Lem. U convex $\Rightarrow U \subset u + C(u) \quad \forall u$

we wish to show that $w \in U \Rightarrow \underbrace{w-u}_{v} \in C(u)$

$$u_n = \left(1 - \frac{1}{n+1}\right) u + \frac{1}{n+1} w \in U \left(\begin{array}{l}\text{convex}\\ \text{combi}\end{array}\right)$$

$n \geq 1$

$\lim\limits_{n\to\infty} u_n = u$, $u_n \neq u \ \forall n$ $\quad u_n - u = \frac{1}{n+2}(w-u)$

$$\frac{u_n - u}{\|u_n - u\|} = \frac{w-u}{\|w-u\|}$$

Hence u_n satisfy the cond. of the def. of $C(u)$.

Thm (necessary cond. for a constrained min)

V ~~and~~ normed vector space $\quad U \neq \emptyset \quad U \subset V$

$F : \mathcal{O}(U) \to \mathbb{R}$ differentiable at $\hat{u}$

If F has a constrained min. at $\hat{u}$

$$\Rightarrow \langle \nabla F(\hat{u}), \underbrace{u - \hat{u}}_{v} \rangle \geq 0 \quad \forall u \in \{\hat{u} + C(\hat{u})\}$$

$\forall v \in C(\hat{u})$

Proof. $v = u - \hat{u} \in C(\hat{u})$

from the def. of $C(\hat{u})$ $\exists (u_k)$, $u_k \in U \setminus \{\hat{u}\}$ s.t

$u_k \xrightarrow[k\to\infty]{} u \qquad \lim\limits_{k\to\infty} \frac{u_k - \hat{u}}{\|u_k - \hat{u}\|} = \frac{v}{\|v\|} \in V$

$F(\hat{u}) \leq F(u_k) \ \forall k$ because min at $\hat{u}$

from the lim $u_k - u$ ...

$$\underline{u_k - \hat{u}} = \frac{v}{\|v\|} \|u_k - \hat{u}\| + \delta_k \|u_k - \hat{u}\|$$

$\in V$ and $\delta_k \xrightarrow{k\to\infty} 0$

$$0 \le F(u_k) - F(\hat{u}) = \langle \nabla F(\hat{u}), \underline{u_k - \hat{u}} \rangle + \varepsilon_k \|u_k - \hat{u}\|$$

1st order expansion $\qquad \lim_{k\to\infty} \varepsilon_k = 0$

$$= \|u_k - \hat{u}\| \left(\langle \nabla F(\hat{u}), \frac{u_k - \hat{u}}{\|u_k - \hat{u}\|} \rangle + \varepsilon_k \right)$$

$$= \frac{v}{\|v\|} + \delta_k = \frac{v + \delta_k \|v\|}{\|v\|}$$

$$= \frac{\|u_k - \hat{u}\|}{\|v\|} \left(\langle \nabla F(\hat{u}), v \rangle + \langle \nabla F(\hat{u}), \delta_k \rangle \|v\| + \varepsilon_k \|v\| \right)$$

$\to 0 \qquad \to 0$

$$\Rightarrow \langle \nabla F(\hat{u}), v \rangle \ge 0$$

$\Rightarrow$ If U is convex $\quad \langle \nabla F(\hat{u}), u - \hat{u} \rangle \ge 0 \quad \forall u \in U$

(because $U \subset u + C(u) \ \forall u \in U$)

$$\bigcap_{i\in I} \ker f_i \overset{\text{linear}}{\subset} \ker h \Leftrightarrow \exists \alpha_i \text{ st. } h = \sum_{i\in I} \alpha_i f_i$$

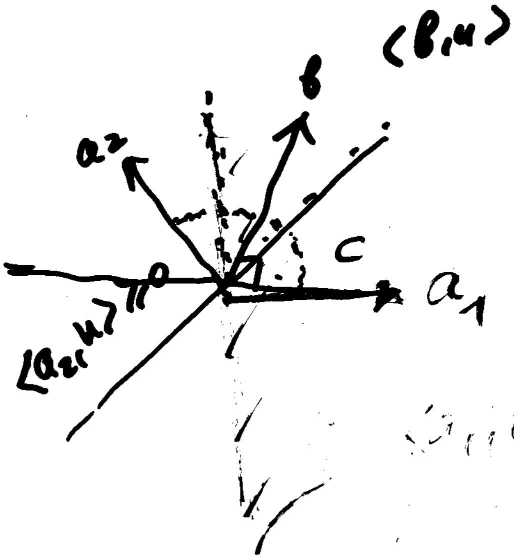
generalisation of this result:

Farkas-Minkowski Lemma

V – Hilbert space, I finite set of indexes $a_i \in V \ \forall i \in V$ and $b \in V$.

$$\{u \in V : \langle a_i, u \rangle \ge 0 \ \forall i \in I\} \subset \{u \in V : \langle b, u \rangle \ge 0\}$$

$$\Leftrightarrow \exists \lambda_i \ge 0 \ \forall i \in I : b = \sum_{i\in I} \lambda_i a_i$$



$\Leftarrow$ evident

$b = \sum \underset{\geq 0}{\lambda_i} a_i$

$\langle b, u \rangle = \sum \lambda_i \overset{\geq 0}{\langle a_i, u \rangle} \geq 0$

$\Rightarrow$: the proof in 3 parts.

define $C = \{ \sum_{i \in I} \lambda_i a_i \in V : \lambda_i \geq 0 \ i \in I \}$

~~we wish to sh~~ the lemma says that $b \in C$.

C is a closed convex cone with apex 0.

convexity

$\theta \sum_{i \in I} \underset{\geq 0}{\lambda_i} a_i + (1-\theta) \sum_{i \in I} \underset{\geq 0}{\mu_i} a_i = \sum_{i \in I} \underbrace{(\theta \lambda_i + (1-\theta) \mu_i)}_{\geq 0} a_i \in C$ hence

closeness.

- $\{a_i : i \in I\}$ linearly independent

$(v_k)_{k \geq 0} \subset C \qquad v_k = \sum_{i \in C} \underset{\geq 0}{\lambda_i^k} a_i \in C$ determined unique way

if v_k converges $\Rightarrow \lambda_i^k \xrightarrow[k \to \infty]{} \lambda_i \ \forall i \in I$

$\lim_{k \to \infty} v_k = \sum (\lim \lambda_i^k) a_i \in C$ hence C closed.

- $\{a_i \ i \in I\}$ lin. dependent:

v_k converge $\not\Leftrightarrow \lambda_i^k$ converge

$\exists \mu_i, i \in I$ s.t $\sum_{i \in I} \mu_i a_i = 0 \quad J = \{ i \in I : \mu_i < 0 \} \neq \emptyset$

$C \ni v = \sum_{i \in I} \lambda_i a_i = \sum_{i \in I} (\lambda_i + t \mu_i) a_i \quad t = \min_J - \frac{\lambda_i}{\mu_i} \geq 0 = \lambda_j / \mu_j$

at least $\lambda_j + t\mu_j = 0$ eg $\text{rank}(a_1 \dots a_n) = n-1$.

$$C = \bigcup_{j\in I} \Big\{ \sum_{i\in I\setminus j} \lambda_i a_i \quad \lambda_i \geq 0 \quad i \in I - \{j\} \Big\}$$

↑ if rank $\{a_i : I\setminus j\} < \#I - 1$
do the same

C is finite union of closed sets associated with ~~...~~ subsets of a_i that are linearly independent.

$\Rightarrow$ C is closed indeed.

Suppose $b \notin C$ then we will find a contradiction.

(b) Suppose $b \notin C$ $\quad b \in V\setminus C$

$\Rightarrow \exists\, h \in V \;\exists\, \alpha \in \mathbb{R}$ $\begin{cases} \langle u, h\rangle > \alpha & \forall u \in C \\ \langle b, h\rangle < \alpha \; * \end{cases}$ strict separation.

(Hahn-Banach thm)

(c) $\quad 0 \in C \quad 0 = \langle 0, h\rangle > \alpha \quad$ hence $\alpha < 0$

$\forall \lambda > 0$ because $\lambda u \in C$

$\langle \lambda u, h\rangle > \alpha$

$\langle u, h\rangle > \dfrac{\alpha}{\lambda} \quad \forall \lambda > 0 \quad \lambda \to \infty \quad \langle u, h\rangle \geq 0$

$a_i \in C \qquad \langle h, a_i\rangle \geq 0 \quad \forall\, i \in I$

But by $*$ $\langle b, h\rangle < 0$

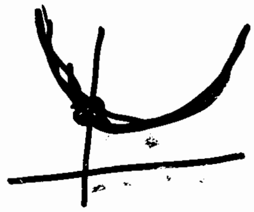
contradicts the inclusion assump. in the Lemma.

If $\{a_i : i \in I\}$ are linearly independent
then $\{\lambda_i : i \in I\}$ are determined in a unique way.

Def. $u \in U$ $\quad I(u)$ the active indexes

$$I(u) = \{1 \le i \le q : h_i(u) = 0\}$$

Remind $U = \{u : h_i(u) \le 0 \quad i = 1 \dots q\}$



$\overline{U}$

Def. The constraints are qualified at $u \in U$ if one of the conditions hold:

- $\exists\, w \in V \setminus \{0\}$ such that $\langle \nabla h_i(u), w\rangle \le 0$ where the inequality is strict if h_i is not affine
- h_i affine $\forall i \in I(u)$

not easy to handle, depends on u.

$$C^*(u) = \{v \in V : \langle \nabla h_i(u), v\rangle \le 0 \quad \forall i \in I(u)\}$$

cone ($\lambda v \quad \lambda \ge 0$)

convex bc intersection of half planes.

Lemma: $C(u) \subset C^*(u)$ (h_i differentiable)

Proof $i \in I(u)$ $\quad h_i$ reaches a maximum at u w.r.t U

from the necessary cond. for a min $\quad \langle -\nabla h_i(u), v\rangle \ge 0 \quad \forall v \in C(u)$

$\Rightarrow \langle \nabla h_i(u), v\rangle \le 0$ hence $v \in C^*(u)$

Thm. If h_i differentiable $\forall i \in I(u)$
h_i Continuous (u) $\forall i \in I - I(u)$
Constraints qualified at u

$\Rightarrow \quad C(u) = C^*(u)$