$

To calculate the **variance**, we can use

$$\text{Var}(Y) = \frac{\text{Var}(Z)}{d^2} = \frac{{}^2A_{x:\bar{n}|} - (A_{x:\bar{n}|})^2}{d^2}.$$

**N-year deferred whole life annuity-due** of 1 payable at the beginning of each year while ( $x$ ) survives from age  $x + n$  onward, the present-value random variable is

$$Y = \begin{cases} 0, & 0 \leq K < n \\ n|\ddot{a}_{\overline{K+1-n}|}, & K \geq n \end{cases}$$

Its actuarial present value and variance are

$$\text{E}[Y] = n|\ddot{a}_x = {}_n\text{E}_x\ddot{a}_{x+n} = \ddot{a}_x - \ddot{a}_{x:\bar{n}|} = \sum_{k=n}^{\infty} v^k {}_k p_x.$$

$$\text{Var}(Y) = \frac{2}{d} v^{2n} {}_n p_x (\ddot{a}_{x+n} - {}^2\ddot{a}_{x+n}) + n|\ddot{a}_x - (n|\ddot{a})^2.$$

## N-year certain and life annuity-due

$$Y = \begin{cases} \ddot{a}_{\overline{n}|}, & 0 \leq K < n \\ \ddot{a}_{\overline{K+1}|}, & K \geq n \end{cases}$$

Its actuarial present value is

$$\ddot{a}_{\overline{x:\overline{n}|}} = \mathbb{E}[Y] = \ddot{a}_{\overline{n}|} {}_nq_x + \sum_{k=n}^{\infty} \ddot{a}_{\overline{k+1}|} {}_k p_x q_{x+k} = \ddot{a}_{\overline{n}|} + \sum_{k=n}^{\infty} v^k {}_k p_x = \ddot{a}_{\overline{n}|} + \ddot{a}_x - \ddot{a}_{x:\overline{n}|}.$$

The procedures used above for annuities-due can be adapted for annuities immediate where payments are made at the ends of the payment period. For instance, for a **whole life annuity-immediate**, the present-value random variable is  $Y = a_{\overline{K}|}$ . Then

$$a_x = \mathbb{E}[Y] = \sum_{k=0}^{\infty} {}_k p_x q_{x+k} a_{\overline{k}|} = \sum_{k=1}^{\infty} v^k {}_k p_x = \frac{1 - (1+i)A_x}{i}.$$

**Summary of Discrete Life Annuities [Annuity of 1 Per Annum Payable at the Beginning of Each Year (Annuity-Due) or at the End of Each Year (Annuity-Immediate)]**

Annuity Name	Present-Value Random Variable $Y$	Actuarial Present Value $E[Y]$ Equal to
Whole Life Annuity		
• Due	$\ddot{a}_{\overline{K+1} }$ $K \geq 0$	$\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_k p_x$
• Immediate	$a_{\overline{K} }$ $K \geq 0$	$a_x = \sum_{k=1}^{\infty} v^k {}_k p_x$
$n$ -Year Temporary Life Annuity		
• Due	$\begin{cases} \ddot{a}_{\overline{K+1} } & 0 \leq K < n \\ \ddot{a}_{\overline{n} } & K \geq n \end{cases}$	$\ddot{a}_{x:\overline{n} } = \sum_{k=0}^{n-1} v^k {}_k p_x$
• Immediate	$\begin{cases} a_{\overline{K} } & 0 \leq K < n \\ a_{\overline{n} } & K \geq n \end{cases}$	$a_{x:\overline{n} } = \sum_{k=1}^n v^k {}_k p_x$
$n$ -Year Deferred Whole Life Annuity		
• Due	$\begin{cases} 0 & 0 \leq K < n \\ \ddot{a}_{\overline{K+1} } - \ddot{a}_{\overline{n} } & K \geq n \end{cases}$	${}_n \ddot{a}_x = \sum_{k=n}^{\infty} v^k {}_k p_x$
• Immediate	$\begin{cases} 0 & 0 \leq K < n \\ a_{\overline{K} } - a_{\overline{n} } & K \geq n \end{cases}$	${}_n a_x = \sum_{k=n+1}^{\infty} v^k {}_k p_x$
$n$ -Year Certain and Whole Life Annuity		
• Due	$\begin{cases} \ddot{a}_{\overline{n} } & 0 \leq K < n \\ \ddot{a}_{\overline{K+1} } & K \geq n \end{cases}$	$\ddot{a}_{x:\overline{n} } = \ddot{a}_{\overline{n} } + \sum_{k=n}^{\infty} v^k {}_k p_x$
• Immediate	$\begin{cases} a_{\overline{n} } & 0 \leq K < n \\ a_{\overline{K} } & K \geq n \end{cases}$	$a_{x:\overline{n} } = a_{\overline{n} } + \sum_{k=n+1}^{\infty} v^k {}_k p_x$

Additional relations are

$$\begin{aligned}
 & \bullet 1 = d \ddot{a}_x + A_x & \bullet A_{x:\overline{n}|} &= v \ddot{a}_{x:\overline{n}|} - a_{x:\overline{n-1}|} \\
 & \bullet A_x = v \ddot{a}_x - a_x & \bullet {}_n|\ddot{a}_x &= \ddot{a}_x - \ddot{a}_{x:\overline{n}|} \\
 & \bullet 1 = d \ddot{a}_{x:\overline{n}|} + A_{x:\overline{n}|} & \bullet \ddot{a}_{x:\overline{n}|} &= \ddot{a}_{\overline{n}|} + \ddot{a}_x - \ddot{a}_{x:\overline{n}|} \\
 & \bullet \ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|} & \bullet \ddot{s}_{x:\overline{n}|} &= \frac{\ddot{a}_{x:\overline{n}|}}{{}_n E_x} \\
 & \bullet A_{x:\overline{n}|}^1 = v \ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|} & &= \sum_{k=0}^{n-1} (1+i)^{n-k} \frac{l_{x+k}}{l_{x+n}}
 \end{aligned}$$

## 4. Benefit Premiums

### Introduction

In practice individual life is usually purchased by a life annuity of **contract premiums**—the insurance contract specifies the premium to be paid.

Contract premiums provides for

- benefit
- expense of initiating and maintaining the insurance,
- and margins for profit and for offsetting possible unfavorable experience.

Determination of the insurance premium requires the adoption of a **premium principle**. Example 1 illustrates the application of three such premium principles.

**Example 1.** An insurer is planning to issue a policy to a life age 0 whose curtate-future-lifetime,  $K$  is governed by the p.f.

$${}_k|q_0 = 0.2, \quad k = 0, 1, 2, 3, 4.$$

The policy will pay 1 unit at the end of the year of death in exchange for the payment of a premium  $P$  at the beginning of each year, provided the life survives. Find the annual premium,  $P$ , as determined by:

- Principle I:  $P$  will be the least annual premium such that the insurer has probability of a positive financial loss of at most 0.25.
- Principle II:  $P$  will be the annual premium such that the insurer, using a utility of wealth function  $u(x) = x$  will be indifferent between accepting and not accepting the risk.
- Principle III:  $P$  will be the annual premium such that the insurer, using a utility of wealth function  $u(x) = -e^{-0.1x}$ , will be indifferent between accepting and not accepting the risk.

For all three parts assume the insurer will use an annual effective interest rate of  $i = 0.06$ .

Outcome $k$	Probability ${}_k q_0$	General Formula	Present Value of Financial Loss When Premium Is by Principle			$\exp(0.1L)$ III
			I	II	III	
0	0.2	$v^1 - P \ddot{a}_{\overline{1} }$	0.48544	0.64067	0.63712	1.06579
1	0.2	$v^2 - P \ddot{a}_{\overline{2} }$	0	0.30169	0.29477	1.02992
2	0.2	$v^3 - P \ddot{a}_{\overline{3} }$	-0.45796	-0.01811	-0.02819	0.99718
3	0.2	$v^4 - P \ddot{a}_{\overline{4} }$	-0.89000	-0.31981	-0.33287	0.96726
4	0.2	$v^5 - P \ddot{a}_{\overline{5} }$	-1.29758	-0.60443	-0.62031	0.93985
	Premium		0.45796	0.30272	0.30628	
	Expected Value		-0.43202	0.00000	-0.12020	1.00000

- **Percentile premiums** Although the principle is attractive on surface, it is easy to show that it can lead to quite unsatisfactory premiums.
- **Equivalence principle** It has many application. If we define the insurer's loss,  $L$ , as the random variable of the present value of benefit to be paid by insurer less the annuity of premiums to be paid by the insured. The requirement of this principle is that

$$E[L] = 0.$$

### **Benefit premiums**

$$E[\text{present value of benefits}] = E[\text{present value of benefit premiums}]$$

**Single benefit premium** When the equivalence principle is used to determine a single premium at policy issue for a life insurance or a life annuity, the premium is equal to the actuarial present value of the benefit payments.

- **Exponential premiums** These premiums are nonproportional in the sense that the premium for the policy with a benefit level of 10 is more than 10 times the premium for a policy with a benefit level of 1. This is consistent for a risk averse utility function.



## Fully Continuous Premiums

- The equivalence principle.
- Fully continuous level annual benefit premium for a unit whole life insurance payable immediately on the death of  $(x)$ .
- For any continuously paid premium,  $\bar{P}$ , consider

$$l(t) = v^t - \bar{P}\bar{a}_{\bar{t}|},$$

the present value of the loss to the issuer if death occurs at time  $t$ .

- $l(0) = 1$ .
- $l(t) \rightarrow -\bar{P}/\delta$  as  $t \rightarrow \infty$ .
- if  $l(t_0) = 0$ , death before  $t_0$  results in a positive loss, whereas death after  $t_0$  produces a negative loss.

Consider the loss random variable

$$L = l(T) = v^T - \bar{P}\bar{a}_{\bar{T}|}.$$

By the equivalence principle, the premium is denoted by  $\bar{P}(\bar{A}_x)$  and is such that

$$\mathbb{E}[L] = 0.$$

Then

$$\bar{A}_x - \bar{P}(\bar{A}_x)\bar{a}_x = 0 \implies \bar{P}(\bar{A}_x) = \frac{\bar{A}_x}{\bar{a}_x}.$$

The variance of  $L$  can be used as a measure of the variability of losses on an individual whole life insurance due to the random nature of time-until-death.

$$\begin{aligned} \text{Var}(L) = \mathbb{E}[L^2] &= \text{Var}(v^T - \bar{P}\bar{a}_{\bar{T}|}) = \text{Var}\left[v^T - \frac{\bar{P}(1 - v^T)}{\delta}\right] \\ &= \text{Var}\left[v^T \left(1 + \frac{\bar{P}}{\delta}\right) - \frac{\bar{P}}{\delta}\right] = \text{Var}(v^T) \left(1 + \frac{\bar{P}}{\delta}\right)^2 \\ &= [{}^2\bar{A}_x - (\bar{A}_x)^2] \left(1 + \frac{\bar{P}}{\delta}\right)^2 = \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{(\delta\bar{a}_x)^2}. \end{aligned}$$

## Fully Continuous Benefit Premiums

Plan	Loss Components		Premium Formula
	$b_T v_T$	$\bar{P} Y$ Where $Y$ Is	$\bar{P} = \frac{E[b_T v_T]}{E[Y]}$
Whole life insurance	$1 v^T$	$\bar{a}_{\bar{T}}$	$\bar{P}(\bar{A}_x) = \frac{\bar{A}_x}{\bar{a}_x}$
$n$ -Year term insurance	$1 v^T$ $0$	$\bar{a}_{\bar{T}}, T \leq n$ $\bar{a}_{\bar{n}}, T > n$	$\bar{P}(\bar{A}_{x:\bar{n}}^1) = \frac{\bar{A}_{x:\bar{n}}^1}{\bar{a}_{x:\bar{n}}}$
$n$ -Year endowment insurance	$1 v^T$ $1 v^n$	$\bar{a}_{\bar{T}}, T \leq n$ $\bar{a}_{\bar{n}}, T > n$	$\bar{P}(\bar{A}_{x:\bar{n}}) = \frac{\bar{A}_{x:\bar{n}}}{\bar{a}_{x:\bar{n}}}$
$h$ -Payment* whole life insurance	$1 v^T$ $1 v^T$	$\bar{a}_{\bar{T}}, T \leq h$ $\bar{a}_{\bar{n}}, T > h$	${}_h\bar{P}(\bar{A}_x) = \frac{\bar{A}_x}{\bar{a}_{x:\bar{h}}}$
$h$ -Payment,* $n$ -year endowment insurance	$1 v^T$ $1 v^T$ $1 v^n$	$\bar{a}_{\bar{T}}, T \leq h$ $\bar{a}_{\bar{h}}, h < T \leq n$ $\bar{a}_{\bar{n}}, T > n$	${}_h\bar{P}(\bar{A}_{x:\bar{n}}) = \frac{\bar{A}_{x:\bar{n}}}{\bar{a}_{x:\bar{h}}}$
$n$ -Year pure endowment	$0$ $1 v^n$	$\bar{a}_{\bar{T}}, T \leq n$ $\bar{a}_{\bar{n}}, T > n$	$\bar{P}(A_{x:\bar{n}}^1) = \frac{A_{x:\bar{n}}^1}{\bar{a}_{x:\bar{n}}}$
$n$ -Year † deferred whole life annuity	$0$ $\bar{a}_{\bar{T}-\bar{n}} v^n$	$\bar{a}_{\bar{T}}, T \leq n$ $\bar{a}_{\bar{n}}, T > n$	$\bar{P}({}_n\bar{a}_x) = \frac{A_{x:\bar{n}}^1 \bar{a}_{x+n}}{\bar{a}_{x:\bar{n}}}$

\*The insurances described in the fourth and fifth rows provide for a premium paying period that is shorter than the period over which death benefits are paid.

†The annuity product described above provides no death benefits and has a level premium with premiums payable for  $n$  years. A different, perhaps more realistic, design for an  $n$ -year level premium-deferred annuity is given in Example 6.6.2.

**Example 2.** Calculate  $\bar{P}(\bar{A}_x)$  and  $\text{Var}(L)$  with the assumptions that the force of mortality is a constant  $\mu = 0.04$  and the force of interest  $\delta = 0.06$ .

**Solution:** These assumptions yields  $\bar{a}_x = 10$ ,  $\bar{A}_x = 0.4$ , and  ${}^2\bar{A}_x = 0.25$ .  
Then we obtain

$$\bar{P}(\bar{A}_x) = \frac{\bar{A}_x}{\bar{a}_x} = 0.04,$$

and

$$\text{Var}(L) = \frac{0.25 - 0.16}{(0.6)^2} = 0.25.$$

- Under the constant force of mortality assumption,

$$\bar{P}(\bar{A}_x) = \frac{\mu(\mu + \delta)^{-1}}{(\mu + \delta)^{-1}} = \mu,$$

which does not depend on the force of interest or the age at issue.

- For a variety of fully continuous life insurances, the formulas for annual premiums can be determined by

$$b_T v_T - \bar{P}Y = Z - \bar{P}Y \text{ and } E[b_T v_T - \bar{P}Y] = 0 \implies \bar{P} = \frac{E[b_T v_T]}{E[Y]}.$$

- For an  $n$ -year deferred whole life annuity of 1 per year payable continuously. In this case  $b_T v_T = 0, T \leq n$  and  $b_T v_T = \bar{a}_{\overline{T-n}|} v^n, T > n$  Then

$$E[b_T v_T] = {}_n p_x E[\bar{a}_{\overline{T-n}|} v^n | T > n] = v^n {}_n p_x \bar{a}_{x+n} = A_{x:\overline{n}|}^1 \bar{a}_{x+n}.$$

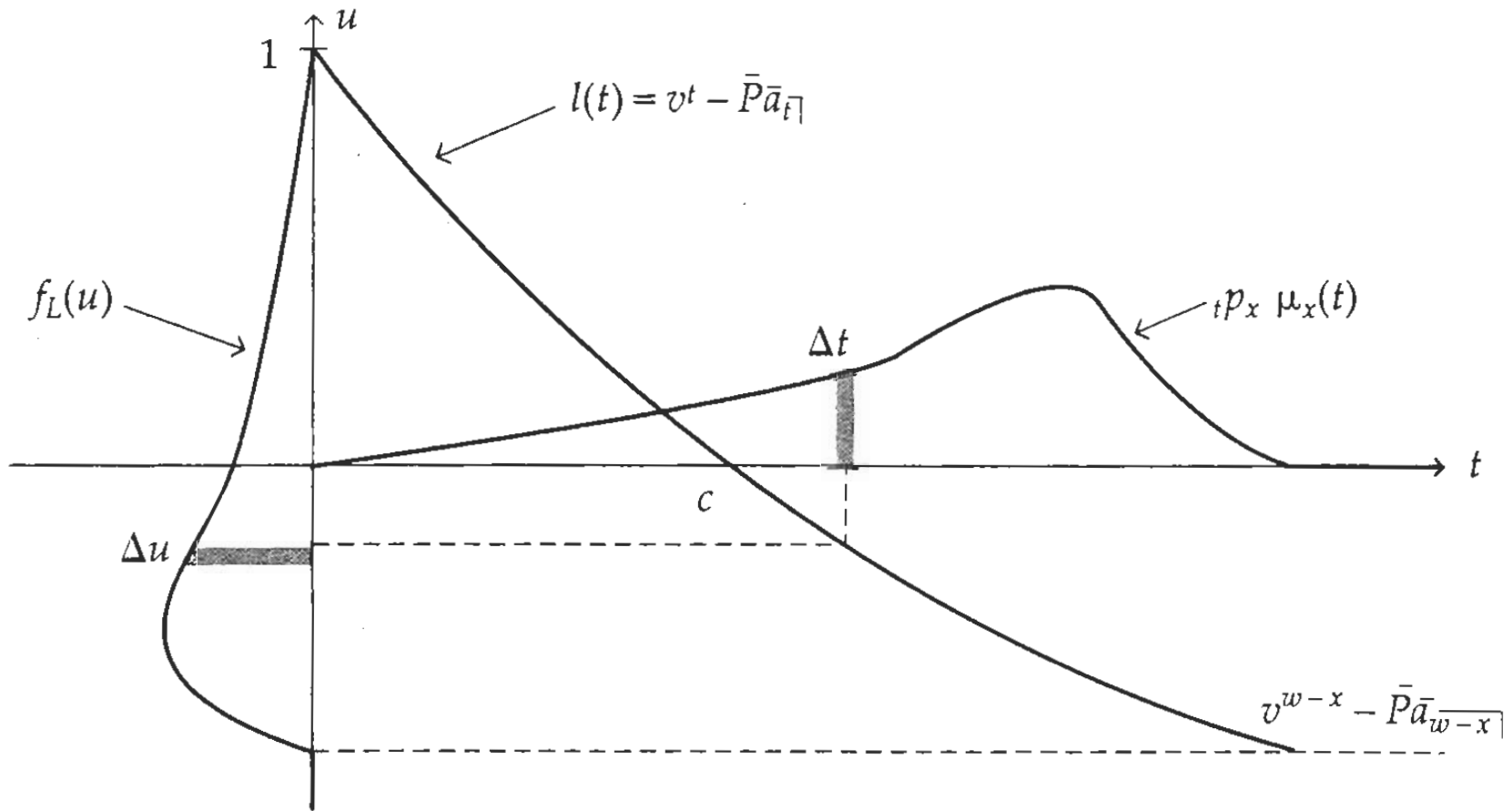
**Example 3.** Express the variance of the Loss  $L$ , associated with an  $n$ -year endowment insurance, in terms of actuarial present values.

**Example 4.** Find the 25th percentile premium for an insured age 55 for the following plans of insurances:

- a. 20-year endowment
- b. 20-year term
- c. 10-year term.

Assume a fully continuous basis with a force of interest,  $\delta = 0.06$  and mortality following the illustrative life Table.

## Schematic Diagrams of $I(t)$ and the p.d.f.'s of $T(x)$ and $L$



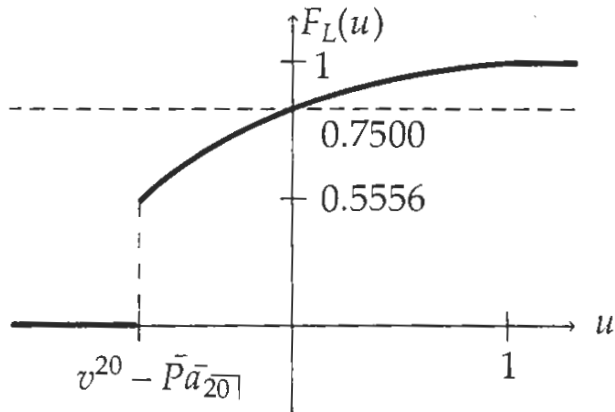
**Example 5.** This example builds on Example 4. except that  $T(55)$  has a De Moivre distribution with p.d.f

$${}_t p_{55} \mu_{55}(t) = 1/45, 0 < t < 45.$$

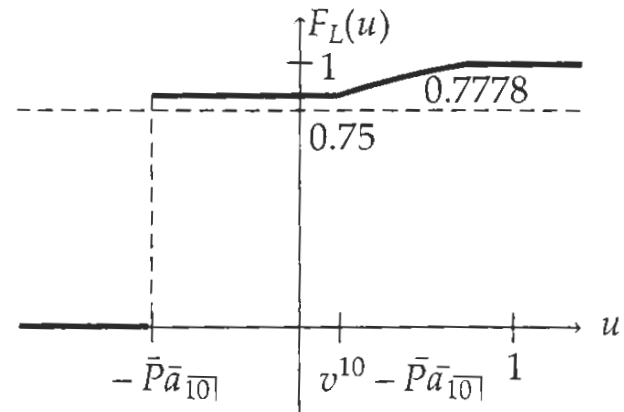
For the three loss variables, display the d.f. of  $L$  and determine the parameter  $\bar{P}$  as the smallest non-negative number such that  $\Pr(L > 0) \leq 0.25$ .



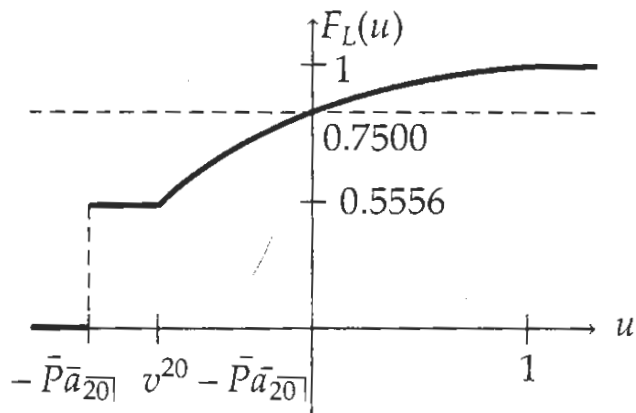
## Distribution Functions of $L$ Developed in Example 6.2.4



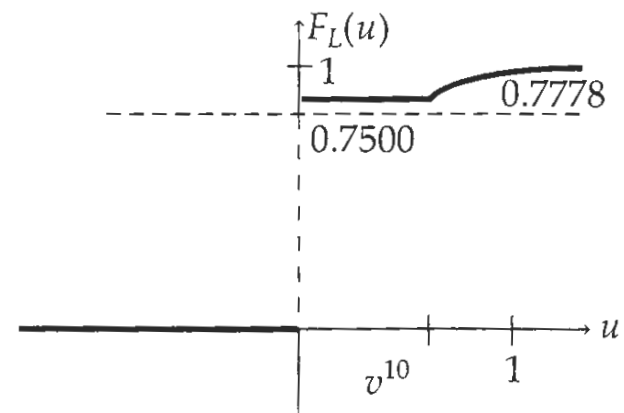
(a) d.f. of  $L$ , 20-year endowment insurance  $\bar{P} = 0.06244$



(c) d.f. of 10-year term insurance  $\bar{P} = 0.06244$



(b) d.f. of  $L$ , 20-year term insurance  $\bar{P} = 0.06244$



(d) d.f. of 10-year term insurance  $\bar{P} = 0$

## Fully Discrete Premiums

- $P_x$ , the level annual benefit premium for a unit whole life insurance.
- the absence of  $(\bar{A}_x)$  means that the insurance is payable at the end of policy year of death.
- The loss of this insurance is

$$L = v^{K+1} - P_x \ddot{a}_{\overline{K+1}|}, \quad k = 0, 1, 2, \dots$$

The equivalence principle requires that  $\mathbf{E}[L] = 0$ , or

$$\mathbf{E}[v^{K+1}] - P_x \mathbf{E}[\ddot{a}_{\overline{K+1}|}] = 0$$

which yields

$$P_x = \frac{A_x}{\ddot{a}_x}$$

and

$$\text{Var}(L) = \frac{{}^2A_x - (A_x)^2}{(d\ddot{a}_x)^2}.$$

## Fully Discrete Annual Benefit Premiums

Plan	Loss Components		Premium Formula
	$b_{K+1}v_{K+1}$	$P Y$ Where $Y$ Is	$P = \frac{E[b_{K+1}v_{K+1}]}{E[Y]}$
Whole life insurance	$1 v^{K+1}$	$\ddot{a}_{\overline{K+1} }, K = 0, 1, 2, \dots$	$P_x = \frac{A_x}{\ddot{a}_x}$
$n$ -Year term insurance	$1 v^{K+1}$ $0$	$\ddot{a}_{\overline{K+1} }, K = 0, 1, \dots, n-1$ $\ddot{a}_{\overline{n} }, K = n, n+1, \dots$	$P_{x:\overline{n} }^1 = \frac{A_{x:\overline{n} }^1}{\ddot{a}_{x:\overline{n} }}$
$n$ -Year endowment insurance	$1 v^{K+1}$ $1 v^n$	$\ddot{a}_{\overline{K+1} }, K = 0, 1, \dots, n-1$ $\ddot{a}_{\overline{n} }, K = n, n+1, \dots$	$P_{x:\overline{n} } = \frac{A_{x:\overline{n} }}{\ddot{a}_{x:\overline{n} }}$
$h$ -Payment whole life insurance	$1 v^{K+1}$ $1 v^{K+1}$	$\ddot{a}_{\overline{K+1} }, K = 0, 1, \dots, h-1$ $\ddot{a}_{\overline{h} }, K = h, h+1, \dots$	${}_hP_x = \frac{A_x}{\ddot{a}_{x:\overline{h} }}$
$h$ -Payment, $hP_{x:\overline{n} }$ $n$ -year endowment insurance $A_{x:\overline{n} }$	$1 v^{K+1}$ $1 v^{K+1}$ $1 v^n$	$\ddot{a}_{\overline{K+1} }, K = 0, 1, \dots, h-1$ $\ddot{a}_{\overline{h} }, K = h, \dots, n-1$ $\ddot{a}_{\overline{n} }, K = n, n+1, \dots$	${}_hP_{x:\overline{n} } = \frac{A_{x:\overline{n} }}{\ddot{a}_{x:\overline{h} }}$
$n$ -Year pure endowment	$0$ $1 v^n$	$\ddot{a}_{\overline{K+1} }, K = 0, 1, \dots, n-1$ $\ddot{a}_{\overline{n} }, K = n, n+1, \dots$	$P_{x:\overline{n} }^1 = \frac{A_{x:\overline{n} }^1}{\ddot{a}_{x:\overline{n} }}$
$n$ -Year deferred whole life annuity	$0$ $\ddot{a}_{\overline{K+1-n} }v^n$	$\ddot{a}_{\overline{K+1} }, K = 0, 1, \dots, n-1$ $\ddot{a}_{\overline{n} }, K = n, n+1, \dots$	$P_{(n \ddot{a}_x)} = \frac{A_{x:\overline{n} }^1 \ddot{a}_{x+n}}{\ddot{a}_{x:\overline{n} }}$

**Example 6.** If  ${}_k|q_x = c(0.96)^{k+1}$ ,  $k = 0, 1, 2, \dots$  where  $c = 0.04/0.96$  and  $i = 0.06$ , calculate  $P_x$  and  $\text{Var}(L)$ .

Continuing to use the equivalence principle, we can determine formulas for annual benefit premiums for a variety fully discrete life insurances.

$$\text{E}[b_{K+1}v_{K+1} - PY] = 0 \implies P = \frac{\text{E}[b_{K+1}v_{K+1}]}{\text{E}[Y]}$$

where  $P$  is a general symbol for an annual premium paid at the beginning of each policy year during the premium paying period while insured survives and  $Y$  is a discrete annuity random variable as defined in Section 3.

**Example 7.** Express the variance of the Loss  $L$ , associated with an  $n$ -year endowment insurance, in terms of actuarial present values.

**Example 8.** Consider a 10,000 fully discrete whole life insurance. Let  $\pi$  denote an annual premium for this policy and  $L(\pi)$  denote the loss-at issue random variable for one such policy on the basis of the Illustrative Life Table, 6% interest and issue age 35

- a. Determine the premium,  $\pi_a$ , such that the distribution of  $L(\pi_a)$  has mean 0. Calculate the variance of  $L(\pi_a)$ .
- b. Approximate the smallest non-negative premium,  $\pi_b$ , such that the probability is less than 0.5 that the loss  $L(\pi_b)$  is positive. Find the variance of  $L(\pi_b)$ .
- c. Determine the premium,  $\pi_c$ , such that the probability of a positive total loss on 100 such independent policies is 0.05 by the normal approximation.