we have

$\theta^{**} = \theta$ $\qquad$ $\theta(t) = \sup_{b} \left(bt - \theta^*(b) \right) = \theta^{**}(t)$

Calculate θ^{**} at t^2:

$$-\varphi(t) = \theta(t^2) = \sup_{b \leq 0} \left(bt^2 - \theta^*(b) \right)$$

replace b by $-\frac{1}{2}b$

(if $b > 0$ $\theta^*(b) = +\infty$)

$$= \sup_{b \geq 0} \left(-\frac{1}{2} b t^2 - \psi(b) \right)$$

$$\boxed{\begin{aligned} &\varphi(t) = \inf_{b \geq 0} \left(\underline{\tfrac{1}{2} b t^2 + \psi(b)} \right) \\ &\text{where } \psi(b) = \sup_{t \geq 0} \left(-\tfrac{1}{2} b t^2 + \varphi(t) \right) \end{aligned}}$$

$$\underline{\inf_{b \in \mathbb{R}^2} \mathcal{F}(u,b)} = \|Au - v\|^2 + \beta \sum_{i=1}^{r} \underbrace{\inf_{b[i]} \left(\underline{\frac{b[i]}{2} \|D_i u\|^2 + \psi(b[i])} \right)}_{\underline{\varphi(\|D_i u\|)}}$$

$= \underline{F(u)}$

$\exists (\hat{u}, \hat{b}) \implies \hat{u}$ minimizes $F(u)$

We will compute the value $\hat{b} \in \mathbb{R}_+$ such that

$\varphi(t) = \frac{1}{2} \hat{b} t^2 + \psi(\hat{b})$ (i.e. for which the inf is reached)

$\hat{b} = \sigma(t)$

Assumptions

φ is C^1 on $\mathbb{R}_+$, $\varphi'(0) = 0$

$\lim_{t \to \infty} \varphi(t)/t^2 = 0$ (classical assumption for edge-preserving regu)

$$\sigma(t) = \begin{cases} \dfrac{\varphi'(t)}{t} & t>0 \\ \varphi''(0^+) & t=0 \end{cases}$$

$$\mathcal{F}(u, \hat{b}) = F(u) \quad \text{if} \quad \hat{b}[i] = \sigma(\|D_i u\|)$$

$\mathcal{F}$ in matrix form

$$\mathcal{F}(u,b) = \|Au - v\|^2 + \frac{\beta}{2}(Du)^T \mathbb{B}\, Du + \beta \sum \psi(b[i])$$

where $B = \mathrm{diag}\big(b[1]\, \overset{[1,1] \text{ or } 1}{\mathbb{1}_s}, \dots, b[r]\, \mathbb{1}_s\big)$

s – the dim of $D_i u$ (e.g. $s = 1$ or 2)

$$H \qquad \begin{bmatrix} b[1] & & & \\ & b[1] & & \\ & & b[2] & \\ & & & b[2] \end{bmatrix} \qquad \begin{bmatrix} b(1) & & \\ & b(2) & \\ & & \ddots \end{bmatrix}$$

$$\mathcal{H}(b) = 2A^TA + \beta D^T B D$$ invertible if $\mathrm{rk}\, A = p$ OR $b>0$ and $\ker D \cap \ker A = (0)$

for b fixed the min of $\mathcal{F}(u,b)$ is reached at $\hat{u}$ satisfying $\nabla_u \mathcal{F}(\hat{u}, b) = 0$ and hence

$$\mathcal{H}(b)\, \hat{u} = 2A^T v$$

$$\nabla_u \mathcal{F} = 2A^TAu - 2A^Tv + \beta D^TBDu = 0$$

The alternate min scheme is

$$\begin{cases} u_k = \big(\mathcal{H}(b_{k-1})\big)^{-1} 2A^T v \\ b_k[i] = \sigma(\|D_i u_k\|) \quad 1 \le i \le r \end{cases}$$

$\hat{b} > 0$ define $f_{\hat{b}} : \mathbb{R}_+ \longrightarrow \mathbb{R}$

convex

$$f_{\hat{b}}(t) = \frac{1}{2}\hat{b}\,t^2 + \theta(t)$$

then $\psi(\hat{b}) = \underbrace{-\inf_{t \geq 0} f_{\hat{b}}(t)}$

$f_{\hat{b}}$ convex, $f_{\hat{b}}(0) = 0$, $f_{\hat{b}}(t) \xrightarrow[t \to \infty]{} +\infty$

$\Rightarrow$ $f_{\hat{b}}$ has a unique min on $[0, +\infty)$

let $\hat{t} > 0$ denote the point where it is reached.

$$\psi(\hat{b}) = -\frac{1}{2}\hat{b}\,\hat{t}^2 + \varphi(\hat{t}) = -\inf_t f_{\hat{b}}(t)$$

$$\Leftrightarrow \quad \varphi(\hat{t}) = \frac{1}{2}\hat{b}\,\hat{t}^2 + \psi(\hat{b}) = \inf(\dots)$$

$\theta'(t) = -\dfrac{\varphi'(\sqrt{t})}{2\sqrt{t}}$ and $\lim_{t \to 0} \theta'(t) = -\frac{1}{2}\varphi''(0^+)$

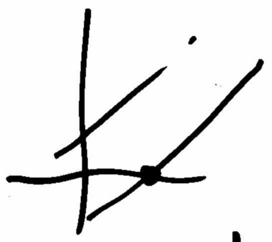
$f'_{\hat{b}}(t) = \frac{1}{2}\hat{b} + \theta'(t)$ increasing because $f_{\hat{b}}$ convex

we are looking for a solution constrained on $\mathbb{R}_+$

If $f'_{\hat{b}}(0^+) \geq 0$ then the min is at $\hat{t} = 0$

$\Leftrightarrow \hat{b} \geq \varphi''(0^+)$

otherwise $\hat{t} > 0$ satisfies



$f'_{\hat{b}}(\hat{t}) = 0 \Leftrightarrow \hat{b} = -2\theta'(\hat{t})$

back to φ ... $\hat{b} = -2\theta'(\hat{t}^2)$ ...

Newton: $\varphi''(\|D_i u\|)$

$$H(u) = \mathcal{H}\left(G(\|D_i u\|)_{i=1}^{r}\right) - -$$

$$= 2A^TA + \beta \sum_{i=1}^{r} \frac{\varphi'(\|D_i u\|)}{\|D_i u\|} D_i^T D_i$$

The * method amounts to

$$u_K = (H(u_{K-1}))^{-1} 2A^T v = \cdots$$

$$= u_{K-1} - \underbrace{(H(u_{K-1}))^{-1}}\nabla F(u_{K-1})$$

~~8/2/~~6

(b) Additive form of HQ regularization.

assumptions
- φ is convex, continuous
- $\frac{1}{2}t^2 - \varphi(t)$ is convex ($\Leftrightarrow \varphi'(0)=0$)
- $\lim_{t\to\infty} \frac{\varphi(t)}{t^2} < \frac{1}{2}$

$s = 1$ or 2 (each D_i matrix $s \times p$ — dim of u)

$b \in \mathbb{R}^s$ below, $z \in \mathbb{R}^s$ — $D_i u$

$$\psi(b) = \sup_z \left\{\underbrace{-\tfrac{1}{2}\|z-b\|^2 + \varphi(\|z\|)}\right\}$$

$$= \sup_z \left\{-\tfrac{1}{2}\|z\|^2 - \tfrac{1}{2}\|b\|^2 + \langle z,b\rangle + \varphi(\|z\|)\right\}$$

$$\underbrace{\psi(b) + \tfrac{1}{2}\|b\|^2}_{f^*} = \sup_z \left\{\langle z,b\rangle - \Big(\overbrace{\underbrace{\tfrac{1}{2}\|z\|^2 - \varphi(\|z\|)}_{\text{convex continuous}}}^{f}\Big)\right\}$$

$$\begin{cases} f^*(x) = \sup_u(\langle u,x\rangle - f(u)) \quad \text{and } f^{**}(u) = f(u) \\ f^{**}(u) = \sup_x(\langle u,x\rangle - f^*(x)) = f(u) \end{cases}$$

$$\overbrace{\frac{1}{2}\|z\|^2-\varphi(\|z\|)}^{f}=\sup_{b}\left\{\langle z,b\rangle-\overbrace{\left(\psi(b)+\frac{1}{2}\|b\|^2\right)}^{f^*}\right\}$$

$$-\varphi(\|z\|)=\sup_{b}\left\{\underbrace{\langle z,b\rangle-\frac{1}{2}\|z\|^2-\frac{1}{2}\|b\|^2}_{-\|z-b\|^2}-\psi(b)\right\}$$

$$\boxed{\varphi(\|z\|)=\inf_{b}\left\{\frac{1}{2}\|z-b\|^2+\psi(b)\right\}}$$

we can compute $\hat{b}$ for which the inf is reached.

$(\hat{z},\hat{b})$ are such that the inf in the def. of φ is reached $\Longleftrightarrow$ the sup in the def. of ψ is reached

$$\varphi(\|\hat{z}\|)=\frac{1}{2}\|\hat{z}-\hat{b}\|^2+\psi(\hat{b})$$

$$\Longleftrightarrow\quad \psi(\hat{b})=-\frac{1}{2}\|\hat{z}-\hat{b}\|^2+\varphi(\|\hat{z}\|)$$

$\Rightarrow$ the grad with respect to $\hat{z}$ of the function $z\to-\frac{1}{2}\|z-\hat{b}\|^2+\varphi(\|z\|)$ is null.

$$-(\hat{z}-\hat{b})+\varphi'(\|\hat{z}\|)\frac{\hat{z}}{\|\hat{z}\|}=0$$

$$\boxed{\hat{b}=\hat{z}-\varphi'(\|\hat{z}\|)\frac{\hat{z}}{\|\hat{z}\|}}$$

$$F(u)=\|Au-v\|^2+\beta\sum_i\varphi(\|D_iu\|)$$

$b_i\in\mathbb{R}^s$

$$\boxed{\mathcal{F}(u,b)=\|Au-v\|^2+\beta\sum_i\underbrace{\left(\frac{1}{2}\|D_iu-b_i\|^2+\psi(b_i)\right)}_{\inf_{b_i}(\cdot)=\varphi(\|D_iu\|)}}$$

$$\inf_{b}\mathcal{F}(u,b)=F(u)$$

~~For $\hat{u}$ fixed~~

for $\hat{u}$ fixed $\inf_{b} \mathcal{F}(u, b)$ is ~~given~~ reached at

$$\hat{b}_i = D_i u - \varphi'(\|D_i u\|) \frac{D_i u}{\|D_i u\|} \quad 1 \leq i \leq r$$

for b fixed $\inf_{u} \mathcal{F}(u, b)$ is reached at $\hat{u}$

such that $\nabla_u \mathcal{F}(u, b) = 0$

$$D = \begin{bmatrix} D_1 \\ \vdots \\ D_r \end{bmatrix} \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix}$$

$$\mathcal{F}(u, b) = \|Au - v\|^2 + \beta \frac{1}{2} \|Du - b\|^2 + \beta \sum_i \psi(b_i)$$

$$2A^T(Au - v) + \beta D^T(Du - b) = 0$$

$$\underbrace{(2A^TA + \beta D^TD)}_{\mathbf{H}} u = 2A^Tv + \beta D^Tb$$

$$\hat{u} = H^{-1}(2A^Tv + \beta D^Tb)$$

The alternate minimization:

$\forall i \quad b_i^k = D_i u^{k-1} - \varphi'(\|D_i u^{k-1}\|) \frac{D_i u^{k-1}}{\|D_i u^{k-1}\|}$ ← iteration

$$u^k = H^{-1}(2A^Tv + \beta D^T b^k)$$

Quasi-Newton

$$u^k = u^{k-1} - H^{-1} \nabla F(u^{k-1})$$

H^{-1} approx of $\nabla^2 F(u^{k-1})$

corresp. to $\varphi'' = 1$ all the time

$$2A^TA + \beta D^T \underbrace{\mathrm{diag}\begin{pmatrix}\varphi''(-) & & \\ & \ddots & \\ & & \ddots\end{pmatrix}}_{1} D$$

Both forms (* and +) convergence established for φ convex smooth, can be interpreted ~~as~~ in terms of ~~qua~~ majorizing quadratic func. (involve the inversion of a matrix)

convergence speed

* needs less iterations than +

cost per iter. for * > cost per iter. +

overall + is faster.

both are faster than classical methods.

Matrix $H(u_k)$ - inversion.

Truncated versions of HQ regu:

the idea: compute $H(u_k)^{-1}$ only approximately (several iter.)

still convergence proven.