The most commun situation:

Thm: $\mathcal{O} \subset \mathbb{R}^n$ open, $g_i : \mathcal{O} \to \mathbb{R}$, $1 \le i \le p$
C^1 on $\mathcal{O}$

$U = \{u \in \mathcal{O} : g_i(u) = 0 \quad 1 \le i \le p\} \subset \mathcal{O}$

$\hat{u} \in U$ $\{Dg_i(\hat{u}), 1 \le i \le p\}$ lin. indep.

$F : \mathcal{O} \to \mathbb{R}$ differentiable at $\hat{u}$

If F has a min. at $\hat{u}$ constrained to U

Then $\exists \lambda_i(\hat{u}) \in \mathbb{R}$ $1 \le i \le p$ (defined in a unique way) such that

$$DF(\hat{u}) + \sum_{i=1}^{p} \lambda_i(\hat{u}) Dg_i(\hat{u}) = 0$$

Proof

$$DG(\hat{u}) = \begin{bmatrix} Dg_1(\hat{u}) \\ \vdots \\ Dg_p(\hat{u}) \end{bmatrix} = \left(\begin{array}{ccc|c} \frac{\partial g_1}{\partial u_1}(\hat{u}) & \cdots & \frac{\partial g_1}{\partial u_p} & \cdots \frac{\partial g_1}{\partial u_n} \\ \vdots & D_2 G(\hat{u}) & & \\ \frac{\partial g_p}{\partial u_1}(\hat{u}) & \cdots & \frac{\partial g_p}{\partial u_p} & \cdots \frac{\partial g_p}{\partial u_n} \end{array} \right)$$

rank $= p$

invertible $p \times p$ matrix

rank $DG(\hat{u}) = p$.

$\{e_1, e_2, \dots, e_n\}$ the standard basis of $\mathbb{R}^n$

$V_2 = \mathrm{span}(e_i, 1 \le i \le p)$

$V_1 = \mathrm{span}(e_i, p+1 \le i \le n)$

$G : V_1 \times V_2 \to V_2 \in \mathbb{R}^p$

$(u_1, u_2) \to G(u) = \sum_{i=1}^{p} g_i(u) e_i \in V_2$

then $D_2 G(\hat{u}) \in \mathrm{Isom}(V_2)$ invertible submatrix

We apply the previous thm:

$\exists\, \lambda(\hat{u}) \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}) \iff \lambda_i(\hat{u}),\ 1 \le i \le p$

such that $D F(\hat{u}) + \underline{\lambda(\hat{u})}\, \underline{D G(\hat{u})} = 0$

$$n \text{ eqs.} \left[\ DF(\hat{u}) + \sum \lambda_i(\hat{u})\, Dg_i(\hat{u}) = 0 \right. \qquad \square$$

$n+p$ unknowns, $n+p$ nonlin. eq.

These are necessary conditions.
Need to check if minimum.

$\left[\begin{array}{l} g_i(\hat{u}) = 0 \\ p \text{ eq.} \end{array}\right.$

2.4.2. Important applications

(a) $F(u) = \frac{1}{2}\langle Bu, u\rangle - \langle c, u\rangle$

$\boxed{U = \{u \in \mathbb{R}^n : Au = v\}}$ — affine subspace of $\mathbb{R}^n$

$v \in \mathbb{R}^p$, $A \in \mathbb{R}^{p \times n}$

$\underline{\operatorname{rank} A = p} < n$

identification $\quad g_i(u) = \underbrace{A_i^T}_{\text{row } i \text{ of } A} u - v_i$

$\nabla g_i(u) = A_i$

$\nabla F(u) = Bu - c$

$$Bu - c + \underbrace{\sum_{i=1}^{p} \lambda_i A_i}_{A^T\lambda} = 0$$

solve wrt (u, λ): $\left[\begin{array}{l} Bu + A^T\lambda = c \\ Au = v \end{array}\right.$ $\qquad \begin{pmatrix} B & A^T \\ A & 0 \end{pmatrix}\begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} c \\ v \end{pmatrix}$

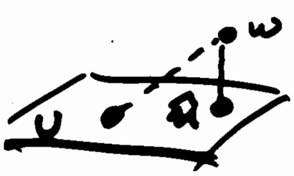
(b) $w \in \mathbb{R}^n$ given find $\hat{u} = \Pi w$ the projection of w on U it solves

$$\inf_{u \in U} \underbrace{\tfrac{1}{2}\|u - w\|^2}_{F(u)}$$

$G(u) = Au - v$

$\nabla G(u) = A$

$\nabla F(u) = u - w$



$\nabla F(u) + \nabla G \lambda = 0$

$$\begin{cases} u - w + A^T \underbrace{\lambda}_{\in \mathbb{R}^p} = 0 \\ Au = v \end{cases}$$

$A^T A$ invertible since rank $A = p$.

$$AA^T \lambda = Aw - \overset{=Au}{\widehat{v}}$$

$$\lambda = (AA^T)^{-1}(Aw - v)$$

$$\hat{u} = w - A^T \lambda$$
$$= w - A^T (AA^T)^{-1}(Aw - v)$$

$$\hat{u} = \underbrace{(I - \overbrace{A^T(AA^T)^{-1}A}^{\text{proj on}(\ker A)^\perp})}_{\text{the proj on}}\, w + A^T(AA^T)^{-1} v$$

$\ker A = \{u : Au = 0\}$

(c) Solving $Au = v$

rank $A = p \le n$

$A : p \times n$

$\hat{u} : \inf_{Au = v} \|u\|$

the minimum norm solution.

$F(u) = \|u\|^2$ st $Au = v$

$$\begin{cases} u + A^T \lambda = 0 \\ Au = v \end{cases}$$

AA^T invertible

$\lambda = -(AA^T)^{-1} v$

$$\boxed{\hat{u} = -A^T \lambda = A^T (AA^T)^{-1} v}$$

rank $A = n \le p$

$p \times n$

in general there is no $u \in R^p$ such that $Au = v$. Instead find $\bar{u}$ yielding

$\inf \|Au - v\|^2$

= the least squares solution. ...

If rank $A < \min(n, p)$ — pseudo-inverse, generalized inverse...

2.5 Problems with inequality constraints

$U = \{u \in V : h_i(u) \le 0 \quad 1 \le i \le q\}$

↑ Hilbert

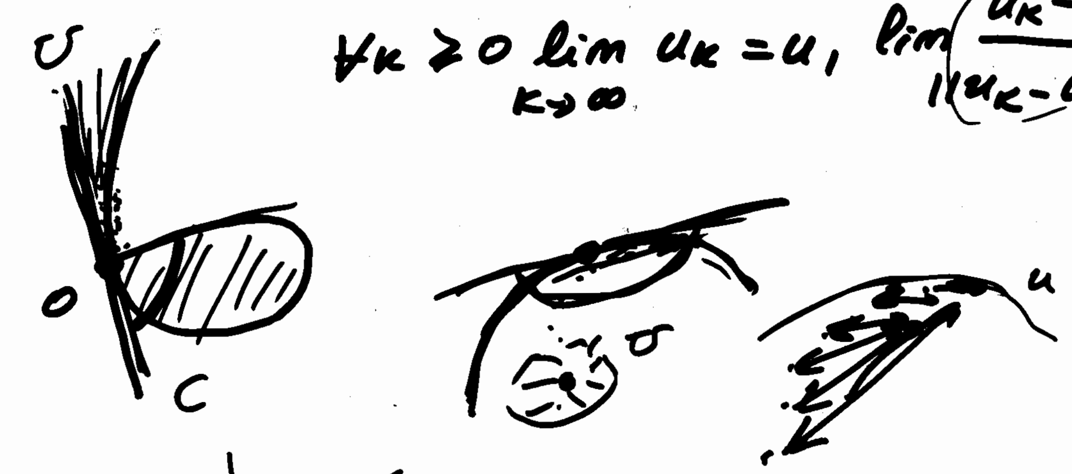
2.5.1 General cond. for a min.

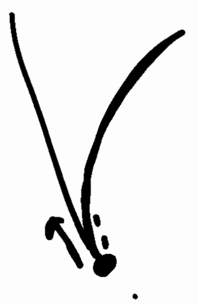
Def. V vector space $U \ne \emptyset$ $U \subset V$

The cone of the acceptable directions at $u \in U$

$C(u) = \{0\} \cup \{v \in V \setminus \{0\} : \exists (u_k) \; u_k \ne u, u_k \in U$

$\forall k \ge 0 \; \lim_{k\to\infty} u_k = u, \; \lim \dfrac{u_k - u}{\|u_k - u\|} = \dfrac{v}{\|v\|}\}$





$C(u)$ is not necessarily convex.

Lemma. $C(u)$ is closed for every $u \in U$

$v_n \in C(u) \; \forall n \quad (v_n)_{n \ge 0}$

$v_n \longrightarrow v \in V \setminus \{0\} \quad v_n \ne 0 \; \forall n$

We have to show that $\underline{v \in C(u)}$. $\Leftrightarrow$

v satisfies the cond. of the definition.

[$v_n \in C(u)$ from the def. $\forall n$ $\exists (u_k^n)_{k \ge 0} \subset U$ $u_k^n \ne u \; \forall k$

$\lim_{k\to\infty} \dfrac{u_k^n - u}{\|u_k^n - u\|} = \dfrac{v^n}{\|v^n\|}, \; \lim_{k\to\infty} u_k^n = u$ *

Let $(\epsilon_n)_{n \geq 0}$ $\quad \epsilon_n > 0 \ \forall n \quad \lim_{n \to 0} \epsilon_n = 0$

$$\begin{array}{l} u_k^1 \to u \\ u_k^2 \to u \\ \vdots \\ u_k^n \xrightarrow[k \to]{} u \\ \vdots \end{array}$$

$\forall n \quad \exists\, k(n) \quad$ s.t. $\quad \| u_{k(n)}^n - u \| \leq \epsilon_n$

$$\left| \frac{u_{k(n)}^n - u}{\| u_{k(n)}^n - u \|} - \frac{v^n}{\| v^n \|} \right| \leq \epsilon_n$$

$$+ \frac{v}{\|v\|} - \frac{v}{\|v\|}$$

Δ inequality

$$\left\| \frac{u_{k(n)}^n - u}{\| u_{k(n)}^n - u \|} - \frac{v}{\|v\|} \right\| \leq \epsilon_n + \underbrace{\left\| \frac{v^n}{\|v^n\|} - \frac{v}{\|v\|} \right\|}_{n \to \infty} \to 0$$

$(u_{k(n)}^n)_n$ satisfies the requirements for $v \in C(u)$.