we have
\[\Theta^{**} = \Theta\]
\[\Theta(t) = \sup_{b \leq 0} (bt - \Theta^*(b)) = \Theta^*(t)\]
calculate \(\Theta^{**}\) at \(t^2\):
\[-\Psi(t) = \Theta(t^2) = \sup_{b \leq 0} (bt^2 - \Theta^*(b))\]
\[= \sup_{b \geq 0} (-\frac{1}{2} bt^2 - \Psi(b))\]
\[= \inf_{b \geq 0} \left(\frac{1}{2} bt^2 + \Psi(b)\right)\]
where \(\Psi(b) = \sup_{t \geq 0} \left(-\frac{1}{2} bt^2 + \Psi(t)\right)\)

\[\inf_{b \in \mathbb{R}^2} F(u, b) = \|Au - v\|^2 + \beta \sum_{i=1}^{2} \inf_{b \in \mathbb{R}^2} \left(\frac{\|b_i\|}{2} \|Db_i u + b_i\|^2 + \Psi(b_i)\right)\]
\[= F(u)\]

\((\hat{u}, \hat{b}) \Rightarrow \hat{u}\) minimizes \(F(u)\)

We will compute the value \(b \in \mathbb{R}\) such that
\[\Psi(t) = \frac{1}{2} bt^2 + \Psi(b)\]
(i.e. for which the inf is reached)
\[\hat{b} = \hat{b}(t)\]

Assumptions
\(\Psi\) is \(C^1\) on \(\mathbb{R}^+\), \(\Psi(0) = 0\)
\[
\lim_{t \to \infty} \frac{\Psi(t)}{t^2} = 0 \quad \text{(classical assumption for edge-preserving regu)}
\]
\[ G(t) = \begin{cases} \frac{\varphi'(t)}{t} & t > 0 \\ \varphi''(0^+) & t = 0 \end{cases} \]

\[ F(u, \mathbf{b}) = F(u) \quad \text{if} \quad \mathbf{b}[i] = \mathbf{c} \left( \| D_i u \| \right) \]

\( F \) in matrix form

\[ F(u, \mathbf{b}) = \| A u - \mathbf{b} \|_2^2 + \frac{1}{\mathbf{b}} \left( \mathbf{D} u \right)^T \mathbf{B} \mathbf{D} u + \mathbf{b} \sum \varphi(\mathbf{b}[i]) \]

where \( \mathbf{B} = \text{diag} \left( \mathbf{b}[1] \mathbf{1}_s, \ldots, \mathbf{b}[r] \mathbf{1}_s \right) \)

\( s = \text{the dim of } D_i u \) (e.g. \( s = 1 \) or \( 2 \))

\[ H = \begin{bmatrix} \mathbf{b}[1] & \mathbf{b}[1] & \cdots & \mathbf{b}[1] \\ \mathbf{b}[2] & \mathbf{b}[2] & \cdots & \mathbf{b}[2] \\ & & \ddots & \end{bmatrix} \]

\( \mathbf{H}(\mathbf{b}) = 2 \mathbf{A}^T \mathbf{A} + \mathbf{b} \mathbf{D}^T \mathbf{B} \mathbf{D} \) is invertible if \( \text{rank} \mathbf{A} = n \) and \( \mathbf{b} > 0 \) and \( \ker \mathbf{D} \cap \ker \mathbf{A} = \{ 0 \} \)

For any fixed \( \mathbf{b} \), the minimum of \( F(u, \mathbf{b}) \) is reached at \( \mathbf{u} \) satisfying \( \mathbf{D}_u F(\mathbf{u}, \mathbf{b}) = 0 \) and hence

\[ \mathbf{H}(\mathbf{b}) \mathbf{u} = 2 \mathbf{A}^T \mathbf{u}, \]

\[ \mathbf{D}_u F = 2 \mathbf{A}^T \mathbf{A} u - 2 \mathbf{A}^T u + \mathbf{b} \mathbf{D}^T \mathbf{B} \mathbf{D} u = 0 \]

The alternate min scheme is

\[ \begin{cases} \mathbf{u}_k = (\mathbf{H}(\mathbf{b}_{k-1}))^{-1} 2 \mathbf{A}^T \mathbf{u} \\ \mathbf{b}_k[i] = \mathbf{c} \left( \| D_i u_k \| \right) \end{cases} \quad 1 \leq i \leq t \]
$\hat{c} > 0$ define $\tilde{f}_ \hat{c} : \mathbb{R}_+ \rightarrow \mathbb{R}$ convex

$\tilde{f}_ \hat{c} (t) = \frac{1}{2} \hat{c} t^2 + \theta (t)$

then $\psi (\hat{c}) = - \inf_{t \geq 0} \tilde{f}_ \hat{c} (t)$

$\tilde{f}_ \hat{c}$ convex, $\tilde{f}_ \hat{c} (0) = 0, \tilde{f}_ \hat{c} (t) \rightarrow +\infty$ as $t \rightarrow +\infty$

$\Rightarrow$ $\tilde{f}_ \hat{c}$ has a unique min on $[0, +\infty)$

let $\hat{c} > 0$ denote the point where it is reached.

$\psi (\hat{c}) = - \frac{1}{2} \hat{c} \hat{c}^2 + \psi (\hat{c}) = - \inf_{t} \tilde{f}_ \hat{c} (t)$

$\Rightarrow \quad \psi (\hat{c}) = \frac{1}{2} \hat{c} \hat{c}^2 + \psi (\hat{c}) = \inf_{t} (\cdots )$

$\theta' (t) = - \frac{\psi' (V_{\hat{c}})}{2 \hat{c}^2}$ and $\lim_{t \rightarrow 0} \theta' (t) = - \frac{1}{2} \psi'' (0^+)$

$\tilde{f}_ \hat{c} (t) = \frac{1}{2} \hat{c} + \theta' (t)$ increasing because $\tilde{f}_ \hat{c}$ convex

we are looking for a solution on $\mathbb{R}_+$

If $\tilde{f}_ \hat{c} (0^+) > 0$ then the min is at $\hat{c} = 0$

$\Rightarrow \quad \hat{c} > \psi'' (0^+)$

Otherwise $\hat{c} > 0$ satisfies

$\frac{d}{d\hat{c}} \tilde{f}_ \hat{c} (\hat{c}) = 0 \Leftrightarrow \hat{c} = -2 \theta' (\hat{c})$

back to $\psi$ ... $\hat{c} = -2 \theta' (\hat{c}^2)$
\[ H(w) = \mathcal{H}(e^{\langle \nu, D_x u \rangle}) = 2A^T A + \beta \sum_{i=1}^n \left( \frac{\rho_i(C \langle \|D_x u\| \rangle)}{\|D_x u\|} \right) D_i^T D_i \]

The method amounts to
\[ u_k = (H(u_{k-1}))^{-1} 2A^T v = \ldots = u_{k-1} - (H(u_{k-1}))^{-1} \nabla F(u_{k-1}) \]

(* Additive form of HQ regularization. *)

Assumptions:

- \( \psi \) is convex and continuous
- \( \frac{1}{2} t^2 - \psi(t) \) is convex \( \Leftrightarrow \psi(0) = 0 \)
- \( \lim_{t \to 0} \frac{\psi(t)}{t} < \frac{1}{2} \)

\( S = 1 \) or \( 2 \) (each \( D_i \) matrix \( S \times P \))

\( \beta \in \mathbb{R}^S \) below, \( z \in \mathbb{R}^S \)

\[ \psi(b) = \sup_{\| z \|} \left\{ -\frac{1}{2} \| z \|_2^2 + \psi(\| z \|_2) \right\} \]

\[ g^*(x) = \sup_{\| z \|} \left\{ \frac{1}{2} \| z \|_2^2 - \frac{1}{2} \| z \|_2^2 + \langle z, x \rangle + \psi(\| z \|_2) \right\} \]

\[ \frac{g^*(x)}{2} = \sup_{\| z \|} \left\{ \langle z, x \rangle - \left( \frac{1}{2} \| z \|_2^2 + \psi(\| z \|_2) \right) \right\} \]

\( g^{**}(u) = \sup_{x} \left\{ \langle u, x \rangle - g^*(x) \right\} \) and \( g^{**}(u) = f(u) \)

\( g^{**}(u) = \sup_{x} \left\{ \langle u, x \rangle - g^*(x) \right\} = g(u) \)
\[ \frac{1}{2} \| z \|^2 - \Psi (\| z \|) = \sup_{\theta} \left\{ \langle z, \theta \rangle - \left( \psi (\theta) + \frac{1}{2} \| z \|^2 \right) \right\} \]

we can compute \( \theta \) for which the inf is reached.

\( (z, \theta) \) are such that the inf in the def. of \( \Psi \) is reached \( \Rightarrow \) the sup in the def. of \( \Psi \) is reached

\[ \Psi (\| z \|) = \frac{1}{2} \| z \|^2 - \theta \| z \|^2 + \psi (\theta) \]

\( \Rightarrow \psi (\theta) = -\frac{1}{2} \| z \|^2 - \theta \| z \|^2 + \psi (\| z \|) \]

\( \Rightarrow \) the grad with respect to \( \theta \) of the function \( z \rightarrow -\frac{1}{2} \| z \|^2 - \theta \| z \|^2 + \psi (\| z \|) \)

is null.

\[ - (\frac{\partial}{\partial \theta} + \psi' (\| z \|)) \frac{\partial}{\partial \theta} = 0 \]

\[ \hat{\theta} = - \frac{\partial}{\partial \theta} \psi' (\| z \|) \frac{\partial}{\partial \theta} \]

\[ F(u) = \| A u - v \|^2 + \beta \sum_{i=1}^m \left( \frac{1}{2} D_i u - b_i \right)^2 + \psi (b_i) \]

\[ \inf_{\theta} F(u, \theta) = F(u) \quad \inf_{\theta} \psi (\theta) = \psi (\| D_i u \|) \]
for $v$ fixed $\inf_b F(v,b)$ is given reached at $v = D_i u - \psi'(||D_i u||) \frac{D_i u}{||D_i u||}$ for $b$ fixed $\inf_u F(u,b)$ is reached at $u$ such that $D_u F(u,b) = 0$

$$D = \begin{bmatrix} D_x^1 \\ D_x^2 \\ \vdots \\ D_x^r \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{bmatrix}$$

$$F(u,b) = ||Au - \psi||^2 + \beta \frac{1}{2} ||Du - b||^2 + p \Sigma \psi(C; \cdot)$$

$$2A^T (Au - \psi) + \beta D^T (Du - b) = 0$$

$$2A^T A + \beta D^T D$$

$$H = \frac{2A^T u + \beta D^T b}{H}$$

$$\hat{u} = H^{-1} (2A^T u + \beta D^T b)$$

The alternate minimization:

$$\forall i \quad \hat{b}_i^{k+1} = D_i \hat{u}^{k-1} - \psi'(||D_i \hat{u}^{k-1}||) \frac{D_i \hat{u}^{k-1}}{||D_i \hat{u}^{k-1}||}$$

$$\hat{u}^{k+1} = \frac{1}{H} \text{approx of } D^2 F(u^{k+1})$$

$$u^{k+1} = u^{k-1} - (H^{-1}) \nabla F(u^{k+1})$$

so resp. to $\psi'' \leq 1$ all the time.
Both forms (\(\#\) and \(\oplus\)) converge established for \(\Phi\) convex smooth, can be interpreted as in terms of \(\Phi\)

majorizing quadratic function.

(Involve the inversion of a matrix)

convergence speed

\(\#\) needs less iterations than \(\oplus\) cost per iter. for \(\#\) \(\geq\) cost per iter. \(\oplus\)

Overall \(\oplus\) is faster.

both are faster than classical methods.

Matrix \(H(\mu)\) - inversion.

Truncated versions of \(H\alpha\) Regu:

the idea: compute \(H(\mu)\)\(^{-1}\) only approximately (several iter.)

\(\therefore\) still convergence proven.