Error analysis for a fast numerical method to the boundary integral equation of the first kind

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For two-dimensional boundary integral equations of the first kind with logarithm kernels, the use of the conventional boundary element methods gives linear algebraic systems with dense matrix. In a recent work [*J. Comput. Math.*, 22 (2004), pp. 287-298], it is demonstrated that the dense matrix can be replaced by a sparse one if appropriate graded meshes are used in the quadrature rules. The numerical experiments also indicate that the proposed numerical methods require less computational time than that of the conventional ones while the formal rate of convergence can be preserved. The purpose of this work is to establish a stability and convergence theory for this fast numerical method. The stability analysis depends on a decomposition of the coefficient matrix for the collocation equation. The formal orders of convergence observed in the numerical experiments are proved rigorously.

Keywords: Boundary integral equation, collocation method, mesh grading

1. Introduction

Consider the first-kind boundary integral equation of the form

$$-\int_{\Gamma} \log|\mathbf{x}-\mathbf{y}|u(\mathbf{y})\,ds_{\mathbf{y}} = f(\mathbf{x}), \qquad \mathbf{x} := (\mathbf{x}_1, \mathbf{x}_2) \in \Gamma, \tag{1.1}$$

where $\Gamma \subset \mathbb{R}^2$ is a smooth and closed curve in the plane, u is a unknown function, f is a given function, $|\mathbf{x}-\mathbf{y}|$ denotes the Euclidean distance and $ds_{\mathbf{y}}$ is the measure of arclength. The boundary integral equation (1.1) arises in connection with the single layer potential:

$$u(\mathbf{x}) = -\int_{\Gamma} \log|\mathbf{x}-\mathbf{y}|u(\mathbf{y})\,ds_{\mathbf{y}}, \qquad \mathbf{x} \in \Omega. \tag{1.2}$$

The applications and some numerical aspects of the boundary integral equation (1.1) can be found in Sloan (1992). A more relevant paper by Bialecki & Yan (1992) introduced a rectangular quadrature

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method for (1.1). More recently, Cheng *et al.* (2004) proposed a new quadrature method for (1.1) based on a *graded mesh* approach. Unlike the quadrature method in Bialecki & Yan (1992) and other traditional numerical methods, the resulting system of equations in Cheng *et al.* (2004) contains a sparse coefficient matrix. It was demonstrated numerically that the proposed approach can not only preserve the formal rate of convergence but also save a significant amount of computational time.

The purpose of this paper is to provide a convergence theory to the efficient method proposed in Cheng *et al.* (2004). To begin with, let Γ be parameterized by the arclength:

$$v : \ [-L/2, L/2] \to \Gamma,$$

where L is the length of Γ,

$$|dv/ds| = 1 \text{ and } v(\sigma) \text{ is a periodic function with period of } L. \tag{1.3}$$

Then the integral equation (1.1) is equivalent to

$$-\int_{-L/2}^{L/2} \log|v(s) - v(\sigma)| u\big(v(\sigma)\big)\, d\sigma = f\big(v(s)\big), \qquad s \in [-L/2, L/2]. \tag{1.4}$$

The conventional way in solving the equation (1.4) is to use n collocation points to obtain n collocation equations. Then for each fixed s the integral in (1.4) is approximated by an appropriate quadrature rule using the information on the n collocation points. This approach will lead to a linear system whose matrix is a full matrix. In Cheng *et al.* (2004), the integral term in (1.4) is approximated by using a subset of the n collocation points. More precisely, we consider the case when the unknown function u is reasonably smooth and the curve Γ is smooth and closed. In this case, some suitable graded-meshes can be used as the quadrature points to handle the logarithmic kernel, which yields a linear system whose matrix is sparse. The graded-mesh concept was proposed by Rice Rice (1969). It was then used to improve the formal order of convergence when solutions have weak singularity, see, e.g., Chandler (1984); Yan & Sloan (1989) for boundary integral equations and Brunner (1985, 2004); Tang (1992, 1993) for weakly *singular* Volterra equations. However, with a smooth solution we just need to use a uniform mesh for the collocation points; while the graded mesh which is a subset of the uniform mesh is employed to evaluate the integrals.

To be more specific of numerical techniques, let us first introduce some notations. Set the *uniform mesh* with the mesh points

$$A := \{\alpha_i\}, \qquad \alpha_i = \frac{2i}{n-1} \cdot \frac{L}{2} \quad (i = -(n-1)/2, \dots, (n-1)/2), \tag{1.5}$$

where n is supposed to be odd; and set the *graded mesh* with the mesh points

$$B := \big\{\beta_j\big\}, \qquad \beta_j = \mathrm{sgn}(j) \left(\frac{2|j|}{m}\right)^q \cdot \frac{L}{2} \quad (j = -m/2, \dots, -1, 1, \dots, m/2), \tag{1.6}$$

where $q \geqslant 1$ is the grading exponent. In Cheng *et al.* (2004), the value of q is set to be 2 and correspondingly to it is assumed that $m = \sqrt{n-1}$. It can be verified that $B \subset A$. Transforming the negative index in (1.5) and (1.6) to positive one, we obtain the equivalent mesh-point sets:

$$\bar{A} := \{\bar{\alpha}_i\}, \qquad \bar{\alpha}_i = \alpha_{(i-1)-(n-1)/2} \quad (i = 1, \dots, n), \tag{1.7}$$

and

$$\bar{B} := \{\bar{\beta}_j\}, \qquad \bar{\beta}_j = \begin{cases} \beta_{(j-1)-m/2}, & j = 1, \ldots, m/2, \\ \beta_{j-m/2}, & j = m/2 + 1, \ldots, m. \end{cases} \tag{1.8}$$

Rewriting equation (1.4) by using a variable substitution $\rho = \sigma - s$ and the periodic property of v gives

$$-\int_{-L/2}^{L/2} \log|v(s) - v(\sigma + s)| u(v(\sigma + s)) \, d\sigma = f(v(s)), \qquad s \in [-L/2, L/2]. \tag{1.9}$$

Applying the trapezoidal rule with the point set $\bar{B}$ to the integral involved in (1.9) and collocating the resulting equation with respect to the point set $\bar{A}$, we obtain the following system of equations:

$$\sum_{j=1}^{m} \mu_{i,j} u_n(v(\bar{\beta}_j + \bar{\alpha}_i)) = f(v(\bar{\alpha}_i)), \qquad i = 1, \ldots, n, \tag{1.10}$$

where $u_n(v(s))$ is the numerical solution to the equation (1.4) (or to its equivalent form (1.9)) for $s \in [-L/2, L/2]$ and the values of $\mu_{i,j}$ ($i = 1, \ldots, n$ and $j = 1, \ldots, m$) are given by

$$\mu_{i,1} = -\frac{1}{2} \log|v(\bar{\beta}_1 + \bar{\alpha}_i) - v(\bar{\alpha}_i)| \cdot (\bar{\beta}_2 - \bar{\beta}_1),$$

$$\mu_{i,m} = -\frac{1}{2} \log|v(\bar{\beta}_m + \bar{\alpha}_i) - v(\bar{\alpha}_i)| \cdot (\bar{\beta}_m - \bar{\beta}_{m-1}),$$

$$\mu_{i,j} = -\frac{1}{2} \log|v(\bar{\beta}_j + \bar{\alpha}_i) - v(\bar{\alpha}_i)| \cdot (\bar{\beta}_{j+1} - \bar{\beta}_{j-1}) \quad (2 \leqslant j \leqslant m - 1).$$

We find that the number of nonzero elements of the coefficient matrix in the (1.10) is equal to $\text{Card}(\bar{B}) \cdot \text{Card}(\bar{A}) = m \cdot n = n \cdot \sqrt{n-1}$.

We finish the introduction by outlining the rest of the paper. In the next section, we will study the stability properties of the numerical method (1.10), which is done by using the kernel-splitting ideas. The convergence results will be established in Section 3.

2. Stability

In this section, we will employ the splitting kernels technique to prove the stability for (1.10). This technique has been used in many cases (c.f., Atkinson (1988), Yan (1990), Bialecki & Yan (1992), and Mclean (1994)). Let us split the kernel in (1.4) into the following form

$$-\log|v(s) - v(\sigma)| = k^{[1]}(s - \sigma) + k^{[2]}(s, \sigma), \tag{2.1}$$

where

$$k^{[1]}(s - \sigma) = -\log|\sin[\pi(s - \sigma)/2L]|, \tag{2.2}$$

$$k^{[2]}(s, \sigma) = \begin{cases} -\log(2L/\pi), & \text{if } s - \sigma = 2jL, \quad j = 0, \pm 1, \ldots \\ -\log|[v(s) - v(\sigma)]/\sin[\pi(s - \sigma)/2L]|, & \text{otherwise.} \end{cases} \tag{2.3}$$

Note that the kernel $k^{[1]}$ is convolutional and the kernel $k^{[2]}$ is symmetric: $k^{[2]}(s, \sigma) = k^{[2]}(\sigma, s)$. Inserting (2.1) into (1.9) yields

$$\int_{-L/2}^{L/2} [k^{[1]}(\sigma) + k^{[2]}(s, \sigma + s)] u(v(\sigma + s)) \, d\sigma = f(v(s)), \qquad s \in [-L/2, L/2]. \tag{2.4}$$

Applying the same process for deriving (1.10) to the equation (2.4) gives

$$\sum_{j=1}^{m} \left(\mu_{i,j}^{[1]} + \mu_{i,j}^{[2]} \right) u_n(v(\bar{\beta}_j + \bar{\alpha}_i)) = f(v(\bar{\alpha}_i)), \qquad i = 1, \dots, n, \tag{2.5}$$

where the values of $\mu_{i,j}^{[1]}$ and $\mu_{i,j}^{[2]}$ ($i = 1, \dots, n$ and $j = 1, \dots, m$) are given by, respectively,

$$\mu_{i,1}^{[1]} = \frac{1}{2} k^{[1]}(\bar{\beta}_1) \cdot (\bar{\beta}_2 - \bar{\beta}_1) = -\frac{1}{2} \log|\sin(\pi \bar{\beta}_1 / 2L)| \cdot (\bar{\beta}_2 - \bar{\beta}_1),$$

$$\mu_{i,m}^{[1]} = \frac{1}{2} k^{[1]}(\bar{\beta}_m) \cdot (\bar{\beta}_m - \bar{\beta}_{m-1}) = -\frac{1}{2} \log|\sin(\pi \bar{\beta}_m / 2L)| \cdot (\bar{\beta}_m - \bar{\beta}_{m-1}),$$

$$\mu_{i,j}^{[1]} = \frac{1}{2} k^{[1]}(\bar{\beta}_j) \cdot (\bar{\beta}_{j+1} - \bar{\beta}_{j-1}) = -\frac{1}{2} \log|\sin(\pi \bar{\beta}_j / 2L)| \cdot (\bar{\beta}_{j+1} - \bar{\beta}_{j-1}) \quad (2 \leqslant j \leqslant m-1),$$

and

$$\mu_{i,1}^{[2]} = \frac{1}{2} k^{[2]} \left(\bar{\alpha}_i, \bar{\beta}_1 + \bar{\alpha}_i \right) \cdot (\bar{\beta}_2 - \bar{\beta}_1),$$

$$\mu_{i,m}^{[2]} = \frac{1}{2} k^{[2]} \left(\bar{\alpha}_i, \bar{\beta}_m + \bar{\alpha}_i \right) \cdot (\bar{\beta}_m - \bar{\beta}_{m-1}),$$

$$\mu_{i,j}^{[2]} = \frac{1}{2} k^{[2]} \left(\bar{\alpha}_i, \bar{\beta}_j + \bar{\alpha}_i \right) \cdot (\bar{\beta}_{j+1} - \bar{\beta}_{j-1}) \quad (2 \leqslant j \leqslant m-1).$$

Write (1.10) and (2.5), respectively, into the matrix forms:

$$\mathbf{D}\mathbf{U} = \mathbf{F}, \tag{2.6}$$

and

$$\left(\mathbf{D}^{[1]} + \mathbf{D}^{[2]} \right) \mathbf{U} = \mathbf{F}, \tag{2.7}$$

where $\mathbf{U} = (u_n(v(\bar{\alpha}_1)), \dots, u_n(v(\bar{\alpha}_n))^{\top}$ and $\mathbf{F} = (f(v(\bar{\alpha}_1)), \dots, f(v(\bar{\alpha}_n)))^{\top}$. The matrices $\mathbf{D}$, $\mathbf{D}^{[1]}$ and $\mathbf{D}^{[2]}$ are sparse with non-zero elements:

$$\bar{d}_{1,j} \neq 0 \quad \text{for } j = \begin{cases} n - 2[(k-1) - m/2]^2, & k = 1, \dots, m/2, \\ (n+1)/2 + 2(k - m/2)^2, & k = m/2 + 1, \dots, m; \end{cases}$$

$$\text{if } \bar{d}_{i,j} \neq 0, \text{ then } \bar{d}_{i+1, j+1 \pmod{n}} \neq 0.$$

Moreover, the matrix $\mathbf{D}^{[1]} := \left(d_{i,j}^{[1]} \right)_{i,j=1,\dots,n}$ is a circulant matrix (see e.g., Davis (1979)) with the elements described by the follows:

- (1): In the first row, $d_{1,j}^{[1]} = \mu_{1,k}^{[1]}$ for

$$j = \begin{cases} n - 2[(k-1) - m/2]^2, & k = 1, \dots, m/2, \\ (n+1)/2 + 2(k - m/2)^2, & k = m/2 + 1, \dots, m, \end{cases}$$

  and $d_{1,j}^{[1]} = 0$ otherwise.

- (2): $d_{i,j}^{[1]} = d_{i+1, j+1 \pmod{n}}^{[1]}$.

It can be verified that

$$\mathbf{D} = \mathbf{D}^{[1]} + \mathbf{D}^{[2]}. \tag{2.8}$$

As to be shown below, the circulant property of the matrix $\mathbf{D}^{[1]}$ helps us to verify that $\mathbf{D}^{[1]}$ is invertible and to derive the upper bound of the condition number. This allows us to rewrite (2.8) into the form

$$\mathbf{D} = \mathbf{D}^{[1]}\left(\mathbf{I} + \left(\mathbf{D}^{[1]}\right)^{-1}\mathbf{D}^{[2]}\right). \tag{2.9}$$

Therefore, the stability of (1.10) is then proved by verifying that the matrix $\mathbf{I} + \left(\mathbf{D}^{[1]}\right)^{-1}\mathbf{D}^{[2]}$ is invertible (cf. Lemma 2.2).

LEMMA 2.1 For the matrix $\boldsymbol{D}^{[1]}$, we have the following estimation for its inverse:

$$\|\left(\boldsymbol{D}^{[1]}\right)^{-1}\|_{\mathscr{F}} \leqslant Cm,$$

where $\|\cdot\|_{\mathscr{F}}$ is the Frobenius norm and the positive constant C is independent of m.

Proof. Since $\mathbf{D}^{[1]}$ is a circulant matrix, it follows from (Davis, 1979, Theorem 3.2.2) that the eigenvalues λ_j are given by

$$\lambda_j = \sum_{\ell=1}^{n} e^{\mathbf{i}(j-1)(\ell-1)2\pi/n} d_{1,\ell}^{[1]} \quad (j=1,\dots,n),$$

where $\mathbf{i}^2 := -1$. Using the expression of $d_{1,\ell}^{[1]}$, we can formulate λ_j as

$$\begin{aligned}
\lambda_j &= \sum_{k=1}^{m/2} e^{\mathbf{i}(j-1)\left(n-2[(k-1)-m/2]^2\right)2\pi/n}\mu_{1,k}^{[1]} + \sum_{k=m/2+1}^{m} e^{\mathbf{i}(j-1)\left((n+1)/2+2(k-m/2)^2\right)2\pi/n}\mu_{1,k}^{[1]} \\
&= \frac{1}{2}\sum_{k=1}^{m/2} e^{\mathbf{i}(j-1)\left(n-2[(k-1)-m/2]^2\right)2\pi/n}\left(-\log|\sin(\pi\bar{\beta}_k/2L)|\cdot(\bar{\beta}_{k+1}-\bar{\beta}_{k-1})\right) \tag{2.10} \\
&\quad + \frac{1}{2}\sum_{k=m/2+1}^{m} e^{\mathbf{i}(j-1)\left((n+1)/2+2(k-m/2)^2\right)2\pi/n}\left(-\log|\sin(\pi\bar{\beta}_k/2L)|\cdot(\bar{\beta}_{k+1}-\bar{\beta}_{k-1})\right),
\end{aligned}$$

where $\bar{\beta}_{-1} := \bar{\beta}_1$ and $\bar{\beta}_{m+1} := \bar{\beta}_m$. The module of λ_j can be bounded from below by

$$\begin{aligned}
|\lambda_j| &\geqslant \frac{1}{2}\min_{k=1,\dots,m}(\bar{\beta}_{k+1}-\bar{\beta}_{k-1})\left(\sum_{k=1}^{m/2} e^{\mathbf{i}(j-1)\left(n-2[(k-1)-m/2]^2\right)2\pi/n}\left(-\log|\sin(\pi\bar{\beta}_k/2L)|\right)\right. \\
&\quad \left. + \sum_{k=m/2+1}^{m} e^{\mathbf{i}(j-1)\left((n+1)/2+2(k-m/2)^2\right)2\pi/n}\left(-\log|\sin(\pi\bar{\beta}_k/2L)|\right)\right). \tag{2.11}
\end{aligned}$$

Using a formula from (Gradshteyn & Ryzhik, 1980, 1.441.2), we have

$$\begin{aligned}
\log|\sin(\pi\bar{\beta}_k/2L)| &= -\log 2 - \sum_{\ell=1}^{\infty} \frac{\cos(\ell\pi\bar{\beta}_k/L)}{\ell} \\
&= -\log 2 - \sum_{\ell=1}^{n-1} \frac{\cos(\ell\pi\bar{\beta}_k/L)}{\ell} - \sum_{p=1}^{\infty}\sum_{\ell=1}^{n-1} \frac{\cos(\ell\pi\bar{\beta}_k/L)}{p(n-1)+\ell}. \tag{2.12}
\end{aligned}$$

Inserting (2.12) into (2.10) and noting

$$\min_{k=1,\dots,m} (\bar{\beta}_{k+1} - \bar{\beta}_{k-1}) = \frac{16L}{m^2}$$

give

$$
\begin{aligned}
|\lambda_j| \;\geqslant\; & \left| \frac{8L\log 2}{m^2} \left(\sum_{k=1}^{m/2} e^{\mathbf{i}(j-1)\left(n-2[(k-1)-m/2]^2\right)2\pi/n} + \sum_{k=m/2+1}^{m} e^{\mathbf{i}(j-1)\left((n+1)/2+2(k-m/2)^2\right)2\pi/n} \right) \right. \\
& \left. + \frac{8L}{m^2} \left(\sum_{\ell=1}^{n-1} \frac{\gamma_{j,\ell}}{\ell} + \sum_{p=1}^{\infty} \sum_{\ell=1}^{n-1} \frac{\gamma_{j,\ell}}{p(n-1)+\ell} \right) \right|,
\end{aligned}
\tag{2.13}
$$

where

$$
\begin{aligned}
\gamma_{j,\ell} \;=\; & \sum_{k=1}^{m/2} e^{\mathbf{i}(j-1)\left(n-2[(k-1)-m/2]^2\right)2\pi/n} \left(e^{\mathbf{i}\ell\pi\bar{\beta}_k/L} + e^{-\mathbf{i}\ell\pi\bar{\beta}_k/L} \right) \\
& + \sum_{k=m/2+1}^{m} e^{\mathbf{i}(j-1)\left((n+1)/2+2(k-m/2)^2\right)2\pi/n} \left(e^{\mathbf{i}\ell\pi\bar{\beta}_k/L} + e^{-\mathbf{i}\ell\pi\bar{\beta}_k/L} \right) \\
\;=\; & \sum_{k=1}^{m/2} e^{\mathbf{i}(j-1)\left(n-2[(k-1)-m/2]^2\right)2\pi/n} \left(e^{\mathbf{i}\ell\pi 2[(k-1)-m/2]^2/(n-1)} + e^{-\mathbf{i}\ell\pi 2[(k-1)-m/2]^2/(n-1)} \right) \\
& + \sum_{k=m/2+1}^{m} e^{\mathbf{i}(j-1)\left((n+1)/2+2(k-m/2)^2\right)2\pi/n} \left(e^{\mathbf{i}\ell\pi 2(k-m/2)^2/(n-1)} + e^{-\mathbf{i}\ell\pi 2(k-m/2)^2/(n-1)} \right).
\end{aligned}
$$

It is straightforward to verify that

$$
\begin{aligned}
& \sum_{k=1}^{m/2} e^{\mathbf{i}(j-1)\left(n-2[(k-1)-m/2]^2\right)2\pi/n} + \sum_{k=m/2+1}^{m} e^{\mathbf{i}(j-1)\left((n+1)/2+2(k-m/2)^2\right)2\pi/n} \\
& \geqslant \sum_{k=1}^{m} e^{\mathbf{i}(j-1)(k-1)2\pi/mn} = \begin{cases} m, & \text{if } j = 0 \bmod m, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
\tag{2.14}
$$

By a simple calculation, (2.14) and (2.13) lead to

$$|\lambda_j| \geqslant C \left(\frac{1}{m} + \frac{1}{j} - \frac{1}{m-j} \right). \tag{2.15}$$

It then follows from the inequality that

$$\frac{1}{j} - \frac{1}{m-j} \geqslant \frac{4}{m}.$$

Thus the proof of Lemma 2.1 is complete. $\qquad\square$

Furthermore, we derive the upper-bound of the inverse of the matrix $\mathbf{I} + \left(\mathbf{D}^{[1]}\right)^{-1} \mathbf{D}^{[2]}$ in the following lemma.

LEMMA 2.2 Let $\boldsymbol{D}^{[1]}$ and $\boldsymbol{D}^{[2]}$ be the matrices involved in (2.9). Then for sufficiently large n and m, we have the estimation:

$$\left\| \left(\boldsymbol{I} + \left(\boldsymbol{D}^{[1]} \right)^{-1} \boldsymbol{D}^{[2]} \right)^{-1} \right\|_{\mathscr{F}} \leqslant C,$$

where the positive constant C is independent of n and m.

Proof. Recall the form of splitting kernel (2.1):

$$- \log |v(s) - v(\sigma)| = k^{[1]}(s - \sigma) + k^{[2]}(s, \sigma),$$

and define the operators $H^{[1]}$ and $H^{[2]}$ by

$$H^{[1]} w(s) := \int_{-L/2}^{L/2} k^{[1]}(s - \sigma) w(\sigma) \, d\sigma,$$

$$H^{[2]} w(s) := \int_{-L/2}^{L/2} k^{[2]}(s, \sigma) w(\sigma) \, d\sigma.$$

Then the original problem (1.4) can be written into it equivalent form

$$\left(H^{[1]} + H^{[2]} \right) w(s) = f(v(s)), \tag{2.16}$$

where $w(s) := u(v(s))$. It is known that the operator $H^{[1]}$ is invertible and $I + (H^{[1]})^{-1} H^{[2]}$ is a compact operator. Hence, (1.4) and (2.16) are also equivalent to

$$(I + (H^{[1]})^{-1} H^{[2]}) w(s) = (H^{[1]})^{-1} f(v(s)). \tag{2.17}$$

The key to the proof of this lemma is to view $\mathbf{I} + \left(\mathbf{D}^{[1]} \right)^{-1} \mathbf{D}^{[2]}$ as the approximation of $I + (H^{[1]})^{-1} H^{[2]}$. Denote $G := (H^{[1]})^{-1} H^{[2]}$. Corresponding to the operator G, the kernel $g(s, \sigma)$ is given by

$$g(s, \sigma) := (H^{[1]})^{-1} k^{[2]}(s, \sigma). \tag{2.18}$$

Let the matrix $\mathbf{T} := \left(t_{i,j} \right)_{i,j=1}^{n}$ be defined by

$$t_{i,j} = \frac{1}{2} g(\bar{\alpha}_i, \bar{\alpha}_j)(\bar{\beta}_{k+1} - \bar{\beta}_{k-1}),$$

$$j = \begin{cases} n - 2[(k-1) - m/2]^2 + (i-1), & k = 1, \ldots, m/2, \\ (n+1)/2 + 2(k - m/2)^2 + (i-1), & k = m/2 + 1, \ldots, m, \end{cases}$$

$$t_{i,j} = 0, \quad \text{otherwise.}$$

We now derive the upper-bound for $\|\mathbf{T} - (\mathbf{D}^{[1]})^{-1} \mathbf{D}^{[2]}\|_{\mathscr{F}}$. Let $\mathbf{t}_\ell$ and $\mathbf{d}_\ell$ represent the ℓ-th row of the matrices $\mathbf{T}$ and $\mathbf{D}^{[2]}$, respectively. Define

$$\bar{g}(s, \bar{\alpha}_\ell) = \begin{cases} \frac{1}{2} g(s, \bar{\alpha}_\ell)(\bar{\beta}_{k+1} - \bar{\beta}_{k-1}), \\ \qquad \ell = \begin{cases} n - 2[(k-1) - m/2]^2 + (i-1), & k = 1, \ldots, m/2, \\ (n+1)/2 + 2(k - m/2)^2 + (i-1), & k = m/2 + 1, \ldots, m, \end{cases} \\ 0, \quad \text{otherwise,} \end{cases}$$

and

$$
\bar{k}^{[2]}(s,\bar{\alpha}_\ell) = \begin{cases} \dfrac{1}{2}k^{[2]}(s,\bar{\alpha}_\ell)(\bar{\beta}_{k+1}-\bar{\beta}_{k-1}), \\ \qquad \ell = \begin{cases} n-2[(k-1)-m/2]^2+(i-1), & k=1,\ldots,m/2, \\ (n+1)/2+2(k-m/2)^2+(i-1), & k=m/2+1,\ldots,m, \end{cases} \\ 0, \quad \text{otherwise.} \end{cases}
$$

Moreover, define a restriction operator $\mathbf{r}$ by

$$
\mathbf{r}v(s) = [v(\bar{\alpha}_1),\ldots,v(\bar{\alpha}_n)]^\top, \qquad v \in C([-L/2,L/2]). \tag{2.19}
$$

It is obvious that $\mathbf{t}_\ell = \mathbf{r}\bar{g}(s,\bar{\alpha}_\ell)$ and $\mathbf{d}_\ell = \mathbf{r}\bar{k}^{[2]}(s,\bar{\alpha}_\ell)$. Then using Lemma 2.1 and Lemma 3.1 (see the next section) gives

$$
\begin{aligned}
\left\| \mathbf{t}_\ell - (\mathbf{D}^{[1]})^{-1}\mathbf{d}_\ell \right\|_2 &= \left\| (\mathbf{D}^{[1]})^{-1}\left(\mathbf{D}^{[1]}\mathbf{t}_\ell - \mathbf{d}_\ell \right) \right\|_2 \\
&= \left\| (\mathbf{D}^{[1]})^{-1}\left(\mathbf{D}^{[1]}\mathbf{r}\bar{g}(s,\bar{\alpha}_\ell) - \mathbf{r}H^{[1]}\bar{g}(s,\bar{\alpha}_\ell) \right) \right\|_2 \leqslant C\frac{\sqrt{n}\log n}{n-1},
\end{aligned}
$$

where $\|\cdot\|_2$ is a vector 2-norm. Therefore,

$$
\left\| \left(\mathbf{T} - (\mathbf{D}^{[1]})^{-1}\mathbf{D}^{[2]} \right)\mathbf{v} \right\|_2 \leqslant \sqrt{\sum_{\ell=1}^n \left\| \mathbf{t}_\ell - (\mathbf{D}^{[1]})^{-1}\mathbf{d}_\ell \right\|_2^2}\,\|\mathbf{v}\|_2 \leqslant C\frac{\sqrt{n}\log n}{n-1}. \tag{2.20}
$$

Now it is ready to derive the lower-bound for $\|\mathbf{I}+\mathbf{T}\|_{\mathscr{F}}$. The following inequality is known from Yan & Sloan (1988):

$$
\|(I+G)v\|_{L^2} \geqslant C\|v\|_{L^2}, \qquad v \in L^2([-L/2,L/2]), \tag{2.21}
$$

where the notation $\|\cdot\|_{L^2}$ stands for $\|\cdot\|_{L^2([-L/2,L/2])}$. The kernel $g(s,\sigma)$ in (2.18) is Lipschitz continuous with respect to the variables s and σ, respectively. Define a map

$$
p_n: \ \mathbb{R}^n \longrightarrow L^2([-L/2,L/2])
$$

by, for $\mathbf{v} = [v_1,\ldots,v_n]^\top$,

$$
(p_n\mathbf{v})(s) = v_i, \quad \text{for } s \in (\alpha_i,\alpha_{i+1}),\ i=1,\ldots,n-1,
$$

i.e., $(p_n(\mathbf{v}))(s)$ is a piecewise constant function. It is easy to verify that

$$
\|\mathbf{v}\|_2 = \|p_n\mathbf{v}\|_{L^2}. \tag{2.22}
$$

Define a matrix $\widetilde{\mathbf{T}} := (\tilde{t}_{i,j})_{i,j=1}^n$, where

$$
\tilde{t}_{i,j} = \begin{cases} \displaystyle\int_{\bar{\beta}_{k-1}}^{\bar{\beta}_{k-1}} g(\bar{\alpha}_i,\sigma)\,d\sigma, \\ \qquad j = \begin{cases} n-2[(k-1)-m/2]^2+(i-1), & k=1,\ldots,m/2, \\ (n+1)/2+2(k-m/2)^2+(i-1), & k=m/2+1,\ldots,m, \end{cases} \\ 0, \quad \text{otherwise.} \end{cases}
$$

Since $g(s,\sigma)$ is Lipschitz continuous with respect to s and σ, respectively, we can verify that

$$\left\| Gp_n\mathbf{v} - p_n(\widetilde{\mathbf{T}}\mathbf{v}) \right\|_{L^2} \leqslant C\frac{1}{n}\|\mathbf{v}\|_2, \tag{2.23}$$

and

$$\left\| (\widetilde{\mathbf{T}} - \mathbf{T})\mathbf{v} \right\|_2 \leqslant \frac{1}{n}\|\mathbf{v}\|_2. \tag{2.24}$$

Applying (2.22) and the triangle inequality, together with (2.16) and (2.17), we derive

$$\left\| (\mathbf{I} + \widetilde{\mathbf{T}})\mathbf{v} \right\|_2 = \left\| p_n\mathbf{v} + p_n\widetilde{\mathbf{T}}\mathbf{v} \right\|_{L^2}$$
$$\geqslant \quad \|(I+G)p_n\mathbf{v}\|_{L^2} - \left\| Gp_n\mathbf{v} - p_n\widetilde{\mathbf{T}}\mathbf{v} \right\|_{L^2} \geqslant C\left(1 - \frac{1}{n}\right)\|\mathbf{v}\|_2.$$

Then it follows again from the triangle inequality that

$$\|(\mathbf{I} + \mathbf{T})\mathbf{v}\|_2 \geqslant \left\| (\mathbf{I} + \widetilde{\mathbf{T}})\mathbf{v} \right\|_2 - \left\| (\widetilde{\mathbf{T}} - \mathbf{T})\mathbf{v} \right\|_2 \geqslant C\|\mathbf{v}\|_2. \tag{2.25}$$

Combining (2.20) and (2.25) leads to

$$\left\| \left(\mathbf{I} + (\mathbf{D}^{[1]})^{-1}\mathbf{D}^{[2]}\right) \right\|_2 \quad \geqslant \quad \|(\mathbf{I}+\mathbf{T})\mathbf{v}\|_2 - \left\| (\mathbf{T} - (\mathbf{D}^{[1]})^{-1}\mathbf{D}^{[2]})\mathbf{v} \right\|_2$$
$$\geqslant \quad C\left(1 - \frac{\sqrt{n}\log n}{n-1}\right)\|\mathbf{v}\|_2. \tag{2.26}$$

This completes the proof of Lemma 2.2. $\qquad\square$

The following stability result follows directly from Lemma 2.1 and Lemma 2.2.

THEOREM 2.1 The numerical method using the graded mesh for the numerical integration, i.e., *(1.10)*, is stable in the sense that the matrix $\boldsymbol{D}$ for the corresponding matrix equation $\boldsymbol{DU} = \boldsymbol{F}$ is non-singular. Furthermore, the sparse matrix $\boldsymbol{D}$ satisfies the following estimate:

$$\left\| \mathbf{D}^{-1} \right\|_{\mathscr{F}} \leqslant C\sqrt{n-1}, \tag{2.27}$$

for sufficiently large n, where n is the total number of collocation points.

3. Convergence

The following two lemmas are important in establishing the convergence result of the scheme (1.10).

LEMMA 3.1 Let $J(s,\sigma) := -\log|v(s) - v(s+\sigma)|$ and $W(s,\sigma) := u(v(s+\sigma))$, where $v(s)$ is subject to the condition (1.3) and assume that $u(v(s)) \in C^4([-L/2, L/2])$. Then the error of the trapezoidal rule is given by

$$Q(s) \quad := \quad \int_{-L/2}^{L/2} J(s,\sigma)W(s,\sigma)\,d\sigma - \frac{1}{2}\sum_{j=1}^{m} J(s,\bar{\beta}_j)W(s,\bar{\beta}_j)(\bar{\beta}_{j+1} - \bar{\beta}_{j-1})$$
$$= \quad G(s) + E(s), \tag{3.1}$$

where

$$G(s) := \frac{1}{12} \sum_{j=1}^{m} \left(J(s,\bar{\beta}_j) W(s,\bar{\beta}_j) \right)_{\sigma\sigma} (\bar{\beta}_{j+1} - \bar{\beta}_j)^3, \tag{3.2}$$

$$E(s) := \frac{1}{2} \sum_{j=1}^{m} \int_{\bar{\beta}_j}^{\bar{\beta}_{j+1}} \left[\frac{\eta - \bar{\beta}_{j+1}}{\bar{\beta}_j - \bar{\beta}_{j+1}} \int_{\bar{\beta}_j}^{\eta} \left(J(s,\sigma) W(s,\sigma) \right)_{\sigma\sigma\sigma} (\sigma - \eta)^2 \, d\sigma \right.$$
$$\left. + \frac{\eta - \bar{\beta}_j}{\bar{\beta}_j - \bar{\beta}_{j+1}} \int_{\eta}^{\bar{\beta}_{j+1}} \left(J(s,\sigma) W(s,\sigma) \right)_{\sigma\sigma\sigma} (\sigma - \eta)^2 \, d\sigma \right] d\eta, \tag{3.3}$$

where $\bar{\beta}_j \in \bar{B}$ for $j = 1, \ldots, m$, with $\bar{B}$ given by (1.8), and we also set $\bar{\beta}_{-1} := \bar{\beta}_1$ and $\bar{\beta}_{m+1} := \bar{\beta}_m$. Furthermore, $G(s)$ and $E(s)$ can be bounded, respectively, by

$$|G(s)| \leqslant C \frac{\log n}{m^2}, \qquad |E(s)| \leqslant C \frac{\log n}{m^3}. \tag{3.4}$$

Proof. Applying the Euler-Maclaurin theorem to the integrand with $\bar{\beta}_j$ and $\bar{\beta}_{j+1}$, respectively, gives

$$J(s,\sigma)W(s,\sigma) = J(s,\bar{\beta}_j)W(s,\bar{\beta}_j) + \left(J(s,\bar{\beta}_j)W(s,\bar{\beta}_j) \right)_{\sigma} (\sigma - \bar{\beta}_j) \tag{3.5}$$
$$+ \frac{\left(J(s,\bar{\beta}_j)W(s,\bar{\beta}_j) \right)_{\sigma\sigma}}{2!} (\sigma - \bar{\beta}_j)^2$$
$$+ \int_{\bar{\beta}_j}^{\sigma} \frac{\left(J(s,\eta)W(s,\eta) \right)_{\eta\eta\eta}}{2!} (\eta - \sigma)^2 \, d\eta,$$

and

$$J(s,\sigma)W(s,\sigma) = J(s,\bar{\beta}_{j+1})W(s,\bar{\beta}_{j+1}) + \left(J(s,\bar{\beta}_{j+1})W(s,\bar{\beta}_{j+1}) \right)_{\sigma} (\sigma - \bar{\beta}_{j+1}) \tag{3.6}$$
$$+ \frac{\left(J(s,\bar{\beta}_{j+1})W(s,\bar{\beta}_{j+1}) \right)_{\sigma\sigma}}{2!} (\sigma - \bar{\beta}_{j+1})^2$$
$$+ \int_{\bar{\beta}_{j+1}}^{\sigma} \frac{\left(J(s,\eta)W(s,\eta) \right)_{\eta\eta\eta}}{2!} (\eta - \sigma)^2 \, d\eta.$$

Multiplying (3.5) and (3.6) by $(\sigma - \bar{\beta}_{j+1})/(\bar{\beta}_j - \bar{\beta}_{j+1})$ and $(\sigma - \bar{\beta}_j)/(\bar{\beta}_{j+1} - \bar{\beta}_j)$ respectively, and adding the resulting quantities, lead to (3.1). Moreover, similar to the proof of Lemma 3 in Cheng *et al.* (2004) we can obtain (3.4). The proof of Lemma 3.1 is thus complete. □

LEMMA 3.2 Assume $\psi(s) \in C^4([0,2\pi])$ and $\psi(s)$ is 2π-periodic. Let vectors $\{\mathbf{e}^j\}$ $(j = 1, \ldots, n)$ be given by

$$[1, \exp(\mathbf{i}2\pi[j-1]/n), \exp(\mathbf{i}2\pi[2(j-1)]/n), \ldots, \exp(\mathbf{i}2\pi[(n-1)(j-1)]/n)]^{\top}.$$

Then

$$\left| < \mathbf{r}\psi''(s), \mathbf{e}^j > \right| \leqslant \begin{cases} Cn^{-2}, & j = 1, \\ C\left(\dfrac{1}{j-1} + \dfrac{1}{n-(j-1)} \right)^2, & j = 2, \ldots, n, \end{cases}$$

where $\mathbf{r}$ is the restriction operator defined in (2.19) and $< \cdot, \cdot >$ denotes the Euclidean inner product.

Proof. It is known from Davis (1979) that the vectors $\mathbf{e}^j$ are the eigenvectors corresponding to the eigenvalues λ_k in (2.10). Then the remaining proof is exactly the same as the one of Lemma 2.4 in Bialecki & Yan (1992). □

Due to the orthogonal property, $<\mathbf{e}^j,\mathbf{e}^k>=2\pi\delta_{j,k}$ $(1\leqslant j,k\leqslant n)$, where $\delta_{j,k}$ is the Kronecker delta function, we may write the following expansion

$$\mathbf{r}\psi''(s) = \frac{1}{2\pi}\sum_{j=1}^{n} <\mathbf{r}\psi''(s),\mathbf{e}^j>\mathbf{e}^j.$$

It then follows from Lemma 3.2 that

$$\left\|(\mathbf{D}^{[1]})^{-1}\mathbf{r}\psi''\right\|_2^2 = \frac{1}{2\pi}\sum_{j=1}^{n}\lambda_j^{-2}\left|<\mathbf{r}\psi'',\mathbf{e}^j>\right|^2. \tag{3.7}$$

Applying Lemma 3.2 and Lemma 2.1 gives

$$\left\|(\mathbf{D}^{[1]})^{-1}\mathbf{r}\psi''\right\|_2 \leqslant C, \tag{3.8}$$

where the constant C is independent of n and m.

To provide error bounds of our numerical schemes, we use a discrete L^2 norm defined by (e.g., Cheng *et al.* (2004))

$$\|u(s)\|_{\text{dis}} := \left[\frac{1}{n}<\mathbf{r}u(s),\mathbf{r}u(s)>\right]^{1/2} = \frac{1}{\sqrt{n}}\|\mathbf{r}u(s)\|_2. \tag{3.9}$$

THEOREM 3.1 Let $w(s) := u(\nu(s))$ and $w_n(s) := u_n(\nu(s))$ be the solutions of (1.4) and (1.10), respectively, where $\nu(s)$ is subject to the condition (1.3). Moreover, assume $w(s) \in C^4([-L/2,L/2])$. Then the a priori error estimate of the scheme (1.10) to the integral equation (1.4) is given by

$$\|w(s) - w_n(s)\|_{dis} \leqslant C\frac{\log n}{n},$$

where the discrete norm $\|\cdot\|_{dis}$ is given in (3.9).

Proof. It is observed that

$$\|w(s) - w_n(s)\|_{\text{dis}} = \frac{1}{\sqrt{n}}\|\mathbf{r}w - \mathbf{r}w_n\|_2 = \frac{1}{\sqrt{n}}\|\mathbf{r}w - \mathbf{U}\|_2, \tag{3.10}$$

where $\mathbf{U}$ is given in (2.6), so it only needs to estimate $\|\mathbf{r}w - \mathbf{U}\|_2$. It follows from $\mathbf{D}(\mathbf{r}w - \mathbf{U}) = \mathbf{r}Q$ that

$$\mathbf{r}w - \mathbf{U} = \mathbf{D}^{-1}\mathbf{r}Q = \mathbf{D}^{-1}\mathbf{r}G(s) + \mathbf{D}^{-1}\mathbf{r}E(s).$$

Therefore, using the inequality (3.8) with the change of the variables $s = \tau\frac{L}{2\pi} - \frac{L}{2}$, together with the stability results Theorem 2.1 and the quadrature error estimates in Lemma 3.1, yield

$$\|\mathbf{r}w - \mathbf{U}\|_2 \leqslant C\frac{\sqrt{n}\log n}{n}.$$

Combining the above estimate and (3.10) completes the proof of this theorem. □

THEOREM 3.2 Assume $w(s) \in C^4([-L/2, L/2])$, where $w(s) := u(v(s))$ is the solution of (1.4). Assume the Simpson quadrature rule is employed to approximate the integral involved in (1.9) and the grading mesh B is used such that $\beta_{j+1/2} := (\beta_j + \beta_{j+1})/2 \in B$ for all $\beta_j, \beta_{j+1} \in B$. Denote the resulting numerical solution by $w_n(s)$. Then the a priori error estimate of the numerical scheme is given by

$$\|w(s) - w_n(s)\|_{\mathrm{dis}} \leqslant C \frac{\log n}{n^2}.$$

Proof.   The proof of the above theorem is quite similar to that of Theorem 3.1 and will be omitted here. The detailed description of the numerical scheme using Simpson rule can be found in Cheng *et al.* (2004).

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