THE NEAREST BISYMMETRIC SOLUTIONS OF LINEAR MATRIX EQUATIONS

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Abstract

The necessary and sufficient conditions for the existence of and the expressions for the bisymmetric solutions of the matrix equations

\[ \begin{align*}
(I) & \quad A_1 X_1 B_1 + A_2 X_2 B_2 + \cdots + A_k X_k B_k = D, \\
(II) & \quad A_1 X B_1 + A_2 X B_2 + \cdots + A_k X B_k = D \quad \text{and} \\
(III) & \quad (A_1 X B_1, A_2 X B_2, \ldots, A_k X B_k) = (D_1, D_2, \ldots, D_k)
\end{align*} \]

are derived by using Kronecker product and Moore-Penrose generalized inverse of matrices. In addition, in corresponding solution set of the matrix equations, the explicit expression of the nearest matrix to a given matrix in the Frobenius norm is given. Numerical methods and numerical experiments of finding the nearest solutions are also provided.

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Key words: Bisymmetric matrix, Matrix equation, Matrix nearness problem, Kronecker product, Frobenius norm, Moore-Penrose generalized inverse.

1. Introduction

Denote by \( R^n \) the set of all real \( n \)-component vectors, \( R^{m \times n} \) the set of all \( m \times n \) real matrices and \( BSR^{n \times n} \) the set of all \( n \times n \) real bisymmetric matrices (A symmetric matrix \( A = (a_{ij}) \in R^{n \times n} \) is called bisymmetric if \( a_{ij} = a_{n+1-j,n+1-i} \) for all \( 1 \leq i, j \leq n \)). \( I_n \) represents the \( n \times n \) identity matrix. \( \|A\|_F, A^+ \) and \( A^T \) stand for the Frobenius norm, Moore-Penrose generalized inverse and transpose of a matrix \( A \), respectively. On \( R^{m \times n} \) we define inner product: \( \langle A, B \rangle = \text{trace}(B^T A) \) for all \( A, B \in R^{m \times n} \), then \( R^{m \times n} \) is a Hilbert inner product space and the norm of a matrix generated by this inner product is Frobenius norm. For \( A = (a_{ij}) \in R^{m \times n}, B = (b_{ij}) \in R^{p \times q} \), let \( A \otimes B \in R^{mp \times nq} \) be the Kronecker product of \( A \) and \( B \).

Various aspects for the solution of linear matrix equations have been investigated. For example, Baksalary and Kala [1], Chu [4], He [8], and Xu, Wei and Zheng [13] considered the nonsymmetric solution of the matrix equation \( AXB + CXD = E \) by using Moore-Penrose generalized inverse and the generalized singular value decomposition of matrices, while Chang and Wang [3], Jameson [9] and Dai [6] considered the symmetric conditions on the solution of the matrix equations: \( AXA^T + BYB^T = C, AX + YA = C, AX = YB \) and \( AXB = C \). Zietak [14, 15] discussed the \( l_p \)-solution and chebyshev-solution of the matrix equation \( AX + YB = C \). Dobovisek [7] discussed the minimal solution of the matrix equation \( AX - YB = 0 \). Chu [5], and Kucera [11] and Jameson [10] are, respectively, studied the nonsymmetric solution of the matrix equation \( AXB + CXD = E \) and its special case \( AX + XB = C \). Mitra [12], Chu [4] and the references therein studied the nonsymmetric solution of the matrix equation \( (AXB, CXD) = (E, F) \).

In this paper, the following problems are considered

\[ \begin{align*}
(I) & \quad A_1 X_1 B_1 + A_2 X_2 B_2 + \cdots + A_k X_k B_k = D, \\
(II) & \quad A_1 X B_1 + A_2 X B_2 + \cdots + A_k X B_k = D \quad \text{and} \\
(III) & \quad (A_1 X B_1, A_2 X B_2, \ldots, A_k X B_k) = (D_1, D_2, \ldots, D_k)
\end{align*} \]

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Problem I. Given \( X_i^* \in R^{n_i \times n_i}, A_i \in R^{p_i \times n_i}, B_i \in R^{n_i \times q_i} \) (\( i = 1, 2, \cdots, k \)) and \( D \in R^{p \times q} \). Let \( H_1 = \{ [X_1, X_2, \cdots, X_k] : A_1X_1B_1 + A_2X_2B_2 + \cdots + A_kX_kB_k = D, X_i \in BSR^{n_i \times n_i} \} \). Find \( \hat{X}_1, \hat{X}_2, \cdots, \hat{X}_k \in H_1 \) such that

\[
\|\hat{X}_1, \cdots, \hat{X}_k] - [X^*_1, \cdots, X^*_k]\|_F = \min_{[X_1, \cdots, X_k] \in H_1} \|X_1, \cdots, X_k] - [X^*_1, \cdots, X^*_k]\|_F.
\] (1.2)

Problem II. Given \( X^* \in R^{n \times n}, A_i \in R^{p \times n}, B_i \in R^{n \times q_i} \) (\( i = 1, 2, \cdots, k \)) and \( D \in R^{p \times q} \). Let \( H_2 = \{ X \in BSR^{n \times n} : A_1XB_1 + A_2XB_2 + \cdots + A_kXB_k = D \} \). Find \( \hat{X} \in H_2 \) such that

\[
\|\hat{X} - X^*\|_F = \min_{X \in H_2} \|X - X^*\|_F.
\] (1.3)

Problem III. Given \( X^* \in R^{n \times n}, A_i \in R^{p_i \times n}, B_i \in R^{n \times q_i} \) and \( D_i \in R^{p_i \times q_i} \) (\( i = 1, 2, \cdots, k \)). Let \( H_3 = \{ X \in BSR^{n \times n} : A_1XB_1 = D_1, A_2XB_2 = D_2, \cdots, A_kXB_k = D_k \} \). Find \( \hat{X} \in H_3 \) such that

\[
\|\hat{X} - X^*\|_F = \min_{X \in H_3} \|X - X^*\|_F.
\] (1.4)

Using Kronecker product and Moore-Penrose generalized inverse of matrices, the necessary and sufficient conditions for the existence of and the explicit expressions for the solution of Problem I, II and III are derived. Numerical methods and numerical experiments of finding the nearest solutions are also provided.

2. Solving Problems I, II and III

For matrix \( A \in R^{m \times n} \), denotes by \( vec(A) \) the following vector containing all the entries of matrix \( A \):

\[
vec(A) = [A(1,:), A(2,:), \cdots, A(n,:)]^T \in R^{mn},
\] (2.1)

where \( A(i,:) \) denote \( i \)th row of matrix \( A \). For vector \( x \in R^{n^2} \), denote by \( vec_n^{-1}(x) \) the following matrix containing all the entries of vector \( x \):

\[
vec_n^{-1}(x) = \begin{pmatrix}
    x(1:n)^T \\
    x(n+1:2n)^T \\
    \vdots \\
    x[(n-1)n+1:n^2]^T
\end{pmatrix} \in R^{n \times n},
\] (2.2)

where \( x(i:j) \) denotes elements \( i \) to \( j \) of vector \( x \).

Let

\[
vec(BSR^{n \times n}) = \{ vec(A) : A \in BSR^{n \times n} \} \subset R^{n^2},
\] (2.3)

then the dimension of the subspace \( vec(\\text{BSR}^{n \times n}) \) is \( r = (n+1)^2/4 \) when \( n \) is odd or \( r = n(n+2)/4 \) when \( n \) is even. Suppose \( w_1, w_2, \cdots, w_r \) is an orthonormal basis-set for \( vec(\\text{BSR}^{n \times n}) \). For example, suitable \( w_i \) for \( n = 3 \) might be \( w_1 = [\frac{1}{\sqrt{2}}, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T \), \( w_2 = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T \), \( w_3 = [0, 0, 0, 0, 0, 0, \sqrt{2}, 0, 0, 0]^T \), \( w_4 = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T \). Consequently

\[
W = [w_1, w_2, \cdots, w_r] \in R^{n^2 \times r}
\] (2.4)
is a basis-matrix for $\text{vec}(\text{BSR}^{n \times n})$ and
\[
W^TW = I_r, \quad R[W] = \text{vec}(\text{BSR}^{n \times n}), \quad R[W^T] = R^r, \quad (2.5)
\]
where $R[W]$ (or $R[W^T]$) denotes the range of $W$ (or $W^T$).

For bisymmetric $A$, let $\bar{\text{vec}}(A)$ denotes the vector of coordinates of $\text{vec}(A)$ with respect to the basis-set $w_1, w_2, \cdots, w_r$, i.e., with respect to the columns of $W$. Then, in view of (2.3),(2.4) and (2.5),
\[
\text{vec}(A) = W\bar{\text{vec}}(A) \in R^n, \quad \bar{\text{vec}}(A) = W^T\text{vec}(A) \in R^r
\]
and
\[
\text{vec}^{-1}_n(Wx) \in \text{BSR}^{n \times n}
\]
for any vector $x \in R^r$.

The following lemma 2.1 and 2.2 are well known results, see, for instance, Baksalary and Kala [1] and Ben-Israel and Greville [2].

**Lemma 2.1.** For any matrices $A$, $B$ and $X$ in suitable size, one gets
\[
\text{vec}(AXB) = (A \otimes B^T)\text{vec}(X). \quad (2.8)
\]

**Lemma 2.2.** The matrix equation $AX = B$, with $A \in R^{p \times m}$ and $B \in R^{p \times n}$, has a solution $X \in R^{m \times n}$ if and only if $AA^+B = B$. In that case it has the general solution $X = A^+B + (I_m - A^+A)G$, where $G \in R^{m \times n}$ is arbitrary matrix.

**Lemma 2.3.** Given $A \in R^{p \times m}$ and $B \in R^{p \times n}$, then the optimal approximation problem
\[
\min_{X \in R^{m \times n}} \| (I_m - A^+A)X - B \|_F \quad (2.9)
\]
has a solution which can be expressed as $X = B + A^+G$, where $G \in R^{p \times n}$ is arbitrary matrix.

**Proof.** Applying the properties of Moore-Penrose generalized inverse and the inner product defined in space $R^{m \times n}$, we have
\[
\| (I_m - A^+A)X - B \|_F^2 = \langle (I_m - A^+A)X - B, (I_m - A^+A)X - B \rangle
\]
\[
= \langle (I_m - A^+A)(X - B), (I_m - A^+A)(X - B) \rangle + \langle A^+AB, A^+AB \rangle
\]
\[
= \| (I_m - A^+A)(X - B) \|_F^2 + \| A^+AB \|_F^2.
\]
Hence,
\[
\min_{X \in R^{m \times n}} \| (I_m - A^+A)X - B \|_F
\]
if and only if
\[
\min_{X \in R^{m \times n}} \| (I_m - A^+A)(X - B) \|_F.
\]
It is clear that $X = B + A^+G$, with $G \in R^{p \times n}$ be arbitrary, is the solution of the above optimal approximation problem. So, the solution of the optimal approximation problem (2.9) can be expressed as $X = B + A^+G$.

The following theorems are the mainly results in this paper.

**Theorem 2.1.** Given $X_i^r \in R^{n_i \times n_i}, A_i \in R^{p \times n_i}, B_i \in R^{n_i \times q} \quad (i = 1, 2, \cdots, k)$ and $D \in R^{p \times q}$. Assume $W_i \in R^{n_i \times r_i}$ is the basis-matrix for subspace $\text{vec}(\text{BSR}^{n_i \times n_i})$, let
\[
A_0 = \left( (A_1 \otimes B_1^T)W_1, (A_2 \otimes B_2^T)W_2, \cdots, (A_k \otimes B_k^T)W_k \right), \quad (2.10)
\]
then the set $H_1$ is nonempty if and only if
\[ A_0 A_0^+ \text{vec}(D) = \text{vec}(D). \] (2.11)

When the condition (2.11) is satisfied, $H_1$ can be expressed as
\[ H_1 = \{ \text{vec}_{n_1}^{-1}(W_1 \alpha(1 : r_1)), \ldots, \text{vec}_{n_k}^{-1}(W_k \alpha(r_1 + \cdots + r_{k-1} + 1 : r_1 + \cdots + r_k)) \} : \]
\[ \alpha = A_0^+ \text{vec}(D) + (I_{r_1+r_2+\cdots+r_k} - A_0^+ A_0) G, \quad G \in \mathbb{R}^{r_1+r_2+\cdots+r_k} \} . \] (2.12)

In $H_1$, there exists an unique $[\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_k]$ that makes (1.2) held and
\[ \hat{X}_i = \text{vec}_{n_1}^{-1}(W_i \hat{\alpha}(r_1 + r_2 + \cdots + r_{i-1} + 1 : r_1 + r_2 + \cdots + r_i)), \quad i = 1, 2, \ldots, k, \] (2.13)
where
\[ \hat{\alpha} = A_0^+ \text{vec}(D) + (I_{r_1+r_2+\cdots+r_k} - A_0^+ A_0) \begin{pmatrix} W_1^T \text{vec}(X_1^1) \\ W_2^T \text{vec}(X_2^2) \\ \vdots \\ W_k^T \text{vec}(X_k^k) \end{pmatrix} . \] (2.14)

Proof. If the set $H_1$ is nonempty, then the matrix equation
\[ A_1 X_1 B_1 + A_2 X_2 B_2 + \cdots + A_k X_k B_k = D \] (2.15)
has a solution $X_i \in \text{BSR}^{(n_i \times n_i)}$ $(i = 1, 2, \ldots, k)$. From Lemma 2.1, we have
\[ (A_1 \otimes B_1^T) \text{vec}(X_1) + (A_2 \otimes B_2^T) \text{vec}(X_2) + \cdots + (A_k \otimes B_k^T) \text{vec}(X_k) = \text{vec}(D), \] (2.16)
which is, in view of (2.6), equivalent to
\[ (A_1 \otimes B_1^T) W_1 \vec{\text{vec}}(X) + (A_2 \otimes B_2^T) W_2 \vec{\text{vec}}(X_2) + (A_k \otimes B_k^T) W_k \vec{\text{vec}}(X_k) = \text{vec}(D), \]
i.e.,
\[ \begin{pmatrix} (A_1 \otimes B_1^T) W_1, (A_2 \otimes B_2^T) W_2, \ldots, (A_k \otimes B_k^T) W_k \end{pmatrix} \begin{pmatrix} \vec{\text{vec}}(X_1) \\ \vec{\text{vec}}(X_2) \\ \vdots \\ \vec{\text{vec}}(X_k) \end{pmatrix} = \text{vec}(D). \] (2.17)
Hence, we have from Lemma 2.2 and (2.10) that (2.11) is held.

Conversely, if (2.11) is held, we have from Lemma 2.2 that (2.17) has a solution which has explicit expression as
\[ \alpha = \begin{pmatrix} \vec{\text{vec}}(X_1) \\ \vec{\text{vec}}(X_2) \\ \vdots \\ \vec{\text{vec}}(X_k) \end{pmatrix} = A_0^+ \text{vec}(D) + (I_{r_1+r_2+\cdots+r_k} - A_0^+ A_0) G, \] (2.18)
where $G \in \mathbb{R}^{r_1+r_2+\cdots+r_k}$ is arbitrary. Hence, $\text{vec}(X_i) = W_i \vec{\text{vec}}(X_i) = W_1 \alpha(r_1 + r_2 + \cdots + r_{i-1} + 1 : r_1 + r_2 + \cdots + r_i)$ $(i = 1, 2, \ldots, k)$, where the first equality gets from (2.6), are the general solution of the equation (2.16), i.e., $X_i = \text{vec}_{n_i}^{-1}(W_i \alpha(r_1 + r_2 + \cdots + r_{i-1} + 1 : r_1 + r_2 + \cdots + r_i)) \in \text{BSR}^{(n_i \times n_i)}$, which gets from (2.7), are the general solution of the matrix equation (2.15). This implies that the set $H_1$ is nonempty.
In addition, $H_1$ is a close convex set, so there is an unique $[\hat{X}_1, \hat{X}_2, \cdots, \hat{X}_k] \in H_1$ that makes (1.2) held. Choose $\hat{W}_i$ such that $(W_i, \hat{W}_i)$ $(i = 1, 2, \cdots, k)$ is an $n_1^2 \times n_2^2$ orthogonal matrix. Using the invariance of the Frobenius norm under orthogonal transformations, we have from (2.6) and (2.18) that

$$
||[X_1, X_2, \cdots, X_k] - [X_1^*, X_2^*, \cdots, X_k^*]\|^2_F
$$

$$
= \left\| \begin{pmatrix} \text{vec}(X_1) \\ \text{vec}(X_2) \\ \vdots \\ \text{vec}(X_k) \end{pmatrix} - \begin{pmatrix} \text{vec}(X_1^*) \\ \text{vec}(X_2^*) \\ \vdots \\ \text{vec}(X_k^*) \end{pmatrix} \right\|_F^2
$$

$$
= \left\| \begin{pmatrix} W_1 & 0 & \cdots & 0 \\ 0 & W_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W_k \end{pmatrix} \begin{pmatrix} \text{vec}(X_1) \\ \text{vec}(X_2) \\ \vdots \\ \text{vec}(X_k) \end{pmatrix} - \begin{pmatrix} \text{vec}(X_1^*) \\ \text{vec}(X_2^*) \\ \vdots \\ \text{vec}(X_k^*) \end{pmatrix} \right\|_F^2
$$

$$
= \left\| (I_{r_1+r_2+\cdots+r_k} - A_0^+ A_0) G + A_0^+ \text{vec}(D) - \begin{pmatrix} W_1^T \text{vec}(X_1^*) \\ W_2^T \text{vec}(X_2^*) \\ \vdots \\ W_k^T \text{vec}(X_k^*) \end{pmatrix} \right\|_F^2.
$$

Hence, there exists $[X_1, X_2, \cdots, X_k] \in H_1$ such that $||[X_1, X_2, \cdots, X_k] - [X_1^*, X_2^*, \cdots, X_k^*]\|^2_F = \min$ is equivalent to exist $G \in R^{r_1+r_2+\cdots+r_k}$ such that

$$
\left\| (I_{r_1+r_2+\cdots+r_k} - A_0^+ A_0) G + A_0^+ \text{vec}(D) - \begin{pmatrix} W_1^T \text{vec}(X_1^*) \\ W_2^T \text{vec}(X_2^*) \\ \vdots \\ W_k^T \text{vec}(X_k^*) \end{pmatrix} \right\|_F = \min.
$$

From Lemma 2.3, we know that the solution of the above optimal approximate problem is

$$
G = \begin{pmatrix} W_1^T \text{vec}(X_1^*) \\ W_2^T \text{vec}(X_2^*) \\ \vdots \\ W_k^T \text{vec}(X_k^*) \end{pmatrix} - A_0^+ \text{vec}(D) + A_0^+ G_0,
$$

where $G_0 \in R^{r_1+r_2+\cdots+r_k}$ is arbitrary. Taking $G$ into (2.18) and furthermore using (2.6) and (2.7) we know that the solution of Problem I can expressed as (2.13).

Similar to the proof of Theorem 2.1, we can prove the following theorems 2.2 and 2.3.

Theorem 2.2. Given $X^* \in R^{n \times n}$, $A_i \in R^{p \times n}$, $B_i \in R^{n \times q}$ $(i = 1, 2, \cdots, k)$ and $D \in R^{p \times q}$. Assume $W \in R^{r \times r}$ is the basis-matrix for subspace vec($BSR^{n \times n}$), let

$$
A_0 = \left( (A_1 \otimes B_1^T) + (A_2 \otimes B_2^T) + \cdots + (A_k \otimes B_k^T) \right) W,
$$

...
then the set $H_2$ is nonempty if and only if
\[
A_0 A_0^T \text{vec}(D) = \text{vec}(D).
\]
Under this condition, $H_2$ can be expressed as
\[
H_2 = \{ \text{vec}_n^{-1}(W \alpha) \in BSR^{n \times n} : \alpha = A_0^+ \text{vec}(D) + (I_r - A_0^+ A_0)G, G \in R^r \}.
\]
In $H_2$, there exists an unique $\hat{X}$ that makes (1.4) hold and
\[
\hat{X} = \text{vec}_n^{-1}[W A_0^+ \text{vec}(D) + W(I_r - A_0^+ A_0)WT \text{vec}(X^*)].
\]

**Theorem 2.3.** Given $X^* \in R^{n \times n}, A_i \in R^{p_i \times n}, B_i \in R^{n \times q_i}$, $(i = 1, 2, \cdots, k)$ and $D_i \in R^{p_i \times q_i}$. Assume $W \in R^{n^2 \times r}$ is the basis-matrix for subspace $\text{vec}(BSR^{n \times n})$, let
\[
A_0 = \begin{pmatrix}
A_1 \otimes B_1^T \\
A_2 \otimes B_2^T \\
\vdots \\
A_k \otimes B_k^T
\end{pmatrix}, \quad D_0 = \begin{pmatrix}
\text{vec}(D_1) \\
\text{vec}(D_2) \\
\vdots \\
\text{vec}(D_k)
\end{pmatrix},
\]
then the set $H_3$ is nonempty if and only if
\[
A_0 A_0^T D_0 = D_0.
\]
Under this condition, $H_3$ can be expressed as
\[
H_3 = \{ \text{vec}_n^{-1}(W \alpha) \in BSR^{n \times n} : \alpha = A_0^+ D_0 + (I_r - A_0^+ A_0)G, G \in R^r \}.
\]
In $H_3$, there exists an unique $\hat{X}$ that makes (1.6) hold and
\[
\hat{X} = \text{vec}_n^{-1}[W A_0^+ D_0 + W(I_r - A_0^+ A_0)WT \text{vec}(X^*)].
\]

### 3. The Algorithm Description and Numerical Examples

The discussion in section 2 provides us with a recipe for finding the solutions of Problem I, II and III. In summary, the method, for example Problem I, is as follows.

1. Taking an orthogonal basis-set $\xi_1, \xi_2, \cdots, \xi_r \in R^{n^2}$ for $\text{vec}(BSR^{n \times n})$ $(i = 1, 2, \cdots, k)$, then construction a basis-matrix $W_i = (\xi_1, \xi_2, \cdots, \xi_r) \in R^{n^2 \times n}$ for $\text{vec}(BSR^{n \times n})$.

2. Let $A_0 = (A_1 \otimes B_1^T)W_1, (A_2 \otimes B_2^T)W_2, \cdots, (A_k \otimes B_k^T)W_k$.

3. If $A_0^+ A_0 \text{vec}(D) = \text{vec}(D)$, go to step (4), otherwise go to step (5).

4. According to (2.14) and (2.15) calculate $\hat{X}_i$ $(i = 1, 2, \cdots, k)$.

5. Stop.

Using the above compute steps for solving Problem I, we now give an example to illustrate that the results obtained in this paper are correct.

**Example 1.** For Problem I, taking $k = 3, p = 4, q = 3, n_1 = 3, n_2 = 4, n_3 = 5$ and

\[
X_1^* = \begin{pmatrix}
-6 & 3 & 1 \\
2 & 5 & 4 \\
1 & 7 & -9
\end{pmatrix}, \quad X_2^* = \begin{pmatrix}
3 & -2 & 3 & 5 \\
-2 & 6 & -7 & 3 \\
3 & -5 & 0 & -2 \\
7 & 3 & -2 & 3
\end{pmatrix}, \quad X_3^* = \begin{pmatrix}
3 & 1 & -7 & 6 & -2 \\
5 & 5 & 7 & 5 & -9 \\
-4 & 3 & 5 & 4 & -1 \\
-5 & 4 & 3 & 1 & 8 \\
-3 & 6 & -1 & 7 & -9
\end{pmatrix}.
\]
A equations: After calculating on the microcomputer through making program, we have an unique \( \hat{A} \).

Matrices will, in general, be very large by using Kronecker products. Of course, this method can only explore the small size of matrices because the size of the resulting matrices will, in general, be very large by using Kronecker products.

\[
A_1 = \begin{pmatrix} 4 & -2 & -3 \\ -2 & 4 & 1 \\ 1 & 3 & -2 \\ -1 & 2 & 3 \end{pmatrix},
A_2 = \begin{pmatrix} 2 & 2 & 3 & -4 \\ 2 & -1 & 2 & 3 \\ 3 & 2 & 1 & -4 \\ 1 & 2 & 3 & -1 \end{pmatrix},
A_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & -5 \\ -2 & 3 & -4 & 5 & 1 \\ 2 & -3 & 2 & 4 & 5 \\ 1 & 2 & 2 & -2 & 2 \end{pmatrix},
B_1 = \begin{pmatrix} 2 & 3 & -1 \\ 2 & 3 & -1 \\ 3 & 5 & 3 \end{pmatrix},
B_2 = \begin{pmatrix} 2 & -2 & 2 \\ 1 & 2 & -1 \\ 4 & 1 & -2 \\ 3 & 2 & -1 \end{pmatrix},
B_3 = \begin{pmatrix} 1 & -2 & -1 \\ 1 & 4 & 3 \\ 2 & 1 & -1 \\ 2 & -3 & 2 \\ 1 & 1 & -1 \end{pmatrix},
D = \begin{pmatrix} 49 & -52 & 29 \\ 84 & 21 & -6 \\ 75 & 6 & 60 \\ 33 & 19 & 32 \end{pmatrix}.
\]

It is easy to verify that these given matrices satisfy the conditions of the Theorem 2.1. After calculating on the microcomputer through making program, we have an unique \( \hat{X}_1 \in BSR^{3 \times 3} \), \( \hat{X}_2 \in BSR^{4 \times 4} \) and \( \hat{X}_3 \in BSR^{5 \times 5} \) as follow:

\[
\hat{X}_1 = \begin{pmatrix} -2.5445 & 2.9599 & -1.4397 \\ 2.9599 & 1.4630 & 2.9599 \\ -1.4397 & 2.9599 & -2.5545 \end{pmatrix},
\hat{X}_2 = \begin{pmatrix} 2.7080 & -2.2338 & 2.2628 & -0.1725 \\ -2.2338 & 2.0599 & -4.0876 & 2.2628 \\ 2.2628 & -4.0876 & 2.0599 & -2.2338 \\ -0.1725 & 2.2628 & -2.2338 & 2.7080 \end{pmatrix},
\]

Conclusions. This paper considered the bisymmetric conditions on the solution of the matrix equations: \( A_1 X B_1 + A_2 X B_2 + \cdots + A_k X B_k = D \), \( A_1 X B_1 + A_2 X B_2 + \cdots + A_k X B_k = D \) and \( (A_1 X B_1, A_2 X B_2, \cdots, A_k X B_k) = (D_1, D_2, \cdots, D_k) \). Using the ideal that any real \( n \times n \) bisymmetric matrix \( A \) can be described by an \((n+1)^2/4\)-component vector when \( n \) is add or an \( n(n+2)/4 \)-component vector when \( n \) is even, together with Kronecker product of matrices, solve these equations can be transformed into solves equation \( A X = B \). This method can also be used to finding Toeplitz, Hankel, Circulant or Browian matrix solutions of these equations. Of course, this method can only explore the small size of matrices because the size of the resulting matrices will, in general, be very large by using Kronecker products.

References


[10] A. Jameson, Solution of the equation $AX + XB = C$ by inversion of an $m \times n$ or $n \times n$ matrix, SIAM J. Appl. Math., 16 (1968), 1020-1023.


