

LEM. $U \text{ convex} \Rightarrow U \subset u + C(u) \quad \forall u$



we wish to

show that $w \in U \Rightarrow \underbrace{w-u}_v \in C(u)$

$$u_n = \left(1 - \frac{1}{n+1}\right)u + \frac{1}{n+1}w \in U \quad \left(\begin{array}{l} \text{convex} \\ \text{combi} \end{array}\right)$$

$n \geq 1$

$$\lim_{n \rightarrow \infty} u_n = u, \quad u_n \neq u \quad \forall n \quad u_n - u = \frac{1}{n+1}(w-u)$$

$$\frac{u_n - u}{\|u_n - u\|} = \frac{w - u}{\|w - u\|}$$

hence u_n satisfy the cond. of the def. of $C(u)$.

Thm (necessary cond. for a constrained min)

V real normed vector space $U \neq \emptyset \quad U \subset V$

$F : \mathcal{O}(U) \rightarrow \mathbb{R}$ differentiable at \hat{u}

If F has a constrained min. at \hat{u}

$$\Rightarrow \langle \nabla F(\hat{u}), \underbrace{u - \hat{u}}_v \rangle \geq 0 \quad \forall u \in \{\hat{u} + C(\hat{u})\} \\ \forall v \in C(\hat{u})$$

Proof. $v = u - \hat{u} \in C(\hat{u})$

from the def. of $C(\hat{u}) \exists (u_k), u_k \in U \setminus \{\hat{u}\}$ s.t

$$\lim_{k \rightarrow \infty} \frac{u_k - \hat{u}}{\|u_k - \hat{u}\|} = \frac{v}{\|v\|} \in V$$

$F(\hat{u}) \leq F(u_k) \quad \forall k$ because min at \hat{u}

from the lim $u_k \rightarrow u \dots$

$$\frac{u_k - \hat{u}}{\|v\|} = \frac{\sigma}{\|v\|} \|u_k - \hat{u}\| + \delta_k \|u_k - \hat{u}\|$$

$\downarrow \in V$ and $\delta_k \xrightarrow{k \rightarrow \infty} 0$

$$0 \leq F(u_k) - F(\hat{u}) = \langle \nabla F(\hat{u}), u_k - \hat{u} \rangle + \epsilon_k \|u_k - \hat{u}\|$$

1st order expansion $\lim_{k \rightarrow \infty} \epsilon_k = 0$

$$= \|u_k - \hat{u}\| \left(\langle \nabla F(\hat{u}), \frac{u_k - \hat{u}}{\|u_k - \hat{u}\|} \rangle + \epsilon_k \right)$$

$$= \frac{\sigma}{\|v\|} + \delta_k = \frac{\sigma + \delta_k \|v\|}{\|v\|} \rightarrow 0$$

$$= \frac{\|u_k - \hat{u}\|}{\|v\|} \left(\langle \nabla F(\hat{u}), v \rangle + \langle \nabla F(\hat{u}), \delta_k \rangle + \epsilon_k \|v\| \right)$$

$$\Rightarrow \langle \nabla F(\hat{u}), v \rangle \geq 0$$

\Rightarrow If U is convex $\langle \nabla F(\hat{u}), u - \hat{u} \rangle \geq 0 \quad \forall u \in U$
 (because $U \subset u + C(u) \quad \forall u \in U$)

$$\bigcap_{i \in I} \ker f_i \subset \ker h \iff \exists \alpha_i \text{ st. } h = \sum_{i \in I} \alpha_i f_i$$

linear

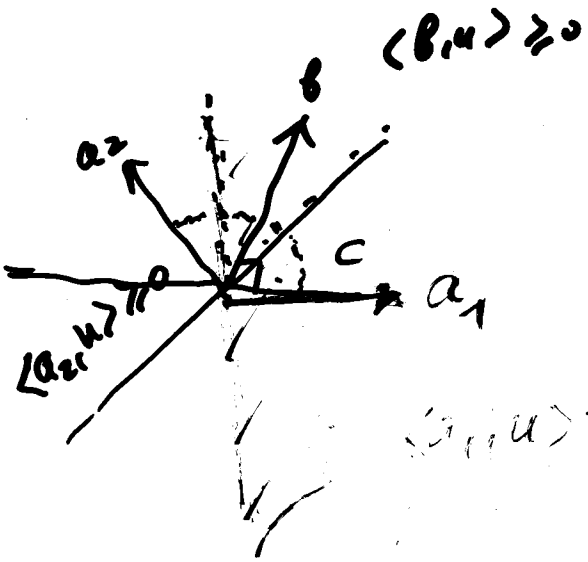
generalization of this result:

Farkes-Minkowski Lemma

V -Hilbert space, I finite set of indexes $a_i \in V \quad \forall i \in I$
 and $b \in V$.

$$\{u \in V : \langle a_i, u \rangle \geq 0 \quad \forall i \in I\} \subset \{u \in V : \langle b, u \rangle \geq 0\}$$

$$\iff \exists \lambda_i \geq 0 \quad \forall i \in I : b = \sum_{i \in I} \lambda_i a_i$$



\Leftarrow evident

$$b = \sum_{i \in I} \lambda_i a_i$$

$$\langle b, a \rangle = \sum_{i \in I} \lambda_i \langle a_i, a \rangle \ge 0$$

$\langle a_i, u \rangle \ge 0 \Rightarrow$: the proof in 3 parts.

define $C = \{ \sum_{i \in I} \lambda_i a_i \in V : \lambda_i \ge 0 \ i \in I \}$

We wish to show the lemma says that $b \in C$.

C is a closed convex cone with apex 0 .

$$\theta \sum_{i \in I} \lambda_i a_i + (1-\theta) \sum_{i \in I} \mu_i a_i = \sum_{i \in I} (\theta \lambda_i + (1-\theta) \mu_i) a_i \in C$$

≥ 0 hence

closedness.

$\{a_i : i \in I\}$ linearly independent determined unique way

$$(\nu_k)_{k \ge 0} \subset C \quad \nu_k = \sum_{i \in I} \lambda_i^k a_i \in C$$

if ν_k converges $\Rightarrow \lambda_i^k \xrightarrow{k \rightarrow \infty} \lambda_i \ \forall i \in I$

$$\lim_{k \rightarrow \infty} \nu_k = \sum (\lim \lambda_i^k) a_i \in C \text{ hence } C \text{ closed.}$$

$\{a_i : i \in I\}$ lin. dependent:

ν_k converge $\not\Rightarrow \lambda_i^k$ converge

$$\exists \mu_i, i \in I \text{ s.t. } \sum_{i \in I} \mu_i a_i = 0 \quad J = \{i \in I : \mu_i < 0\} \neq \emptyset$$

$$C \ni v = \sum_{i \in I} \lambda_i a_i = \sum_{i \in I} (\lambda_i + t \mu_i) a_i \quad t = \min_J - \frac{\lambda_i}{\mu_i} \ge 0$$

at least $\lambda_j + t\mu_j = 0$ $\text{rank}(a_1, \dots, a_n) = n-1$

$$C = \bigcup_{j \in I} \left\{ \sum_{i \in I \setminus j} \lambda_i a_i : \lambda_i \geq 0 \right\}$$

↑ if $\text{rank} \{a_i : i \in I \setminus j\} < \#I - 1$
do the same

C is finite union of closed sets associated with ~~these~~ subsets of a_i that are linearly independent.

\Rightarrow C is closed indeed.

Suppose $b \notin C$ then we will find a contradiction.

(b) Suppose $b \notin C$ $b \in V \setminus C$

$\Rightarrow \exists h \in V \exists \alpha \in \mathbb{R} \left\{ \begin{array}{l} \langle u, h \rangle > \alpha \quad \forall u \in C \\ \langle b, h \rangle < \alpha \end{array} \right. *$ strict separation.

(Hahn-Banach thm)

(c) $0 \in C$ $0 = \langle 0, h \rangle > \alpha$ hence $\alpha < 0$

$\forall \lambda > 0$ $\langle \lambda u, h \rangle > \alpha \quad \forall \lambda > 0$ because $\lambda u \in C$
 $\langle u, h \rangle > \frac{\alpha}{\lambda} \quad \forall \lambda > 0 \quad \lambda \rightarrow \infty \quad \langle u, h \rangle \geq 0$

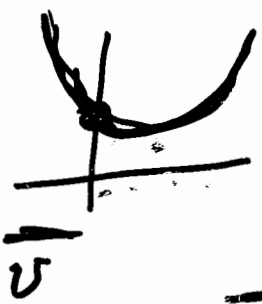
$a_i \in C$ $\langle h, a_i \rangle \geq 0 \quad \forall i \in I$

But by * $\langle b, h \rangle < 0$
contradicts the inclusion assump.
in the Lemma.

If $\{a_i : i \in I\}$ are linearly independent
 then $\{\lambda_i : i \in I\}$ are determined in a unique way.

Def. $u \in U$ $I(u)$ the active indexes
 $I(u) = \{1 \leq i \leq q : h_i(u) = 0\}$

Remind $U = \{u : h_i(u) \leq 0 \quad i = 1, \dots, q\}$



Def. The constraints are qualified at $u \in U$ if one of the conditions hold:

- $\exists w \in V \setminus \{0\}$ such that $\langle \nabla h_i(u), w \rangle \leq 0$ where the inequality is strict if h_i is not affine
- h_i affine $\forall i \in I(u)$

not easy to handle, depends on u .

$C^*(u) = \{v \in V : \langle \nabla h_i(u), v \rangle \leq 0 \quad \forall i \in I(u)\}$

Cone ($\lambda v \quad \lambda \geq 0$)

convex bc intersection of half planes.



Lemma: $C(u) \subset C^*(u)$ (h_i differentiable)

Proof: $i \in I(u)$ h_i reaches a maximum at u w.r.t U
 $\langle -\nabla h_i(u), v \rangle \geq 0 \quad \forall v \in C(u)$
 $\Rightarrow \langle \nabla h_i(u), v \rangle \leq 0 \quad \forall v \in C(u)$
 hence $v \in C^*(u)$

from the necessary cond. for min

Thm. If f_i differentiable $\forall i \in I(u)$
 f_i continuous $(u) \forall i \in I - I(u)$
 Constraints qualified at u

$\Rightarrow C(u) = C^*(u)$

Q.E.D.