

The most common situation:

Thm: $O \subset \mathbb{R}^n$ open, $g_i : O \rightarrow \mathbb{R}, 1 \leq i \leq p$
 C^1 on O

$$U = \{u \in O : g_i(u) = 0 \quad 1 \leq i \leq p\} \subset O$$

$\hat{u} \in U$ $\{Dg_i(\hat{u}), 1 \leq i \leq p\}$ lin. indep.

$F : O \rightarrow \mathbb{R}$ differentiable at \hat{u}

If F has a min. at \hat{u} constrained to U

Then $\exists \lambda_i(\hat{u}) \in \mathbb{R} \quad 1 \leq i \leq p$

(defined in a unique way) such that

$$DF(\hat{u}) + \sum_{i=1}^p \lambda_i(\hat{u}) Dg_i(\hat{u}) = 0$$

Proof

$$DG(\hat{u}) = \begin{bmatrix} Dg_1(\hat{u}) \\ \vdots \\ Dg_p(\hat{u}) \end{bmatrix} = \begin{pmatrix} \frac{\partial g_1}{\partial u_1}(\hat{u}) \dots \frac{\partial g_1}{\partial u_p}(\hat{u}) \dots \frac{\partial g_1}{\partial u_n}(\hat{u}) \\ \vdots \\ \frac{\partial g_p}{\partial u_1}(\hat{u}) \dots \frac{\partial g_p}{\partial u_p}(\hat{u}) \dots \frac{\partial g_p}{\partial u_n}(\hat{u}) \end{pmatrix}$$

rank = p
invertible $p \times p$ matrix

rank $DG(\hat{u}) = p$.

$\{e_1, e_2, \dots, e_n\}$ the standard basis of \mathbb{R}^n

$$V_2 = \text{span}(e_i, 1 \leq i \leq p)$$

$$V_1 = \text{span}(e_i, p+1 \leq i \leq n)$$

$$G : V_1 \times V_2 \rightarrow V_2 \in \mathbb{R}^p$$

$$(u_1, u_2) \rightarrow G(u) = \sum_{i=1}^p g_i(u) e_i \in V_2$$

then $D_2 G(\hat{u}) \in \text{Isom}(V_2)$ invertible submatrix

We apply the previous thm :

$\exists \lambda(\hat{u}) \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}) \iff \lambda_i(\hat{u}), 1 \leq i \leq p$
 such that $DF(\hat{u}) + \lambda(\hat{u}) DG(\hat{u}) = 0$

n eqs. $\left[DF(\hat{u}) + \sum \lambda_i(\hat{u}) Dg_i(\hat{u}) = 0 \right] \quad \square$

$n+p$ unknowns, $n+p$ nonlin. eq.
 These are necessary conditions.

Need to check if minimum. $\left[\begin{array}{l} g_i(\hat{u}) = 0 \\ p \text{ eq.} \end{array} \right.$

2.4.2. Important applications

(a) $F(u) = \frac{1}{2} \langle Bu, u \rangle - \langle c, u \rangle$ — affine subspace of \mathbb{R}^n
 $U = \{u \in \mathbb{R}^n : Au = v\}$ $v \in \mathbb{R}^p, A \in \mathbb{R}^{p \times n}$
 $\text{rank } A = p < n$

identification $g_i(u) = \underbrace{A_i^T}_{\text{row } i \text{ of } A} u - v_i$

$\nabla g_i(u) = A_i$

$\nabla F(u) = Bu - c$

$Bu - c + \sum_{i=1}^p \lambda_i A_i = 0$
 $\underbrace{\hspace{10em}}_{A^T \lambda}$

solve: $\begin{cases} Bu + A^T \lambda = c \\ Au = v \end{cases}$ $\left(\begin{array}{cc|c} B & A^T & u \\ A & 0 & \lambda \end{array} \right) = \left(\begin{array}{c} c \\ v \end{array} \right)$

(b) $w \in \mathbb{R}^n$ given find $\hat{u} = \Pi w$ the projection of w on U it solves



$\min_{u \in U} \frac{1}{2} \|u - w\|^2$
 $\underbrace{\hspace{10em}}_{F(u)}$

$G(u) = Au - v$
 $\nabla G(u) = A$
 $\nabla F(u) = u - w$

$$\nabla F(u) + \nabla G \lambda = 0$$

$$\begin{cases} u - w + A^T \lambda = 0 \\ Au = v \end{cases}$$

$A^T A$ invertible since $\text{rank } A = p$.

$$AA^T \lambda = Aw - v$$

$$\lambda = (AA^T)^{-1} (Aw - v)$$

$$\hat{u} = w - A^T \lambda$$

$$= w - A^T (AA^T)^{-1} (Aw - v)$$

$$\hat{u} = \underbrace{\left(I - A^T (AA^T)^{-1} A \right)}_{\text{proj on } (\ker A)^\perp} w + A^T (AA^T)^{-1} v$$

the proj on

$$\ker A = \{ u : Au = 0 \}$$

(c) Solving $Au = v$

$$\frac{\text{rank } A = p \leq n}{A: p \times n}$$

$\hat{u} : \inf \|u\|$
 $Au = v$
 the minimum norm solution.

$$F(u) = \|u\|^2 \text{ st } Au = v$$

$$\begin{cases} u + A^T \lambda = 0 \\ Au = v \end{cases}$$

AA^T invertible

$$\lambda = -(AA^T)^{-1} v$$

$$\boxed{\hat{u} = -A^T \lambda = A^T (AA^T)^{-1} v}$$

$$\text{rank } A = n \leq p$$

$p \times n$

in general there is no $u \in \mathbb{R}^p$ such that $Au = v$. Instead find \hat{u} yielding $\inf \|Au - v\|^2 =$ the least squares solution. ...

If $\text{rank } A < \min(n, p)$ - pseudo-inverse, generalised inverse...

2.5 Problems with inequality constraints

$$U = \{u \in \mathbb{R}^n : h_i(u) \leq 0 \quad 1 \leq i \leq q\}$$

↑ Hilbert

2.5.1 General cond. for a min.

Def. V vector space $U \neq \emptyset \quad U \subset V$

The cone of the acceptable directions, at $u \in U$

$$C(u) = \{0\} \cup \left\{ v \in V \setminus \{0\} : \exists (u_k)_{k \geq 0} \quad u_k \neq u, u_k \in U \right.$$

$$\left. \forall k \geq 0 \quad \lim_{k \rightarrow \infty} u_k = u, \quad \lim_{k \rightarrow \infty} \frac{u_k - u}{\|u_k - u\|} = \frac{v}{\|v\|} \right\}$$



$C(u)$ is not necessarily convex.

Lemma. $C(u)$ is closed for every $u \in U$

$$\{v_n\} \in C(u) \quad \forall n \quad (v_n)_{n \geq 0}$$

$$v_n \rightarrow v \in V \setminus \{0\} \quad v_n \neq 0 \quad \forall n$$

We have to show that $v \in C(u)$. \Leftrightarrow
 v satisfies the cond. of the definition.

$$\left[\begin{array}{l} v_n \in C(u) \\ \text{from the def. } \forall n \end{array} \right] \exists (u_k^n)_{k \geq 0} \subset U \quad u_k^n \neq u \quad \forall k$$

$$\lim_{k \rightarrow \infty} \frac{u_k^n - u}{\|u_k^n - u\|} = \frac{v^n}{\|v^n\|}, \quad \lim_{k \rightarrow \infty} u_k^n = u \quad \#$$

Let $(\epsilon_n)_{n \geq 0}$ $\epsilon_n > 0 \quad \forall n \quad \lim_{n \rightarrow \infty} \epsilon_n = 0$

$\forall n \exists k(n)$ s.t. $\|u_{k(n)}^n - u\| \leq \epsilon_n$

$$\left| \frac{u_{k(n)}^n - u}{\|u_{k(n)}^n - u\|} - \frac{v^n}{\|v^n\|} \right| \leq \epsilon_n$$

$$+ \frac{v}{\|v\|} - \frac{v}{\|v\|}$$

Δ inequality

$$\left\| \frac{u_{k(n)}^n - u}{\|u_{k(n)}^n - u\|} - \frac{v}{\|v\|} \right\| \leq \epsilon_n + \underbrace{\left\| \frac{v^n}{\|v^n\|} - \frac{v}{\|v\|} \right\|}_{n \rightarrow \infty \rightarrow 0}$$

$(u_{k(n)}^n)_n$ satisfies the requirements for $v \in C(u)$.

$u_{k_1}^1 \rightarrow u$
 $u_{k_2}^2 \rightarrow u$
 \vdots
 $u_{k_k}^k \rightarrow u$
 \vdots