Regularity and Global Structure of Solutions to Hamilton-Jacobi Equations I. Convex Hamiltonian

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Abstract

This paper is concerned with the Hamilton-Jacobi (HJ) equations of multidimensional space variables with convex Hamiltonian. Using Hopf’s formula (I), we will study the differentiability of the HJ solutions. For any given point, we give a sufficient and necessary condition under which the solutions are $C^k$ smooth in some neighborhood of the point. We also study the characteristics of the HJ equations. It is shown that there are only two kinds of characteristics, one never touches the point of singularity, and the other touches the point of singularity in a finite time. The sufficient and necessary condition under which the characteristic never touches the point of singularity is given. Based on these results, we study the global structure of the set of singularity points for the HJ solutions. It is shown that there exists a one-to-one correspondence between the path connected components of the set of singularity points and the path connected components of a set on which the initial function does not attain its minimum. A path connected component of the set of singularity points never terminates at a finite time. Our results are independent of the particular forms of the equations as long as the Hamiltonian is convex.

1 Introduction

Consider the Cauchy problem for the following Hamilton-Jacobi (HJ) equation

\begin{center}
\begin{tikzpicture}
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\end{center}
\[
\begin{aligned}
&\left\{
\begin{array}{ll}
u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\
u = g & \text{on } \mathbb{R}^n \times \{t = 0\},
\end{array}
\right.
\end{aligned}
\]  

where the Hamiltonian \( H : \mathbb{R}^n \to \mathbb{R} \) is \( C^k \) smooth with \( k \geq 2 \), and uniformly convex (with constant \( \alpha > 0 \)):

\[
\sum_{i,j=1}^n H_{p_ip_j}(p)\xi_i\xi_j \geq \alpha |\xi|^2, \quad \forall \ p, \xi \in \mathbb{R}^n;
\]  

\( g : \mathbb{R}^n \to \mathbb{R} \) is \( C^k \) smooth with \( k \geq 2 \) and satisfies

\[
\sup_{y \in \mathbb{R}^n} |Dg(y)| < \infty.
\]

It is known that the solutions to (1.1) are given by the Hopf’s formula (I)\(^1\), see, e.g., [2, 4, 5, 6],

\[
 u(x, t) = \min_{y \in \mathbb{R}^n} \left\{ tL \left( \frac{x-y}{t} \right) + g(y) \right\},
\]  

where \( L \) is the Legendre transform of \( H \), i.e.,

\[
L(p) = \sup_{q \in \mathbb{R}^n} \{ p \cdot q - H(q) \} \quad (p \in \mathbb{R}^n).
\]

It can be verified that

\[
L(DH(p)) = p \cdot DH(p) - H(p).
\]

Note that the mapping \( DH : \mathbb{R}^n \to \mathbb{R}^n \) is one-to-one and onto (since \( H \) is uniform convex). Another equivalent formula for the solutions is given by Kruzkov [7]:

\[
 u(x, t) = \min_{p \in \mathbb{R}^n} F(x, t, p),
\]  

where

\[
F(x, t, p) = tL(DH(p)) + g(x - tDH(p)),
\]

and the initial function \( g \) is assumed to be lower semi-continuous and satisfies

\[
|g(x)| \leq M(|x| + 1).
\]

The regularity of the solutions to the HJ equations has attracted attention of many authors, see, e.g., [1, 3, 5, 6, 7, 13, 14, 15] and references there in. It is known [2] that the solutions \( u(x, t) \) defined by the Hopf’s formula (I) are viscosity solutions of (1.1), which are differentiable a.e. in \( \mathbb{R}^n \times (0, \infty) \). However, in general \( u(x, t) \) is not of class \( C^1 \) in the sense that their gradient may have discontinuities.

\(^1\)It is called the “Lax formula” in [10].
Let \( U \) be the set of all points \((x, t)\) such that \( F(x, t, \bullet) \) has a unique non-degenerate minimizer. Then \( U \) is open on which the solutions are \( C^k \) smooth. We study the properties of characteristics, which are also interesting in their own sake and have other applications. Given \( y_0 \in \mathbb{R}^n \), let
\[
C = \{(x, t) | x = y_0 + DH(Dg(y_0))t, \ t > 0\}. \tag{1.10}
\]
A characteristic segment
\[
\bar{C} = C \cap \{0 < t \leq T_1\}
\]
is said to be valid if \( Dg(y_0) \) is a minimizer for \( F(x, t, \bullet) \) for each \((x, t) \in \bar{C}\). In the case that \( \max T_1 < \infty \), we prove there exists a point \((x_s(y_0), t_s(y_0))\), where
\[
t_s(y_0) = \max T_1, \ x_s(y_0) = y_0 + t_s(y_0)DH(Dg(y_0)), \tag{1.11}
\]
such that \( Dg(y_0) \) is a unique degenerate minimizer or one of the minimizers for \( F(x_s(y_0), t_s(y_0), \bullet) \), while \( Dg(y_0) \) will no longer be a minimizer for \( F(x, t, \bullet) \) for \((x, t) \in C, t > t_s(y_0)\) and \( Dg(y_0) \) is a unique non-degenerate minimizer for \( F(x, t, \bullet) \) for \((x, t) \in C, t < t_s(y_0)\). We define \((x_s(y_0), t_s(y_0))\) as a singularity point. Let \( S \) be the set of singularity points.

We will introduce a singularity mapping based on the properties of the characteristics. A singularity mapping is defined as
\[
\mathcal{S}(y) = (x_s(y), t_s(y)) \tag{1.12}
\]
from \( \mathbb{R}^n \) to \( \mathbb{R}^n \times (0, \infty) \). It will be shown that \( t_s(y_0) \) is finite if and only if
\[
g(y_0) > \inf_{y \in \mathbb{R}^n} g(y). \tag{1.13}
\]
Thus the domain of \( \mathcal{S} \) is
\[
\mathring{\mathbb{R}}^n = \{y \in \mathbb{R}^n | g(y) > \inf_{x \in \mathbb{R}^n} g(x)\}. \tag{1.14}
\]
Furthermore, it will be proved that the singularity mapping is continuous. Thus the singularity mapping is continuous from \( \mathring{\mathbb{R}}^n \subset \mathbb{R}^n \) to \( \mathbb{R}^n \times (0, \infty) \) and
\[
S = \{(x_s(y), t_s(y)) | y \in \mathring{\mathbb{R}}^n\}. \tag{1.10}
\]
The second part of the paper is concerned with the differentiability of the HJ solutions. We prove that \( u(x, t) \) is non-differentiable at \((x_0, t_0)\) if \( F(x_0, t_0, \bullet) \) has more than one minimizer, and \((x_0, t_0)\) is a cluster point of non-differentiable points of the solution \( u(x, t) \) if \( F(x_0, t_0, \bullet) \) has a unique degenerate minimizer. Thus an equivalent definition of the singularity point can be stated as follows: We call a point a singularity point if it is a non-differentiable point of the solution \( u(x, t) \) or a cluster point of non-differentiable points of \( u(x, t) \). We will show that the solution of (1.1) is smooth in some neighborhood of a point \((x_0, t_0)\) if and only if there exists a unique non-degenerate minimizer for \( F(x_0, t_0, \bullet) \).
We are interested in the global structures of $S$. It will be shown that the set of singularity points consists of several path connected components. We prove that there exists a one-to-one correspondence between the path connected components of $\tilde{R}^n$ and the path connected component $S_i$ of the set of singularity points and each path connected component of the set of singularity points never vanishes as $t$ increases. Moreover, our results are independent of the particular forms of the equations as long as the Hamiltonian is convex. It is pointed out that this work is a kind of extension of our earlier work on the solution structures of the entropy solutions to scalar conservation laws [8, 12].

2 Hopf’s formula (I) and characteristics

In this section we will give several lemmas and propositions on characteristics, which play important role in studying the regularity and global structure of the HJ solutions. The fact that solutions $u(x, t)$ of (1.1) are given by Hopf’s formula (I) shows that the minimizers of $F(x, t, \bullet)$ belong to a subset of the set

$$\{Dg(y) | y \in \mathbb{R}^n\},$$

where $y$ is a point from which the characteristic $C$:

$$C : \{(x, t) | x = y + tDH(Dg(y)), t > 0\} \quad (2.1)$$

emanates and passes through $(x, t)$. We will show that $Dg(y)$ is a unique non-degenerate minimizer of $F(x, t, \bullet)$ for $(x, t) \in C, t > 0$ if and only if

$$g(y) = \inf_{y \in \mathbb{R}^n} g(y). \quad (2.2)$$

In the case that

$$g(y) > \inf_{y \in \mathbb{R}^n} g(y),$$

there exists $(x_s(y), t_s(y))$ belonging to $C$ such that $Dg(y)$ is a minimizer of $F(x, t, \bullet)$ for $(x, t) \in C, t \leq t_s(y)$ and $Dg(y)$ is not minimizer of $F(x, t, \bullet)$ for $(x, t) \in C, t > t_s(y)$.

We record here the following relations that will be needed later

$$D_pF(x, t, p) = tD^2H(p) \cdot (p - Dg(x - tDH(p))). \quad (2.3)$$

If $D_pF(x, t, p) = 0$, then

$$D_pF(x, t, p) = p, \quad D_tF(x, t, p) = -H(p), \quad (2.4)$$

$$D^2_pF(x, t, p) = tD^2H(p)[I_n + t \cdot D^2g(x - tDH(p) \cdot D^2H(p)]. \quad (2.5)$$

**Definition 2.1** Let $p_0$ be a minimizer for $F(x_0, t_0, \bullet)$. Then $p_0$ is called non-degenerate (degenerate) if $|D^2_pF(x_0, t_0, p_0)| \neq 0 \ (= 0)$. 4
Lemma 2.1 Let
\[ U = \{(x, t) : \exists \text{ unique non-degenerate minimizer for } F(x, t, \bullet)\}. \] (2.6)
Then \( U \) is an open subset of \( \mathbb{R}^n \times (0, \infty) \), and \( u(x, t) \) is \( C^k \) smooth on \( U \).

Proof: The proof is a straightforward generalization of Lemma 1.1 in [11] for scalar conservation laws in one space dimension.

Lemma 2.2 Suppose \((x_0, t_0) \in \mathbb{R}^n \times (0, \infty)\), \( p_0 = p(x_0, t_0) \) is a minimizer of \( F(x_0, t_0, \bullet) \), \( l \) is an open straight line segment connecting \((x_0, t_0)\) and \((y_0, 0)\), where
\[ y_0 = y(x_0, t_0) = x_0 - t_0DH(p_0). \]
If \((x_1, t_1) \in l\), then there is a unique point \( p(x_1, t_1) = p(x_0, t_0) = p_0 \) which minimizes \( F(x_1, t_1, \bullet) \). Furthermore,
\[ u(x_0, t_0) = u(x_1, t_1) + (t_0 - t_1)L\left(\frac{x_0 - y_0}{t_0}\right). \] (2.7)

Proof: Suppose \( p_1 \) is a minimizer for \( F(x_1, t_1, \bullet) \). Set \( y_1 = x_1 - t_1DH(p_1) \). It follows from the Hopf’s formula (I) that
\[ u(x_1, t_1) = t_1L\left(\frac{x_1 - y_1}{t_1}\right) + g(y_1) \leq t_1L\left(\frac{x_1 - y_0}{t_1}\right) + g(y_0). \] (2.8)
Using the fact that \( L \) is strictly convex, we have from (2.8) that
\[ u(x_0, t_0) = t_0L\left(\frac{x_0 - y_0}{t_0}\right) + g(y_0) \]
\[ = (t_0 - t_1)\frac{x_0 - y_0}{t_0} + t_1L\left(\frac{x_0 - y_0}{t_0}\right) + g(y_0) \]
\[ = (t_0 - t_1)L\left(\frac{x_0 - x_1}{t_0 - t_1}\right) + t_1L\left(\frac{x_1 - y_0}{t_1}\right) + g(y_0) \]
\[ \geq (t_0 - t_1)L\left(\frac{x_0 - x_1}{t_0 - t_1}\right) + t_1L\left(\frac{x_1 - y_1}{t_1}\right) + g(y_1) \]
\[ = t_0\left[t_0 - t_1\right]L\left(\frac{x_0 - x_1}{t_0 - t_1}\right) + t_1L\left(\frac{x_1 - y_1}{t_1}\right) + g(y_1) \]
\[ \geq t_0L\left(\frac{x_0 - y_1}{t_0}\right) + g(y_1). \] (2.9)
Since
\[ u(x_0, t_0) = \min_{y \in \mathbb{R}^n} \left\{ t_0L\left(\frac{x_0 - y}{t_0}\right) + g(y) \right\}, \]
5
we have
\[ u(x_0, t_0) = t_0 L\left(\frac{x_0 - y_1}{t_0}\right) + g(y_1). \]
Consequently,
\[ \frac{t_0 - t_1}{t_0} L\left(\frac{x_0 - x_1}{t_0 - t_1}\right) + \frac{t_1}{t_0} L\left(\frac{x_1 - y_1}{t_1}\right) = L\left(\frac{x_0 - y_1}{t_0}\right), \]
where we have used (2.9). Note that
\[ p_0 = p_1 \iff DH(p_0) = DH(p_1) \iff \frac{x_0 - y_0}{t_0} = \frac{x_1 - y_1}{t_1} \]
\[ \iff \frac{t_0 - t_1}{t_0} L\left(\frac{x_0 - x_1}{t_0 - t_1}\right) + \frac{t_1}{t_0} L\left(\frac{x_1 - y_1}{t_1}\right) = L\left(\frac{x_0 - y_1}{t_0}\right). \]
This, together with (2.10), implies that \( p_0 = p_1 \).

We now discuss the relationship between the critical point of \( F(x, t, \bullet) \) and the characteristic. Suppose \( p_0 \) is a critical point of \( F(x_0, t_0, \bullet) \), i.e.,
\[ D_{p}F(x_0, t_0, p_0) = 0. \]
Then it follows from (2.3) that \( p_0 = Dg(x_0 - t_0 DH(p_0)) \). Let
\[ y_0 = x_0 - t_0 DH(p_0). \]
The characteristic
\[ x = y_0 + t DH(Dg(y_0)) \]
will pass through \((x_0, t_0)\) with the speed \( DH(Dg(y_0)) = DH(p_0) \).
On the other hand, consider a characteristic
\[ C : \quad x = y + t DH(Dg(y)), \quad t > 0. \]
Then \( D_{p}F(x, t, Dg(y)) = 0 \) for \((x, t) \in C\) due to the fact that
\[ Dg(y) = Dg(x - t DH(Dg(y))) \]
and (2.3). This implies that \( Dg(y) \) is a critical point of \( F(x, t, \bullet) \).

It is natural to ask if \( Dg(y) \) is a minimizer of \( F(x, t, \bullet) \) for \((x, t) \in C\). The following lemma gives an answer.

**Lemma 2.3** Let \( y_0 \in \mathbb{R}^n \) and assume the corresponding characteristic \( C \) is given by (1.10). If \( g \in C^k \) is bounded, then precisely one of the following statements must hold:

- **either** \( Dg(y_0) \) is the unique non-degenerate minimizer of \( F(x, t, \bullet) \) for each \((x, t) \in C\);
• or there exits a point \((x_s(y_0), t_s(y_0)) \in C\) such that \(Dg(y_0)\) is either the unique degenerate minimizer of \(F(x_s(y_0), t_s(y_0), \bullet)\) or one of more than one minimizers for \(F(x_s(y_0), t_s(y_0), \bullet)\). Furthermore \(Dg(y_0)\) is a unique non-degenerate minimizer for \(F(x, t, \bullet)\) for each

\[(x, t) \in C^- := C \cap \{(x, t) : t_s(y_0) > t > 0\},\]

while for

\[(x, t) \in C^+ := C \cap \{(x, t) : t > t_s(y_0)\},\]

\(Dg(y_0)\) is no longer the minimizer of \(F(x, t, \bullet)\).

**Proof.** We first show that \(Dg(y_0)\) is no longer a minimizer for \(F(x, t, \bullet)\) for \((x, t) \in C^+\) if there exist more than one minimizer for \(F(x_s(y_0), t_s(y_0), \bullet)\). Otherwise, there exists a point \((\tilde{x}, \tilde{t}) \in C^+\) such that \(Dg(y_0)\) is a minimizer for \(F(\tilde{x}, \tilde{t}, \bullet)\). Consequently, \(Dg(y_0)\) is the unique minimizer for \(F(x_s(y_0), t_s(y_0), \bullet)\) according to Lemma 2.2, which is a contradiction since there are more than one minimizers for \(F(x_s(y_0), t_s(y_0), \bullet)\).

If \(Dg(y_0)\) is a unique degenerate minimizer for \(F(x_s(y_0), t_s(y_0), \bullet)\), i.e.,

\[|D_p^2 F(x_s(y_0), t_s(y_0), Dg(y_0))| = 0. \tag{2.15}\]

Therefore, there exists a non-zero vector \(\xi \in \mathbb{R}^n\) such that

\[\xi^T D_p^2 F(x_s(y_0), t_s(y_0), Dg(y_0))\xi = 0. \tag{2.16}\]

It follows from (2.4) and (2.5) that

\[t_s[\xi^T D^2 H(Dg(y_0))\xi + t_s\xi^T D^2 H(Dg(y_0))D^2 g(y_0)D^2 H(Dg(y_0))\xi] = 0. \tag{2.17}\]

Consider the equation

\[\tilde{g}(t) = \xi^T D^2 H(Dg(y_0))\xi + t\xi^T D^2 H(Dg(y_0))D^2 g(y_0)D^2 H(Dg(y_0))\xi. \tag{2.18}\]

Thus

\[\xi D_p^2 F(x, t, Dg(y_0))\xi^T = t\tilde{g}(t). \tag{2.19}\]

According to Lemma 2.2, \(Dg(y_0)\) is a unique minimizer for \(F(x, t, \bullet)\) for each \((x, t) \in C, t \leq t_s(y_0)\) since \(Dg(y_0)\) is a minimizer for \(F(x_s(y_0), t_s(y_0), \bullet)\). Then

\[\xi D_p^2 F(x, t, Dg(y_0))\xi^T \geq 0, \quad \forall (x, t) \in C, \quad t \leq t_s(y_0). \tag{2.20}\]

On the other hand, (2.18) is a linear equation of \(t\) and has a unique root, \(t = t_s(y_0)\). Thus

\[\tilde{g}(t) < 0 \text{ for } t > t_s(y_0). \tag{2.21}\]

It follows from (2.19) and (2.21) that

\[\xi^T D_p^2 F(x, t, Dg(y_0))\xi < 0 \text{ for } (x, t) \in C, \quad t > t_s(y_0), \tag{2.22}\]

which means that the matrix \(D_p^2 F(x, t, Dg(y_0))\) is negative definite or non-definite. Consequently, \(Dg(y_0)\) can not be a minimizer for \(F(x, t, \bullet)\). This completes the proof of this lemma. □
The analogue of Lemma 2.3 for convex scalar conservation laws was obtained by Li and Wang in [8].

**Definition 2.2** A characteristic segment

\[ C = C \cap \{0 < t \leq T_1\} \]

is said to be valid if there exists \(0 < T_1 \leq \infty\) such that for each \((x, t) \in \bar{C}\), \(Dg(y_0)\) is a minimizer for \(F(x, t, \bullet)\), \(C\) is given by (1.10).

For each point \((x_0, t_0) \in \mathbb{R}^n \times (0, \infty)\), there exists at least one valid characteristic segment \(\bar{C} = C \cap \{0 < t \leq t_0\}\) passing through it. Lemma 2.3 can be used to judge if \(F(x_0, t_0, \bullet)\) has a unique non-degenerate (or degenerate) minimizer or several minimizers by considering the relationship between \(C^-\) and \(\bar{C}\). That is to say: if \(C^- \subset \subset \bar{C}\), then \(F(x_0, t_0, \bullet)\) has a unique degenerate minimizer or more than one minimizer; if \(\bar{C} \subset \subset C^-\), then \(F(x_0, t_0, \bullet)\) has a unique non-degenerate minimizer.

From the above lemma, for \(y \in \mathbb{R}^n\) satisfying \(t_s(y) < \infty\), consider \(C\), a characteristic emanating from \(y\) of the form (2.1). We see that

\[
\begin{cases}
  t_s(y) = \max\{t \mid u(x, t) = F(x, t, Dg(y)), (x, t) \in C, t > 0\}, \\
x_s(y) = y + t_s(y)DH(Dg(y)).
\end{cases}
\]

(2.23)

We define the point \((x_s(y), t_s(y))\) as singularity point of solution \(u(x, t)\) and let \(S\) be the set of singularity points. In order to study the structure of the set of singularity points we introduce a singularity mapping \(\mathcal{S}\) from some subset of \(\mathbb{R}^n\) to \(\mathbb{R}^n \times (0, \infty)\),

\[
\mathcal{S}(y) = (x_s(y), t_s(y)).
\]

(2.24)

In other words, \((x_s(y), t_s(y))\) is the point such that \(F(x_s(y), t_s(y), \bullet)\) has a unique degenerate minimizer or more than one minimizer.

**Lemma 2.4** \(\mathcal{S}\) defined by (2.24) is a continuous map.

**Proof:** We need to prove that \(t_s(y_n) \to t_s(y_0)\) if \(y_n \to y_0\), where \(y_n \in \mathbb{R}^n\). This will be done in two steps.

**Step 1.** We claim

\[
\limsup t_s(y_n) \leq t_s(y_0).
\]

(2.25)

Otherwise there exists a subsequence \(\{t_s(y_{n_k})\}\) of \(\{t_s(y_n)\}\) such that \(t_s(y_{n_k}) \to T_1 > t_s(y_0)\). Then according to the definition of \(t_s(y_{n_k})\), for \(k\) big enough we have

\[
u\left(DH(Dg(y_{n_k}))T_1 + y_{n_k}, T_1\right) = F\left(DH(Dg(y_{n_k}))T_1 + y_{n_k}, T_1, Dg(y_{n_k})\right).
\]

(2.26)
Using the continuity property of $u(x, t)$ and $F(x, t, p)$, we obtain by letting $k \to \infty$ in (2.26) that
\[ u \left( DH(Dg(y_0))T_1 + y_0, T_1 \right) = F \left( DH(Dg(y_0))T_1 + y_0, T_1, Dg(y_0) \right), \]
which contradicts the definition of $t_s(y_0)$ and Lemma 2.3.

Step 2. We claim
\[ \lim \inf t_s(y_n) \geq t_s(y_0). \]  
Otherwise there exists some subsequence $\{t_s(y_{n_k})\}$ of $\{t_s(y_n)\}$ such that $t_s(y_{n_k}) \to T < t_s(y_0)$. Then there exists a neighborhood $U_{(x_1,T)}$ of $(x_1, T)$, where
\[ x_1 = y_0 + TDH(Dg(y_0)). \]
For each $(x, t) \in U_{(x_1,T)}$, there exists a unique non-degenerate minimizer for $F(x, t, \bullet)$. On the other hand, for sufficiently large $k$, we have
\[ \left( DH(Dg(y_{n_k}))t_s(y_{n_k}) + y_{n_k}, t_s(y_{n_k}) \right) \in U_{(x_1,T)}. \]
According to Lemma 2.3, there are more than one minimizers or a unique degenerate point for $F(y_{n_k} + t_s(y_{n_k})DH(Dg(y_{n_k})), t_s(y_{n_k}), \bullet)$, which is a contradiction. $\blacksquare$

Consider a characteristic given by (1.10), we have shown that either $Dg(y_0)$ is a minimizer for $F(x, t, \bullet)$ for each $(x, t) \in C$ (in this case $t_s(y_0) = \infty$) or $Dg(y_0)$ is a minimizer for $F(x, t, \bullet)$ for $(x, t) \in C, t \leq t_s(y_0) < \infty$ while $Dg(y_0)$ will be no longer a minimizer for $F(x, t, \bullet)$ for $(x, t) \in C, t > t_s(y_0)$ in lemma 2.3. How to determine a point $y_0$ whether $t_s(y_0)$ is finite? That is to say what the domain of definition of the singularity mapping $S$ is. The following three propositions on the characteristic, which are also interesting in their own sake and have other applications, provide a criterion. The criterion is dependent on the initial data and independent of the particular forms of the equations as long as the Hamiltonian is convex.

Proposition 2.1 Assume that the initial function $g(y)$ attain its minimum at $y_0$. Then $Dg(y_0)$ must be a unique non-degenerate minimizer for $F(x, t, \bullet)$ for $(x, t) \in C$, where $C$ is defined by (1.10).

Proof: We first show that for each $(x, t) \in C$, any other local minimum of $F(x, t, \bullet)$ is strictly greater than $F(x, t, Dg(y_0))$. Suppose $Dg(y_1)$ is another local minimizer for $F(x, t, \bullet)$. Thus $Dg(y_1) \neq 0$. Let
\[ h(s) = L(DH(sDg(y_1))). \]
Direct computations yield
\[ h(0) = L(DH(0)) = L\left(DH(Dg(y_0))\right), \]
\[ h(1) = L\left(DH(Dg(y_1))\right). \]
and
\[
h'(s) = sDg(y_1)D^2H(sDg(y_1))Dg(y_1)^T \\
\geq s\alpha|Dg(y_1)|^2 > 0,
\] (2.31)
where we have used the fact that \(Dg(y_0) = 0\) and \(Dg(y_1) \neq 0\). Consequently, \(h(s)\) is strictly increasing, which gives \(h(1) > h(0)\), i.e.,
\[
L\left(DH(Dg(y_1))\right) > L\left(DH(Dg(y_0))\right).
\] (2.32)
Since \(g\) attains its minimum at \(y_0\), we have
\[
g\left(x - tDH(Dg(y_1))\right) - g\left(x - tDH(Dg(y_0))\right) = g\left(x - tDH(Dg(y_1))\right) - g(y_0) \geq 0.
\] (2.33)
Therefore, we have
\[
F(x, t, Dg(y_1)) - F(x, t, Dg(y_0)) = t\left[L\left(DH(Dg(y_1))\right) - L\left(DH(Dg(y_0))\right)\right] \\
+ g\left(x - tDH(Dg(y_1))\right) - g\left(x - tDH(Dg(y_0))\right) > 0.
\] (2.34)
The above result indicates that \(Dg(y_0)\) is a unique non-degenerate minimizer for \(F(x, t, \bullet)\). This completes the proof of the lemma. ■

**Proposition 2.2** Assume that \(Dg(y) \to 0\) as \(|y| \to \infty\), \(g(y)\) does not attain its minimum at \(y_0\) and \(Dg(y_0) \neq 0\). Let \(C\) be given by (1.10). Then there exists \((\tilde{x}, \tilde{t}) \in C\) such that \(Dg(y_0)\) is not the minimizer for \(F(\tilde{x}, \tilde{t}, \bullet)\).

**Proof.** Choose \(y_n \in \partial B(y_0, r_n)\) with \(r_n \to \infty\) as \(n \to \infty\). Then we have
\[
Dg(y_n) \to 0, \text{ as } n \to \infty.
\]
For \(n \geq 0\), let
\[
C_n = \{(x, t) : x = y_n + DH(Dg(y_n))t, t > 0\}.
\]
Now we claim that there exists a point \(y_n \in \partial B(y_0, r_n)\) for \(n\) sufficiently large such that the characteristics \(C_n\) intersects with the characteristic \(C\). In fact, we only need to show that there exists a solution \(y_n\) for
\[
y_n - y_0 = t\left(DH(Dg(y_0)) - DH(Dg(y_n))\right).
\] (2.35)
Let \(f : \partial B(0, r_n) \to \partial B(0, r_n)\) be the mapping
\[
\hat{y}_n \to \frac{DH(Dg(y_0)) - DH(Dg(\hat{y}_n + y_0))}{|DH(Dg(y_0)) - DH(Dg(\hat{y}_n + y_0))|}r_n,
\]
where $\tilde{y}_n = y_n - y_0$ and $y_n \in \partial B(y_0, r_n)$. It is obvious that $f$ is continuous and $f(\partial B(0, r_n)) \neq \partial B(0, r_n)$ for sufficiently large $n$ since

$$DH(Dg(y_n)) \rightarrow DH(0) \neq DH(Dg(y_0)), \quad \text{as } n \rightarrow \infty.$$  

Then $f$ has a fixed point according to an equivalent form of Brouwer’s fixed point theorem. This implies that Eq. (2.35) has a solution.

Denote $(x_n, t_n)$ with $x_n = y_n + t_n DH(Dg(y_n))$ being the intersection point of $C_n$ and $C$. Then

$$y_0 + t_n DH(Dg(y_0)) = y_n + t_n DH(Dg(y_n)),$$

which gives

$$|y_n - y_0| = |DH(Dg(x_0)) - DH(Dg(y_n))| t_n. \quad (2.37)$$

Thus $t_n \rightarrow \infty$ as $n \rightarrow \infty$ by the fact that $|y_n - y_0| = r_n \rightarrow \infty$ and $|DH(Dg(x_0)) - DH(Dg(y_n))| \rightarrow |DH(Dg(x_0)) - DH(0)| \neq 0$ as $n \rightarrow \infty$. We also have

$$F(x_n, t_n, Dg(y_0)) - F(x_n, t_n, Dg(y_n)) = t_n \left[ L(DH(Dg(y_0))) - L(DH(Dg(y_n))) \right] + g(x_n - t_n DH(Dg(y_0))) - g(x_n - t_n DH(Dg(y_n))) \quad (2.36)$$

where we have used the facts that $t_n \rightarrow \infty$ and

$$L \left( DH(Dg(y_0)) \right) - L \left( DH(Dg(y_n)) \right) \rightarrow L \left( DH(Dg(y_0)) \right) - L(DH(0)) > 0. \quad (2.38)$$

The proof of (2.38) is as same as (2.32). This indicates that $Dg(y_0)$ is no longer the minimizer for $F(x_n, t_n, \bullet)$. ■

**Remark 2.1** It is worth pointing out that the condition $Dg(y) \rightarrow 0$ as $|y| \rightarrow \infty$ is necessary in general. For example, it can be verified that there exists a global smooth solution of (1.1) if the initial function $g$ is convex on $\mathbb{R}^n$.

**Proposition 2.3** Assume that $g(y)$ does not attain its minimum at $y_0$ and $Dg(y_0) = 0$. Let $C$ be defined by (1.10). Then there exist $(\tilde{x}, \tilde{t}) \in C$ such that $Dg(y_0)$ is not the minimizer for $F(\tilde{x}, \tilde{t}, \bullet)$

**Proof**: Note that

$$DL(DH(p)) = p, \quad \forall \ p \in \mathbb{R}^n. \quad (2.39)$$

Differentiating (2.39) with respect to $p$ gives

$$D^2L(DH(p))D^2H(p) = I_n, \quad (2.40)$$

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where $I_n$ is an identity matrix. Consequently,

$$D^2L(DH(p)) = (D^2H(p))^{-1} \leq \frac{1}{\alpha}I_n,$$

(2.41)

where we have used the facts that $D^2H(p)$ is positive definite and $D^2H(p) \geq \alpha I_n$. Using (2.41) and $Dg(y_0) = 0$ gives

$$L(DH(p)) - L(DH(Dg(y_0))) = DL(DH(p)) \cdot V(p; y_0)$$

$$+ \int_0^1 \{ sV(p; y_0)^T D^2L[sDH(p) + (1 - s)DH(Dg(y_0))]V(p; y_0) \} ds$$

$$\leq \frac{1}{2\alpha} |V(p; y_0)|^2,$$

(2.42)

where for ease of notations, we let $V(x; y) = DH(x) - DH(Dg(y))$.

For each point $(x, t) \in C$, i.e., $x = y_0 + tDH(Dg(y_0))$, we have

$$F(x, t, Dg(y_0)) - F(x, t, p)$$

$$= g(y_0) - g(y_0 - tV(p; y_0) - t \left[ L(DH(p)) - L(DH(Dg(y_0))) \right]$$

$$\geq g(y_0) - g(y_0 - tV(p; y_0) - \frac{t}{2\alpha} |V(p; y_0)|^2,$$

(2.44)

where in the last step we have used (2.42). Let $m = \inf_{y \in \mathbb{R}^n} g(y)$. We have $g(y_0) - m > 0$ since $g(y)$ does not attain its minimum at point $y_0$. There are two cases to be considered.

- **Case 1: there exists $y_1$ such that $m = g(y_1)$.** Then set

$$y_0 - tV(p; y_0) = y_1,$$

or equivalently,

$$\frac{y_0 - y_1}{t} = V(p; y_0),$$

(2.45)

where $V$ is defined by (2.43). Select $t_n > 0$ with $t_n \to \infty$ and set $c_1 = |y_0 - y_1|$. Since the map $DH$ is one-to-one and onto, we can find $p_n$ satisfying

$$\frac{y_0 - y_1}{t_n} = V(p_n; y_0).$$

(2.46)

Consequently, $|V(p_n; y_0)| = c_1/t_n$, which gives that

$$\frac{t_n}{2\alpha} |V(p_n; y_0)|^2 = \frac{c_2}{t_n},$$

(2.47)

where $c_2 = c_1^2/2\alpha$. For $t_n$ sufficiently large, we have

$$\frac{c_2}{t_n} \leq \frac{1}{2} \left( g(y_0) - g(y_1) \right),$$

(2.48)
Combining (2.42)-(2.48), we have, for sufficiently large $n$,

$$\begin{align*}
F(x_n, t_n, Dg(y_0)) - F(x_n, t_n, p_n) \\
\geq g(y_0) - g(y_1) - \frac{c_2}{t_n} \\
\geq \frac{1}{2} \left( g(y_0) - g(y_1) \right) > 0,
\end{align*}$$

(2.49)

where $x_n = y_0 + t_n DH(Dg(y_0))$. This implies that for $t_n$ sufficiently large, $Dg(y_0)$ is no longer a minimizer for $F(x_n, t_n, \cdot)$.

- Case 2: there does not exist $y_1$ such that $m = g(y_1)$. Then we can find a sequence $\{\tilde{y}_n\}_{n \geq 1}$, such that $|\tilde{y}_n| \to \infty$, and $g(\tilde{y}_n) \to m$, as $n \to \infty$. Set

$$\frac{y_0 - \tilde{y}_n}{t_n} = V(p_n; y_0), \quad t_n = |y_0 - \tilde{y}_n|^3.$$  

(2.50)

Then

$$\frac{1}{2\alpha} t_n^3 |V(p_n; y_0)|^2 = \frac{1}{2\alpha} t_n^{-\frac{1}{3}}.$$  

(2.51)

By the definition of $\tilde{y}_n$, we have for $|\tilde{y}_n|$ sufficiently large,

$$0 < g(\tilde{y}_n) - m \leq \frac{g(y_0) - m}{3},$$

which gives

$$g(y_0) - g(\tilde{y}_n) \geq \frac{2}{3} \left( g(y_0) - m \right).$$

(2.52)

Consequently,

$$\frac{1}{2\alpha} \frac{1}{t_n^{\frac{1}{3}}} \leq \frac{1}{3} \left( g(y_0) - m \right).$$

(2.53)

Using (2.50)-(2.53), we have, for $x_n = y_0 + t_n DH(Dg(y_0))$,

$$\begin{align*}
F(x_n, t_n, Dg(y_0)) - F(x_n, t_n, p_n) \\
\geq g(y_0) - g(\tilde{y}_n) - \frac{1}{2\alpha} t_n^{\frac{1}{3}} \\
\geq \frac{2}{3} [g(y_0) - m] - \frac{1}{3} [g(y_0) - m] \\
= \frac{1}{3} [g(y_0) - m] > 0,
\end{align*}$$

(2.54)

which implies that

$$F(x_n, t_n, Dg(y_0)) - F(x_n, t_n, p_n) > 0.$$  

(2.55)

The above result indicates that for $t_n$ sufficiently large, $Dg(y_0)$ is no longer a minimizer for $F(x_n, t_n, \cdot)$. The proof is then complete.

From Lemmas 2.3-2.3, we see that $Dg(y) \to 0$ as $|y| \to \infty$, and that the domain of definition of the singularity mapping $S$ is given by

$$\bar{\mathbb{R}}^n = \{ y \in \mathbb{R}^n | g(y) > \inf_{x \in \mathbb{R}^n} g(x) \}.$$  

(2.56)
Then the singularity mapping $S$ is continuous from $\tilde{\mathbb{R}}^n$ to $\mathbb{R}^n \times (0, \infty)$ and the set of singularity points can be written in the form

$$S = \{(x_s(y), t_s(y)) \mid y \in \tilde{\mathbb{R}}^n\}$$

(2.57)

where $(x_s(y), t_s(y))$ are defined by (2.23).

### 3 Regularity and global structure of solutions

This section is concerned with the regularity of the solutions and global structure of the set of singularity points $S$ of the solutions in the upper half space $\mathbb{R}^n \times (0, \infty)$. We will show that $S$, as the complementary set of the set $U$ in Lemma 2.1, is a closure of the set consisting of points at which the solution is non-differentiable. Then as a corollary we have the result that the solution $u(x, t)$ is $C^k$ smooth in some neighborhood of $(x_0, t_0)$ if and only if there is a unique non-degenerate minimizer for $F(x_0, t_0, \bullet)$. The set of singularity points consists of several path connected components $S_i$. We will show that there exists one-to-one correspondence between the path connected components $S_i$ of the set of singularity points and path connected components $R_i$ of the subset of $\tilde{\mathbb{R}}^n$ on which initial function does not attain its minimum. A singularity never terminates as $t$ increases.

**Lemma 3.1** If $F(x_0, t_0, \bullet)$ has a unique degenerate minimizer or more than one minimizer, then $u(x, t) = \min_{p \in \mathbb{R}^n} F(x, t, p)$ is not differentiable in any neighborhood $U(x_0, t_0)$ of $(x_0, t_0)$.

**Proof.** Without loss of generality we assume that $Dg(y) \to 0$ as $|y| \to \infty$ since we are concerned with local properties of solutions $u(x, t)$.

The assertion that the solution $u(x, t)$ is not differentiable at $(x_0, t_0)$ if there are more than one minimizer for $F(x_0, t_0, \bullet)$ can be deduced from Theorem 2.1 of Hoang [6].

For the case that there is a unique degenerate minimizer $p_0 = Dg(y_0)$ for $F(x_0, t_0, \bullet)$, we only need show that for any neighborhood $U(x_0, t_0)$ of $(x_0, t_0)$,

there exists a point $(x, t) \in U(x_0, t_0)$ such that

there are more than one minimizer for $F(x, t, \bullet)$.

(3.1)

Otherwise,

there exists a neighborhood $U(x_0, t_0)$ of $(x_0, t_0)$ such that for each point

$(x, t) \in U(x_0, t_0)$ there is a unique minimizer for $F(x, t, \bullet)$.

(3.2)

It follows from Lemma 2.1 that $g(y_0) > \inf_{y \in \mathbb{R}^n} g(y)$. Then there exists a neighborhood $U_{y_0} \subset \mathbb{R}^n \times \{t = 0\}$ of $y_0$ such that $g(y) > \inf_{y \in \mathbb{R}^n} g(y)$ for each $y \in U_{y_0}$. By Lemmas 2.2 and 2.3, we conclude that $t_s(y)$ is finite for each $y \in U_{y_0}$. Then by Lemma 2.4 we have

$$\mathcal{S}(U_{y_0}) = \{ (x_s(y), t_s(y)) \mid y \in U_{y_0} \}$$

(3.3)
is a continuous manifold passing through \((x_0, t_0)\).

It follows from (3.2) that the singularity mapping \(S\) from \(U_{y_0}\) to \(S(U_{y_0})\) is one-to-one if \(S(U_{y_0}) \subset U_{(x_0, t_0)}\) provided that \(U_{y_0}\) is small enough. Then \(S(U_{y_0})\) is a \(n\)-dimensional continuous hyper-surface. Moreover, there is a unique intersection point for each characteristic from \(U_{y_0}\) and the hypersurface, and \((x_0, t_0)\) is an interior point of the hypersurface. Therefore, there exists a neighborhood of \((x_0, t_0)\) as a subset of \(U_{(x_0, t_0)}\) (denoted again by \(U_{(x_0, t_0)}\)) which can be divided by the hypersurface \(S(U_{y_0})\) into three parts. More precisely,

\[
U_{(x_0, t_0)} = U_{(x_0, t_0)}^+ \cup U_{(x_0, t_0)}^- \cup S(U_{y_0}). \tag{3.4}
\]

where

\[
U_{(x_0, t_0)}^+ = U_{(x_0, t_0)} \cap \{t > t_s(y)\}, \quad U_{(x_0, t_0)}^- = U_{(x_0, t_0)} \cap \{t < t_s(y)\}.
\]

Consider a sequence \((\bar{x}_n, \bar{t}_n) \in U_{(x_0, t_0)}^+\), which is convergent to \((x_0, t_0)\). Assume \(C_n^-\) is the valid segment of the characteristic emanating from \(y_n\) and passing through \((\bar{x}_n, \bar{t}_n)\), here the assumption (3.2) is used. Then \(C_n^-\) must meet \(S(U_{y_0})\) for \(n\) sufficiently large since there is a unique minimizer for \(F(x_0, t_0, \bullet)\), which implies that \(t_n > t_s(y_n)\). This contradicts Lemma 2.3. Thus assertion (3.1) is true. \(\blacksquare\)

Let

\[
S^2 = \{(x, t) \in \mathbb{R}^n \times (0, \infty) \mid F(x, t, \bullet) \text{ has more than one minimizer}\},
\]

\[
S^1 = \{(x, t) \in \mathbb{R}^n \times (0, \infty) \mid F(x, t, \bullet) \text{ has a unique degenerate minimizer}\}. \tag{3.5}
\]

It follows from the proof of Lemma 3.1 that \((x, t)\) is a non-differentiable point of the solution \(u(x, t)\) for \((x, t) \in S^2\) and \((x, t)\) is a cluster point of non-differentiable points of the solution \(u(x, t)\) for \((x, t) \in S^1\). This implies each point of \(S^1\) is a cluster point of the elements of \(S^2\). Furthermore, the set of singularity points \(S\) is a closure of \(S^2\). Then another definition of a singularity point can be given: a point is called a singularity point if it is a non-differentiable point of the solution \(u(x, t)\) or a cluster point of non-differentiable points of the solution \(u(x, t)\).

The following theorem follows from Lemmas 2.1-3.1.

**Theorem 3.1** The solution \(u(x, t)\) is \(C^k\) smooth in some neighborhood of \((x_0, t_0)\) if and only if there is a unique non-degenerate minimizer for \(F(x_0, t_0, \bullet)\).

It is known that

\[
\tilde{\mathbb{R}}^n = \{y \in \mathbb{R}^n \mid g(y) > \inf_{x \in \mathbb{R}^n} g(x)\}
\]

is an open subset and is the union of path connected components \(R_i\), i.e.,

\[
\tilde{\mathbb{R}}^n = \{y \in \mathbb{R}^n \mid g(y) > \inf_{x \in \mathbb{R}^n} g(x)\} = \bigcup R_i. \tag{3.6}
\]
Theorem 3.2 Assume $g \in C^k$ and $|Dg(y)| \to 0$ as $|y| \to \infty$. Let $R_i$ be the path connected component of $\mathbb{R}^n$ on which initial function does not attain its minimum. Then $S_i = \mathcal{S}(R_i)$ is a path connected component of the set singularity points $S$ and never vanishes for $t > t_i$, where $t_i$ is the formation time of $S_i$. Moreover, there exists one-to-one correspondence $S_i \leftrightarrow R_i$ and $S = \bigcup S_i$.

Proof: We have $g(y) = \inf_{x \in \mathbb{R}^n} g(x)$ and the characteristic emanating from $y$ has slope $DH(0)$ for $y \in \partial R_i$, where $\partial R_i$ is the boundary of $R_i$. It follows from Lemma 2.1 that for $y \in \partial R_i Dg(y)$ is the unique non-degenerate minimizer for $F(x,t,\cdot)$, where $x = y + tDH(0)$. Note that $R_i$ and $R_j$ are both path connected components of $\mathbb{R}^n$, and so $R_i$ and $R_j$ disjoint. Therefore, a valid segment of characteristic from $R_i$ and a valid one from $R_j (i \neq j)$ can not intersect with each other.

For each $y \in \mathbb{R}^n$, it is known that $\mathcal{S}(y) = (x_s(y), t_s(y))$, where

$$x_s(y) = y + t_s(y)DH(Dg(y)),$$
$$t_s(y) = \max \{t \mid u(x,t) = F(x,t,Dg(y)), (x,t) \in C\},$$

where $C = \{(x,t) \mid x = y + tDH(Dg(y))\}$. Furthermore

$$\mathcal{S}(\mathbb{R}^n) = \{(x_s(y), t_s(y)) \mid x_s(y) = y + t_s(y)DH(Dg(y)), y \in \mathbb{R}^n\}$$
$$= \bigcup \mathcal{S}(R_i) = \bigcup S_i = S, \tag{3.7}$$

where $S_i = \mathcal{S}(R_i)$. We have $S_i \cap S_j = \emptyset$ ($i \neq j$) since valid segments of characteristic from $R_i$ and $R_j$ can not intersect with each other. Thus $S_i = \mathcal{S}(R_i)$ is a path connected component of the set of the singularity points since the singularity mapping $\mathcal{S}$ is continuous and $R_i$ is path connected.

Next we will show each path connected component $S_i$ never vanishes as $t$ increases. Suppose $S_i$ vanishes before $t = T < \infty$. Then

for each point $(x,t) \in \Pi_i \cap \{t \geq T\}$, there exists a valid segment of characteristic passing through $(x,t), \tag{3.8}$

where

$$\Pi_i = \{(x,t) \mid x = y + tDH(Dg(y)), y \in R_i, t > 0\}.$$

We see $\Pi_i \cap \Pi_j = \emptyset$ if $i \neq j$ since valid segments of characteristic from $R_i$ and $R_j$ can not intersect with each other. Using this fact, we claim that

the valid segment of characteristic must emanate from a point $y$ satisfying $g(y) = \inf_{y \in \mathbb{R}^n} g(y). \tag{3.9}$

Otherwise, it follows from Lemmas 2.2 and 2.3 that the characteristic emanating from $y$ will meet at the singularity point $(x_s(y), t_s(y))$. This contradicts to the fact that $S_i$ is a path connected component.

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The assertion (3.9) suggests that all valid segments of characteristics passing through \( \Pi_i \cap \{(x,t), t = T\} \) have slope \( DH(0) \). Therefore, the valid segments of characteristics mentioned above will cover \( \Pi_i \cap \{(x,t), 0 < t < T\} \) and therefore meet \( S_i \). This contradicts to the fact that \( S_i \) is a set of singularity points. Hence the theorem is proved. 

The above result is dependent on the initial data, while it is independent of the particular forms of the equations as long as the Hamiltonian is convex. Using Theorem 3.2 we have the following corollary.

**Corollary 3.1** The domain of dependence of a point \((x,t) \in \Pi_i\) is \( R_i \cap B(x, r t) \), where \( r = \sup_{y \in \mathbb{R}^n} |DH(Dg(y))| \). The domain of influence of a point \( y \in \mathbb{R}_i \) is

\[ \Pi_i \cap \{(y + \xi t, t) : |\xi| \leq r\} \]

### 4 Concluding Remarks

This paper is concerned with the Hamilton-Jacobi equations of multidimensional space variables with convex Hamiltonian. Using Hopf's formula (I), we studied the differentiability of the solutions. For any given point, we give a sufficient and necessary condition under which the solutions are \( C^k \) smooth in some neighborhood of the point. We also study the characteristics of the HJ equations. It is shown that there are only two kinds of characteristics, one never touches the singularity, and the other touches the singularity in a finite time. The sufficient and necessary condition under which the characteristic never touches the point of singularity is given. It is also shown that there exists an one-to-one correspondence between the path connected components of the set of singularity points and the path connected components of the subset of \( \mathbb{R}^n \times \{t = 0\} \) on which the initial function does not attain its minimum. A path connected component of the set of singularity points never terminates at a finite time.

In the second paper [16], we will consider the regularity and global structure of the HJ solutions with convex initial data under the assumptions that the initial function \( g \in C^k \) is strictly convex with general Hamiltonian \( H \). The solutions to the Hamilton-Jacobi equations are given by Hopf's formula (II) as analogue to Hopf's formula (I). In this case, we also obtain the regularity results on the differentiability of the solutions similar to the ones obtained in this work. We will show that there exists a one-to-one correspondence between the path connected components of set of singularity points and the path connected components of the set

\[ \{(Dg(y), \text{conv}H(Dg(y))) | y \in \mathbb{R}^n\} \setminus \{(Dg(y), H(Dg(y))) | y \in \mathbb{R}^n\} \]

where \( \text{conv}H \) is the convex hull of \( H \),

\[ \text{conv}H(x) = \inf \left\{ \sum_{i=1}^m \lambda_i H(x_i) \mid \sum_{i=1}^m \lambda_i x_i = x, \sum_{i=1}^m \lambda_i = 1, \lambda_i \geq 0 \right\} \]
That each path connected component of the set of singularity points never terminates at a finite time is also proved.

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