Optimization for image processing

I Generalities

II Differentiable problems

- without constraints
- with general constraints
- constraints equality
  inequality

III Nonsmooth optimization

1.1. Optimization problems

Find $\hat{u} \in \underset{\substack{\nearrow \\ \text{normed vect. space}}}{V}$ such that

$$\hat{u} \in \underset{\substack{\uparrow \\ \text{subset of } V}}{U} \quad \text{and} \quad F(\hat{u}) = \inf_{u \in V} F(v)$$

$$U = \{ u \in \mathbb{R}^n : g_i(u) = 0 \quad 1 \leq i \leq n \}$$

$$\dots \qquad h_i(u) \leq 0 \quad \dots$$

$\mathcal{O}(v)$ neigh. of v

minimum of F $\qquad F(\hat{u})$

minimizer of F is $\hat{u}$

Standard form

$$F(u) = \frac{1}{2} \langle \underset{\substack{\uparrow \\ \text{matrix } n \times n}}{B} u, u \rangle - \langle c, u \rangle$$

2. ~~Rest~~ Image restoration via the minimization of an energy

- Observation model $\underset{\text{data}}{v} = Au + \underset{\text{noise}}{n}$

A – Blur, degradation ...
(integral operator ...)

After sampling A matrix $m \times m$

Solution : $A^{-1}v$ cannot work

e.g. $A = I$. $v = u + n$.
if we do $A^{-1}v$ we get the noisy data.

Objective function

$$F(u) = \underset{\substack{\text{data fidelity term}\\ (A\hat{u} \simeq v)}}{\Psi(u,v)} + \overset{\text{param.} > 0}{\beta}\, \overset{\text{regularization}}{\underbrace{\Phi(u)}}$$

e.g. $\Phi(u) = \sum_i \varphi(\|D_i u\|)$

discrete approx of the grad. of the image.

$\varphi(t) = \sqrt{t^2 + \alpha}\,,\ |t|^\alpha, \ldots$

$\Psi(u,v) = \|Au - v\|^2$. the most classical.

Quadratic regularization

$$\Phi(u) = \|D_i u\|^2$$

Constraints $U = \{ u \in V : Au = v \}$

$\{ u \in V : \|Au - v\|^2 - \varepsilon^2 \leq 0 \}$

$\hat{u} \in \bigcap_{i \geq 1} U_i$ U_i - properties required by $\bar{u}$.

1.3. Analysis of the solution.

a) Existence, unicity

Def. F is coercive if $\lim_{\|u\| \to \infty} F(u) \to \infty$

Thm. V normed vector space. finite dim.

$U \subset V$ non empty closed

$F : V \to \mathbb{R}$ continuous.

If U is not bounded we suppose that F is coercive.

$\Rightarrow \exists \hat{u} \in U$ Such that $F(\hat{u}) = \inf F(u)$

Proof. $\overline{U}$ bounded $\Rightarrow$ U is compact $\left.\begin{matrix} \\ \end{matrix}\right\}$ $\Rightarrow$
F continuous

$\Rightarrow$ Thm. Weierstrass – existence of $\hat{u}$.

$u_0 \in U$

F coercive $\Rightarrow$ $\exists\, r > 0$ such that

$\|u\| > r \quad \Rightarrow \quad F(u_0) < F(u)$

$\Rightarrow$ $\hat{u}$ satisfies

$$F(\hat{u}) = \inf \left\{ F(u) : u \in \overline{B(u_0, r)} \cap U \right\}$$

compact

closed ball, radius r.

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he result can be extended to general Hilbert spaces. (separable)

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alternative proof with minimizing equence $u_k \in \overline{U}$

$$\lim_{k \to \infty} F(u_k) = \inf F(u)$$

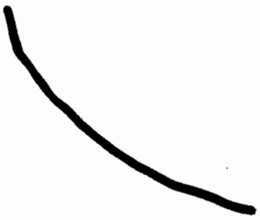
Feasable optim. pb. if F, U convex

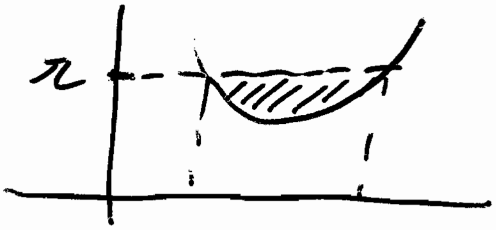
Def. $F : U \to R$ convex if $\forall u, v \in U$

$\forall\ \theta \in (0,1)$ $F(\theta u + (1-\theta) v) \leq$

$$\theta F(u) + (1-\theta) F(v)$$

F can be convex without being coercive





Def. Epigraph

$\text{epi } F = \{ (u,r) \in V \times \mathbb{R} : F(u) \leq r \}$

- F convex $\Longleftrightarrow$ epi F convex

  F lower semi-continuous $\Longleftrightarrow$ epi F closed in $V \times \mathbb{R}$.

sublevel sets

$(u,r) \in \text{epi } F \Longleftrightarrow u \in S_r \{ v : F(v) \leq r \}$

- F convex $\Longrightarrow$ S_r convex

  $\not\Longleftarrow$

Thm. $F : U \to R$ convex, continuous

If F has a local minimum at $\hat{u} \in U$

$\Longrightarrow$ $\hat{u}$ is a global min. / U

$$\hat{U} = \{ \hat{u} \in U : F(\hat{u}) = \inf_{u \in U} F(u) \}$$

is closed and convex.

F strictly convex $\Longrightarrow$ F has at most one minimum and it is strict.

$\exists\, r > 0$ such that $F(\hat{u}) \leq F(u) \quad \forall u \in \overline{B(\hat{u}, r)} \cap U$

By the def. of local min.

$u \in U \setminus B(\hat{u}, r)$ arbitrary.

$\theta = \dfrac{r}{\|u - \hat{u}\|} \Rightarrow \theta \in (0, 1)$

$u_\theta = (1-\theta)\hat{u} + \theta u \quad \Longrightarrow u_\theta \in \overline{B(\bar{u}, r)} \cap U$

because $u_\theta - \hat{u} = \theta (u - \hat{u})$

$= r \dfrac{u - \hat{u}}{\|u - \hat{u}\|}$

$$F(\hat{u}) \leq F(u_\theta) = F((1-\theta)\bar{u} + \theta u)$$

$$\leq (1-\theta) F(\hat{u}) + \theta F(u)$$

$$F(\hat{u}) - \theta F(\hat{u})$$

$\cancel{\theta} F(\bar{u}) \leq \cancel{\theta} F(u) \quad \Longrightarrow \bar{u}$ global min.

$\hat{U} = U \cap \{ u \in V : F(u) \leq F(\bar{u}) \}$

convex since closed since F continuous

(b) Characterize the solution

$\hat{u}$ - implicitely defined in general.

1.4. Algorithms to approximate $\hat{u}$

(a) Iterative schemes

$(u_k)_{k \geq 1}$ starting from u_0 and converging to $\hat{u}$

$$u_k = \underset{\nearrow \text{iterative scheme}}{G}(u_{k-1}) \qquad V \to V$$

G use ∇F, $F(u_k)$, $g(u_k)$ . . .

$\hat{u}$ is a fixed point for G iff

$$\boxed{G(\hat{u}) = \hat{u}.}$$

Def. G is a contraction if $\exists\, \gamma \in (0,1)$ such that

$$d\left(G(u_1), G(u_2)\right) \leq \gamma\, d\left(u_1, u_2\right)$$

$\uparrow$ distance (e.g. $\|u_1 - u_2\|$)

$\Rightarrow$ G is Lip. $\Rightarrow$ uniformly continuous.

Re Fixed point theorem: (V complete)

G is contraction $\Longrightarrow$ G has a fixed point $\hat{u}$

oof . Unicity

Suppose 2 fixed points $\hat{u}_1$ and $\hat{u}_2$.

$$\underline{0 <} \underline{d(\hat{u}_1, \hat{u}_2)} = d(G\hat{u}_1, G\hat{u}_2) \leq$$

$$\leq \gamma\, d(\hat{u}_1, \hat{u}_2) \underline{< d(\hat{u}_1, \hat{u}_2)}$$

$\Rightarrow \hat{u}_1 = \hat{u}_2$.

Existence

$u_0 \in V$ arbitrary

$$u_1 = G(u_0), \; \dots \quad u_k = G(u_{k-1}) \dots$$

we will show that (u_k) is a Cauchy sequence.

$$d(u_2, u_1) \leq \gamma\, d(u_1, u_0)$$

$$d(u_3, u_2) \leq \gamma\, d(u_2, u_1) \leq \gamma^2 d(u_1, u_0)$$

$$\dots\dots$$

$$d(u_{k+1}, u_k) \leq \gamma^k d(u_1, u_0)$$

$$\underline{d(u_{k+l}, u_k)} \overset{\Delta \text{ inequality}}{\leq} d(u_{k+l}, u_{k+l-1}) + \dots\dots$$

$$+ d(u_{k+1}, u_k)$$

$$\leq \gamma^k (\gamma^{l-1} + \dots\dots + \gamma^0)\, d(u_1, u_0)$$

$$\leq \frac{\gamma^k}{1-\gamma} d(u_1, u_0) \xrightarrow[k \to \infty]{} 0$$

nce (u_k) is Cauchy.

V complete $\Rightarrow \exists\, \hat{u} \in V \quad u_k \to \hat{u}$

G continuous $\Rightarrow \quad G(u_k) \to G(\hat{u})$

$$0 = \lim \underbrace{\left(u_{k+1} - G(u_k)\right)}_{\searrow\, 0} = \hat{u} - G(\bar{u}).$$

(b) Convergence speed.

- Q convergence factor (quotient)

$$Q((u_n)_k) = \limsup_{k\to\infty} \frac{\|u_{k+1} - \hat{u}\|}{\|u_n - \hat{u}\|}$$

the worst case $\underline{Q}(G, \hat{u}) = \sup\limits_{\text{all seq.}} Q(\{u_k\})$

Compare G_1 and G_2 converging to $\hat{u}$

speed is Q superlinear $\underline{Q} = 0$

linear $\underline{Q}$ finite

sublinear $\underline{Q}$ ∞

- R convergence factor (root)

$$R((u_n)_{n\geq 1}) = \limsup_{k\to\infty} \|u_n - \hat{u}\|^{\frac{1}{k}}$$

R indep. of the norm.

the worst case $\underline{R}(G, \hat{u}) = \sup R((u_n)_{k\geq})$

R- superlinear $\underline{R} = 0$

linear $0 < \underline{R} < 1$

Sublinear $\underline{R} = 1$

the Ostrowski theorem

$G: \mathbb{R}^n \to \mathbb{R}^n$ differentiable en at $\hat{u}$

and $G(\hat{u}) = \hat{u}$ (fixed point)

$\max_{1 \le i \le n} |\lambda_i| < 1$ where λ_i is an

eigenvalue of $DG(\hat{u})$

$\Rightarrow$ $O(\hat{u})$ neighborhood such that

$\forall u_0 \in O(\hat{u})$, $u_k \to \hat{u}$.

The linear convergence thm.

Under the same conditions

$$\max_{1 \le i \le n} |\lambda_i| = R(G, \hat{u})$$

to analyze the conv. speed we should check the eigenvalues of DG.

II Differentiable optim. problems.

1. Preliminaries.

$f: V \to Y$ is differentiable at $u \in V$ if

$\exists$ $Df(u)$ linear continuous app. from V to Y such that $\varepsilon(v) \xrightarrow[\|v\| \to 0]{} 0$

$$f(u+v) = f(u) + Df(u)v + \|v\| \cdot \varepsilon(v)$$

$$F(\underline{u}) \geq F(\overset{\text{the } u_0 \text{ of the definition}}{\overbrace{u + \theta(u_0 - u)}}) - \theta \langle \nabla F(u + \theta(u_0 - u)), u_0 - u \rangle \times (1-\theta)$$

$$F(\underline{u_0}) \geq F(u + \theta(u_0 - u)) + (1-\theta) \langle \nabla F(u + \theta(u_0 - u)), u_0 - u \rangle \times \theta \quad / +$$

$$(1-\theta) F(u) + \theta F(u_0) \geq F(\theta u_0 + (1-\theta) u) \quad \left(\begin{array}{c}\text{the def of}\\ \text{convexity}\end{array}\right)$$

strict convexity!

$(\Rightarrow)$ $F(u_0 + \theta(u - u_0)) - F(u_0) < \theta \left(F(u) - F(u_0) \right)$

~~∇F~~ the nonstrict ineq. holds true.

$$\langle \nabla F(u_0), \theta (u - u_0) \rangle \leq F(u_0 + \theta(u - u_0)) - F(u_0)$$
$$< \theta \left(F(u) - F(u_0) \right).$$

$\Leftarrow$ change. $>$.

identification $Df(u)$ and $\nabla f(u)$

/ $\langle \nabla f(u), v \rangle = Df(u)\, v.$

Property. V - n.v.s. U convex

F differentiable in $\mathcal{O}(U)$

) F convex $\Longleftrightarrow$ $F(u) \geq F(u_0) + \langle \nabla F(u_0), u-u_0 \rangle$

$\forall u, u_0 \in U$

) ... strictly ... $\Longleftrightarrow$ ... $>$ ... $u \neq u_0$

roof. $\Rightarrow$ F convex take $u, u_0 \in U$ $\theta \in (0,1)$

$(\theta u + (1-\theta) u_0) \leq \theta F(u) + (1-\theta) F(u_0)$

$\in U$ since U convex

$(\theta u + (1-\theta) u_0) - F(u_0) \leq \theta \big(F(u) - F(u_0)\big)$ $\frac{1}{\theta}$

$u_0 + \theta (u - u_0)$

lim for $\theta \to 0$

$$\lim_{\theta \to 0} \frac{F(u_0 + \theta(u-u_0)) - F(u_0)}{\theta} \leq F(u) - F(u_0)$$

$= \langle \nabla F(u_0), u - u_0 \rangle$

$\Leftarrow)$ $F(u) \geq F(u + \theta(u_0 - u)) - \theta \nabla$

F 2 times differentiable in $\mathcal{O}(U)$

) F convex U $\Longleftrightarrow$ $\nabla^2 F(u_0)(u-u_0, u-u_0) \geq 0$

$\forall u_0, u \in U$ ($\nabla^2 F$ is semi positive definite

) if strict ineq. $u \neq u_0$ $\Longrightarrow$ F strictly convex

Def. F is elliptic if : $F \sim C^1$ and $\exists\, \mu > 0$ such that $\langle \nabla F(u) - \nabla F(u_0), u - u_0 \rangle \geq \mu \|u - u_0\|^2$ $\forall u, u_0$

$F(u) = \frac{1}{2} \langle Bu, u \rangle - \langle c, u \rangle$

$\uparrow$ elliptic if $B = B^*$ (symmetric) and $\lambda_i(B) > 0$ $\forall i$

$\nearrow$ eigenvalues

Property. Suppose F differentiable V

- F elliptic $\Longrightarrow$ F strictly convex and coercive

$$F(u) - F(u_0) \geq \langle \nabla F(u_0), u - u_0 \rangle + \frac{\mu}{2} \|u - u_0\|^2 \quad \forall u, u_0 \in V$$

Let F be 2 times differentiable

$$F \text{ elliptic} \Longleftrightarrow \langle \nabla^2 F(u_0) u, u \rangle \geq \mu \|u\|^2 \quad \forall u \in V$$

2. Problems without constraints

- stopping rules $\|\nabla F(u_k)\| \approx 0$
- descent direction $-d \in V$ at u_k
  $\langle \nabla F(u_k), d \rangle > 0$
- line search – finding ρ_k such that
  $F(u_k + \rho_k d_k) = \inf_{\rho} F(u_k + \rho\, d_k)$