On spectral methods for Volterra type integral equations and the convergence analysis

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March 23, 2007

Abstract

The main purpose of this work is to provide a novel numerical approach for the Volterra integral equations based on a spectral approach. A Legendre-collocation method is proposed to solve the Volterra integral equations of the second kind. We provide a rigorous error analysis for the proposed method, which indicate that the numerical errors (in the infinity norm) will decay exponentially provided that the kernel function and the source function are sufficiently smooth. Numerical results are presented, which confirm the theoretical prediction of the exponential rate of convergence. The result in this work seems to be the first successful spectral approach (with theoretical justification) for the Volterra type equations.

AMS(MOS) subject classification. 35Q99, 35R35, 65M12, 65M70

Key Words. Legendre-spectral method, second kind Volterra integral equations, convergence analysis.

Submitted to *Journal of Computational Mathematics*

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1 Introduction

This paper is concerned with the second kind Volterra integral equations

$$u(x) + \int_a^x k(x, s, u(s))ds = g(x), \qquad x \in [a, b], \tag{1.1}$$

where $k(\cdot, \cdot, \cdot)$ is called kernel function. We will consider the case that the solutions of (1.1) are sufficiently smooth – in this case it is necessary to consider very high-order numerical methods such as spectral methods for the solutions of (1.1).

There are many existing numerical methods for solving the Volterra equation (1.1), such as collocation methods, product integration methods, see, e.g., Brunner [1] and references therein. However, very few works touched the spectral approximations to (1.1). In [6], Chebyshev spectral methods are developed to solve nonlinear Volterra-Hammerstein integral equations, and in [7], Chebyshev spectral methods are investigated for Fredholm integral equations of the first kind under multiple-precision arithmetic. However, no theoretical analysis is provided to justify the high accuracy obtained. In [11], a spectral method is developed for solving (1.1), but unfortunately spectral accuracy is not observed for most of computations.

It is known that the Fredholm type equations behave more or less like a boundary value problem (see, e.g., [5]). As a result, some efficient numerical methods useful for boundary values problems (such as spectral methods) can be used directly to handle the Fredholm type equations (again see [5]). However, the Volterra equation (1.1) behaves like an initial value problem. Therefore, it is very difficult and unpopular to apply the spectral approximations to the Volterra type equations. The main reason is that the (1.1) is a local equation while the spectral methods use global basis functions. The main difficulty is how to implement the method so that spectral accuracy can be eventually obtained. On the other hand, the numerical methods for the equation (1.1) may be different with those for the standard initial values problems in the sense that the former requires storage of all values at grid points while the latter only requires information at a fixed number of previous grid points. The storage requirement for (1.1) also makes the use of the global basis functions of the spectral methods more acceptable.

The main purpose of this work is to provide a novel numerical approach for the Volterra integral equations based on a spectral approach. We will provide a rigorous error analysis which theoretically justify the spectral rate of convergence for the Volterra integral equations. This paper is organized as follows. In section 2, we introduce the spectral approaches for the Volterra integral equations of second kind. The convergence analysis is provided in Section 3. Numerical experiments are carried out in Section 4, which will be used to verify the theoretical results obtained in Section 3.

2 Legendre-collocation method

Without lose of generality, we assume that the solution domain is $[-1, 1]$. The second kind linear integral equations in one dimension is of the form

$$u(x) + \int_{-1}^{x} K(x, s)u(s)ds = g(x), \qquad x \in [-1, 1]. \tag{2.1}$$

Set the collocation points as the set of $(N + 1)$ Legendre Gauss, or Gauss-Radau, or Gauss-Lobatto points, $\{x_i\}_{i=0}^{N}$. Assume that Eq. (2.1) holds at x_i:

$$u(x_i) + \int_{-1}^{x_i} K(x_i, s)u(s)ds = g(x_i), \qquad 0 \le i \le N. \tag{2.2}$$

The main difficulty in obtaining high order of accuracy is to compute the integral term in (2.2). In particular, for small values of x_i, there is little information available for $u(s)$. To overcome this difficulty, we will transfer the integral interval $[-1, x_i]$ to a fixed interval $[-1, 1]$ and i then make use some appropriate quadrature rule. More precisely, we first make a simple linear transformation:

$$s(x, \theta) = \frac{1 + x}{2}\theta + \frac{x - 1}{2}, \qquad -1 \le \theta \le 1. \tag{2.3}$$

Then (2.2) becomes

$$u(x_i) + \int_{-1}^{1} \tilde{K}(x_i, s(x_i, \theta))u(s(x_i, \theta))d\theta = g(x_i), \qquad 0 \le i \le N, \tag{2.4}$$

where

$$\tilde{K}(x_i, s(x_i, \theta)) = \frac{1 + x_i}{2}K(x_i, s(x_i, \theta)). \tag{2.5}$$

Using a $(N + 1)$-point Gauss quadrature formula relative to the Legendre weights $\{\omega_k\}$ gives

$$u(x_i) + \sum_{j=0}^{N} \tilde{K}(x_i, s(x_i, \theta_j))u(s(x_i, \theta_j))\omega_j = g(x_i), \qquad 0 \le i \le N, \tag{2.6}$$

where the set $\{\theta_j\}_{j=0}^{N}$ coincide with the collocation points $\{x_j\}_{j=0}^{N}$.

We now need to represent $u(s(x_i, \theta_j))$ using u_i, $0 \le i \le N$, i.e., the values at all the grid points. To this end, we expand u using Lagrange interpolation polynomials, i.e.,

$$u(\sigma) \approx \sum_{j=0}^{N} u_j F_j(\sigma),$$

where F_j is the j-th Lagrange basis function. Combining the above equation and (2.6) yields

$$u_i + \sum_{j=0}^{N} u_j \left(\sum_{j=0}^{N} \tilde{K}(x_i, s(x_i, \theta_j))F_j(s(x_i, \theta_j))\omega_j \right) = g(x_i), \qquad 0 \le i \le N. \tag{2.7}$$

Remark 2.1 It is seen from the numerical scheme (2.7) that to compute the approximation to $u(x_i)$, we require the entire solution information of $\{u(x_j)\}_{j=0}^N$ and the semi-local information of $\{K(x_i, s(x_i, \theta_j)\}_{j=0}^i$. Here, $-1 \le s(x_i, \theta_j) \le x_i$. This is different with the collocation methods or product integration methods which use the semi-local information of both the solution and K, namely, $\{u(x_j)\}_{j=0}^i$ and $\{K(x_i, \beta_j\}$ where $-1 \le \beta_j \le x_i$ are some collocation points. It is because of this difference that we will be able to obtain, as to be demonstrated in the next section, a spectral rate of accuracy instead of an algebraic order of accuracy for the proposed scheme (2.7).

2.1 Implementation of the spectral collocation algorithm

Denoting $U_N = [u_0, u_1, \cdots, u_N]^T$ and $g_N = [g(x_0), g(x_1), \cdots, g(x_N)]^T$, we can obtain an equation of the matrix form:

$$U_N + AU_N = F_N, \tag{2.8}$$

where the entries of the matrix A is given by

$$A_{i,j} = \sum_{p=0}^{N} \tilde{K}(x_i, s(x_i, \theta_p)) F_j(s(x_i, \theta_p))\omega_p.$$

We now discuss an efficient computation of $F_j(s(x_i, \theta_p))$. The idea is to express $F_j(s)$ in terms of the Legendre functions:

$$F_j(s) = \sum_{p=0}^{N} \alpha_{p,j} L_p(s), \tag{2.9}$$

where $\alpha_{p,j}$ is called the discrete polynomial coefficients of F_j. The inverse relation is (see, e.g., [4]):

$$\alpha_{p,j} = \frac{1}{\gamma_p} \sum_{i=0}^{N} F_j(x_i) L_p(x_i)\omega_i = L_p(x_j)/\gamma_p, \tag{2.10}$$

where

$$\gamma_p = \sum_{i=0}^{N} L_p^2(x_i)\omega_i = (p + \frac{1}{2})^{-1}, \quad \text{for} \quad p < N \tag{2.11}$$

and $\gamma_N = (N + 1/2)^{-1}$ for the Gauss and Gauss-Radau formulas, and $\gamma_N = 2/N$ for the Gauss-Lobatto formula. It follows from (2.9) and (2.10) that

$$F_j(s) = \sum_{p=0}^{N} L_p(x_j) L_p(s)/\gamma_p, \tag{2.12}$$

which, together with the known recurrence formulas for $L_p(s)$, can be used to evaluate $F_j(s(x_i, \theta_p))$ in an efficient way.

2.2 Nonlinear equations

In practice, many Volterra equations are usually nonlinear. However, the nonlinearity adds rather little to the difficulty of obtaining a numerical solution. The methods described above remain applicable. To demonstrate this, we consider the Volterra equations of the second kind with the form

$$u(x) + \int_{-1}^{x} k(x, s, u(s))ds = g(x), \qquad x \in [-1, 1]. \tag{2.13}$$

Similar to the linear case, we assume that (2.8) holds at the Legendre collocation points and transform the interval $[-1, x]$ to $[-1, 1]$. This gives

$$u(x_i) + \frac{1 + x_i}{2} \int_{-1}^{1} k(x_i, s(x_i, \theta), u(s(x_i, \theta)))d\theta = g(x_i), \qquad 0 \le i \le N. \tag{2.14}$$

Similar to (2.7), we obtain

$$u_i + \frac{1 + x_i}{2} \sum_{j=0}^{N} k\left(x_i, s(x_i, \theta_j), \sum_{j=0}^{N} u_j F_j(s(x_i, \theta_j))\right) \omega_j = g(x_i), \qquad 0 \le i \le N. \tag{2.15}$$

Equation (2.15) can then be solved by some methods suitable for solving the nonlinear systems.

2.3 Two-dimensional extension

Consider the second-kind Volterra equations in 2D:

$$u(x, y) + \int_{a}^{x} \int_{a}^{y} k(x, y, s, t, u(s, t))dsdt = g(x, y), \qquad (x, y) \in [a, b]^2. \tag{2.16}$$

Letting the above equation hold at the Legendre point pairs (x_i, y_j), and then using the linear transformation and tricks used in 1D case yields

$$u_{i,j} + \frac{1 + x_i}{2} \frac{1 + x_j}{2} \sum_{p=0}^{N} \sum_{l=0}^{N} k\left(x_i, y_j, s(x_i, \theta_p), s(x_i, \theta_l), u(s(x_i, \theta_p), s(x_i, \theta_l))\right) \omega_p \omega_l = g(x_i, y_j). \tag{2.17}$$

The values of $u(s(x_i, \theta_p), s(x_i, \theta_l))$ can also be approximated by $u_{i,j}$ with the use of the relationship between the Lagrange interpolation polynomials associate with the Legendre collocation points, as demonstrated in the one-dimensional case.

3 Convergence analysis

In this section, a convergence analysis for the numerical schemes for the Volterra equation (2.1) will be provided. The goal is to show that the rate of convergence is exponential, i.e., the spectral accuracy can be obtained for the proposed spectral approximations.

Lemma 3.1 ([4], p.290. Integration error from Gauss quadrature.) Assume that a $(N+1)$-point Gauss, or Gauss-Radau, or Gauss-Lobatto quadrature formula relative to the Legendre weight is used to integrate the product $u\phi$, where $u \in H^m(I)$ with $I := (-1, 1)$ for some $m \geq 1$ and $\phi \in \mathcal{P}_N$. Then there exists a constant C independent of N such that

$$\left| \int_{-1}^{1} u(x)\phi(x)dx - (u, \phi)_N \right| \leq CN^{-m} |u|_{\tilde{H}_{m,N}(I)} \|\phi\|_{L^2(I)}, \tag{3.1}$$

where

$$|u|_{\tilde{H}_{m,N}(I)} = \left(\sum_{j=\min(m,N)}^{m} \|u^{(j)}\|^2_{L^2(I)} \right)^{1/2}, \tag{3.2}$$

$$(u, \phi)_N = \sum_{j=0}^{N} \omega_j u(x_j)\phi(x_j). \tag{3.3}$$

Lemma 3.2 ([4], p. 289. Estimates for the interpolation error.) Assume that $u \in H^m(I)$ and denote $I_N u$ its interpolation polynomial associated with the $(N+1)$-point Gauss, or Gauss-Radau, or Gauss-Lobatto points $\{x_j\}_{j=0}^N$, namely,

$$I_N u = \sum_{i=0}^{N} u(x_i) F_i(x). \tag{3.4}$$

Then

$$\|u - I_N u\|_{L^2(I)} \leq CN^{-m} |u|_{\tilde{H}_{m,N}(I)}. \tag{3.5}$$

Lemma 3.3 ([9] Lebesgue constant for the Legendre series.) Assume that $F_j(x)$ is the N-th Lagrange interpolation polynomials associated with the Gauss, or Gauss-Radau, or Gauss-Lobatto points. Then

$$\max_{x \in (-1,1)} \sum_{j=0}^{N} |F_j(x)| = 1 + \Lambda_N^{leg}, \tag{3.6}$$

where

$$\Lambda_N^{leg} = \frac{2^{3/2}}{\sqrt{\pi}} N^{1/2} + B_0 + \mathcal{O}(N^{-1/2})$$

and B_0 is a bounded constant.

Lemma 3.4 (Gronwall inequality.) If a non-negative integrable function $E(t)$ satisfies

$$E(t) \le C_1 \int_{-1}^{t} E(s)ds + G(t), \quad -1 < t \le 1, \tag{3.7}$$

where $G(t)$ is an integrable function, then

$$\|E\|_{L^\infty(I)} \le C\|G\|_{L^1(I)}. \tag{3.8}$$

Theorem 1 *Let u be the exact solution of the Volterra equation* (2.1) *and assume that*

$$U(x) = \sum_{j=0}^{N} u_j F_j(x), \tag{3.9}$$

where u_j is given by (2.7) *and $F_j(x)$ is the j-th Lagrange basis function associated with the Gauss-points $\{x_j\}_{j=0}^N$. If $u \in H^m(I)$, then for $m \ge 1$,*

$$\begin{aligned} & \|u - U\|_{L^\infty(I)} \\ \le\ & CN^{1/2-m} \max_{1\le i\le N} \left|\tilde{K}(x_i, s(x_i,\cdot))\right|_{\tilde{H}_{m,N}(I)} \|u\|_{L^2(I)} + CN^{-m}|u|_{\tilde{H}_{m,N}(I)}, \end{aligned} \tag{3.10}$$

provided that N is sufficiently large, where $s(x_i,\theta)$ is defined by (2.5) *and C is a constant independent of N.*

Proof. Following the notations of (3.3), we let

$$(\tilde{K}(x,s), \phi(s))_{N,s} = \sum_{j=0}^{N} \tilde{K}(x, s(x,\theta_j))\phi(s(x,\theta_j))\omega_j.$$

Then the numerical scheme (2.7) can be written as

$$u_i + (\tilde{K}(x_i,s), U(s))_{N,s} = g(x_i), \tag{3.11}$$

which gives

$$u_i + \int_{-1}^{1} \tilde{K}(x_i, s(x_i,\theta))U(s(x_i,\theta))d\theta = g(x_i) + I_{i,1}, \ \ 1 \le i \le N, \tag{3.12}$$

where

$$I_{i,1} = \int_{-1}^{1} \tilde{K}(x_i, s(x_i,\theta))U(s(x_i,\theta))d\theta - (K(x_i,s), U(s))_{N,s}.$$

It follows from (3.12), (2.2) and (2.4) that

$$u_i + \int_{-1}^{x_i} K(x_i,s)U(s)ds = g(x_i) + I_{i,1}, \ \ 1 \le i \le N. \tag{3.13}$$

Using Lemma 3.1 gives

$$|I_{i,1}| \leq CN^{-m}|s\tilde{K}K(x_i, s(x_i,\cdot))|_{\tilde{H}_{m,N}(I)}\|U\|_{L^2(I)}. \tag{3.14}$$

Multiplying $F_j(x)$ on both sides of (3.13) and summing up from 0 to N yield

$$U(x) + I_N\left(\int_{-1}^{x} K(x,s)u(s)ds\right) + I_N\left(\int_{-1}^{x} K(x,s)e(s)ds\right) = I_N(g) + J_1(x), \tag{3.15}$$

where U is defined by (3.9), the interpolation operator I_N is defined by (3.4), e denotes the error function, i.e.,

$$e(x) = U(x) - u(x), \qquad x \in [-1,1],$$

and the remainder function $J_1(x)$ is defined by

$$J_1(x) = \sum_{j=0}^{N} I_{i,1}F_j(x).$$

It follows from (3.15) and (2.1) that

$$U(x) + I_N(g-u) + I_N\left(\int_{-1}^{x} K(x,s)e(s)ds\right) = I_N(g) + J_1(x),$$

which gives

$$e(x) + (u - I_N u)(x) + I_N\left(\int_{-1}^{x} K(x,s)e(s)ds\right) = J_1(x).$$

Consequently,

$$e(x) + \int_{-1}^{x} K(x,s)e(s)ds = J_1(x) + J_2(x) + J_3(x), \tag{3.16}$$

where

$$\begin{aligned} J_2 &= I_N u(x) - u(x), \\ J_3 &= \int_{-1}^{x} K(x,s)e(s)ds - I_N\left(\int_{-1}^{x} K(x,s)e(s)ds\right). \end{aligned}$$

It follows from the Gronwall inequality (see Lemma 3.4) that

$$\|e\|_{L^\infty(I)} \leq C\Big(\|J_1\|_{L^1(I)} + \|J_2\|_{L^1(I)} + \|J_3\|_{L^1(I)}\Big). \tag{3.17}$$

Using (3.14) and Lemma 3.3 gives

$$\begin{aligned} \|J_1\|_{L^1(I)} &\leq C\|J_1\|_{L^2(I)} \\ &\leq CN^{-m}\max_{1\leq i\leq N}\left|\tilde{K}(x_i, s(x_i,\cdot))\right|_{\tilde{H}_{m,N}(I)}\|U\|_{L^2(I)}\max_{x\in I}\sum_{j=0}^{N}|F_j(x)| \\ &\leq CN^{1/2-m}\max_{1\leq i\leq N}\left|\tilde{K}(x_i, s(x_i,\cdot))\right|_{\tilde{H}_{m,N}(I)}\|U\|_{L^2(I)} \\ &\leq CN^{1/2-m}\max_{1\leq i\leq N}\left|\tilde{K}(x_i, s(x_i,\cdot))\right|_{\tilde{H}_{m,N}(I)}\left(\|e\|_{L^\infty} + \|u\|_{L^2(I)}\right). \end{aligned} \tag{3.18}$$

Using the L^2-error bounds for the interpolation polynomials (i.e., Lemma 3.2) gives

$$\|J_2\|_{L^1(I)} \leq C\|J_2\|_{L^2(I)} \leq CN^{-m}|u|_{\tilde{H}_{m,N}(I)}, \tag{3.19}$$

and

$$\begin{aligned}\|J_3\|_{L^1(I)} &\leq C\|J_3\|_{L^2(I)} \\ &\leq CN^{-1}\left\|K(x,x)e(x) + \int_{-1}^{x} K_x(x,s)e(s)ds\right\|_{L^2(I)} \\ &\leq CN^{-1}\|e\|_{L^\infty(I)}. \end{aligned} \tag{3.20}$$

The above estimates, together with (3.17), yield

$$\begin{aligned}\|e(x)\|_{L^\infty(I)} \leq CN^{1/2-m} \max_{1\leq i\leq N}\left|\tilde{K}(x_i, s(x_i,\cdot))\right|_{\tilde{H}_{m,N}(I)} \left(\|e\|_{L^\infty} + \|u\|_{L^2(I)}\right) \\ +CN^{-m}|u|_{\tilde{H}_{m,N}(I)} + CN^{-1}\|e\|_{L^\infty(I)},\end{aligned} \tag{3.21}$$

which leads to (3.10), provided that N is sufficiently large. This completes the proof of this theorem. $\square$

Remark 3.1 It can be verified that

$$\max_{1\leq i\leq N}\left|\tilde{K}(x_i, s(x_i,\cdot))\right|_{\tilde{H}_{m,N}(I)} \leq \max_{x\in I}\left(\sum_{j=\min(m,N)}^{m}\int_{-1}^{x}\left|\partial_s^{(j)}K(x,s)\right|^2 ds\right).$$

Therefore, the corresponding term in (3.10) can be replaced by the right hand side of the above inequality.

4 Numerical experiments

Without lose of generality, we will only use the Legendre-Gauss-Lobatto points (i.e., the zeros of $L_{N+1}(x)$) as the collocation points. Our numerical evidences show that the other two kinds of Legendre-Gauss points produce results with similar accuracy. For the Legendre-Gauss-Lobatto points, the corresponding weights are

$$\omega_j = \frac{2}{(1-x_j^2)[L'_{N+1}(x_j)]^2}, \quad 0 \leq j \leq N.$$

Example 4.1 Our first example is concerned with an one-dimensional Volterra equation of the second kind. More precisely, consider the Volterra equation (2.1) with

$$K(x,s) = e^{xs}, \quad g(x) = e^{4x} + \frac{1}{x+4}(e^{x(x+4)} - e^{-(x+4)}).$$

The corresponding exact solution is given by $u(x) = e^{4x}$.

N	6	8	10	12	14
error	3.66e-01	1.88e-02	6.57e-04	1.65e-05	3.11e-07
N	16	18	20	22	24
error	4.57e-09	5.37e-11	5.19e-13	5.68e-14	4.26e-14

Table 1: Example 4.1: The maximum point-wise error.

N	6	8	10	12	14
error	2.33e-02	7.22e-04	1.82e-05	3.15e-07	4.06e-09
N	16	18	20	22	24
error	3.98e-11	3.05e-13	3.86e-15	3.33e-15	3.98e-15

Table 2: Example 4.2: The maximum point-wise error.

N	5	6	7	8	9
error	6.21e-04	2.02e-04	8.16e-006	1.78e-06	7.77e-08
N	12	14	16	18	20
error	1.73e-10	2.89e-13	1.30e-14	2.94e-15	1.67e-15

Table 3: Example 4.3: The maximum point-wise error for the 2D linear problem.

We use the numerical scheme (2.7). Numerical errors with several values of N are displayed in Table 1 and Fig. 1. These results indicate that the desired spectral accuracy is obtained.

Example 4.2 Our second example is about a nonlinear problem in one-dimension. Consider the Volterra equation (2.13) with

$$g(x) = -\frac{1}{2(1+36\pi^2)}\Big(e^{-x} + 36\pi^2 e^{-x} - e^{-x}\cos 6\pi x + 6\pi e^{-x}\sin 6\pi x$$
$$-36e\pi^2)e^x + e^x \sin 3\pi x, \quad K(x,s,u(s)) = e^{x-3s}u^2(s).$$

The exact solution is $u(x) = e^x \sin 3\pi x$.

This is a nonlinear problem. Although our convergence theory does not cover this case, it should be quite straightforward to establish a convergence result similar to Theorem 1 provided that the kernel k in (2.13) is Lipschitz continuous with its third argument. A similar technique for the collocation methods to the nonlinear Volterra equations was used by Brunner and Tang [3].

On the other hand, the numerical scheme (2.15) leads to a nonlinear system for $\{u_i\}_{i=1}^N$, and a proper solver for the nonlinear system (e.g., Newton method) should be used. In our computations, we just use a simple Jacobi type iteration method to solve the nonlinear system, which takes about 5 to 6 iterations. The numerical results can be seen from Table 2 and Figure 2. Again the exponential rate of convergence is observed for the nonlinear problem.

Example 4.3 The third example is concerned with a 2D linear Volterra equation with second kind. Consider the equation (2.16) with

$$K(x, y, s, t, u(s, t)) = e^{x+y} \cos(2s + t)u(s, t),$$
$$g(x, y) = \frac{1}{16} e^{x+y}(\sin(4x + 2y) + \sin(2y - 4) + \sin(4x - 2) + \sin 6) + \sin(2x + y).$$

This problem has a unique solution: $u(x, y) = \sin(2x + y)$.

There are some recent studies for using the collocation methods and the product integration methods for solving multi-dimensional Volterra integral equations, see, e.g., [2, 8]. Again, the ideas in these analysis can be used, together with the ideas in proving Theorem 1, to extend Theorem 1 to obtain a spectral convergence rate for (2.17).

Table 3 and Figure 3 present the maximum pointwise errors with difference values of N. Again, it is observed clearly that the errors decay exponentially.

5 Conclusion

This paper proposes a numerical method for the Volterra type equations based on spectral methods. The most important contribution of this work is that we are able to demonstrate rigorously that the errors of the spectral approximations decay exponentially. More precisely, it is proved that if the kernel function and solutions of the underlying Volterra equations are smooth, then errors obtained by the proposed spectral method decay exponentially which is a desired feature for a spectral method.

This work seems to be the first successful numerical method with exponential rate of convergence, which is justified theoretically, for the numerical approximations of the Volterra type equations. The tools used in establishing the error estimates include the standard estimates for the quadrature rule and the L^2-error bounds for the interpolation function.

We close by pointing out that the rate of convergence in Theorem 1 seems not optimal, which should be $\mathcal{O}(N^{-m})$ instead of $\mathcal{O}(N^{1/2-m})$ as given in (3.10). This would be true if

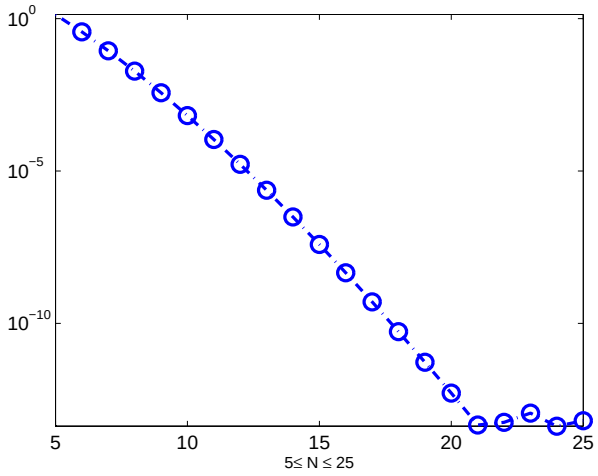

Figure 1: Example 4.1: maximum error for the 1-D linear Volterra equation.

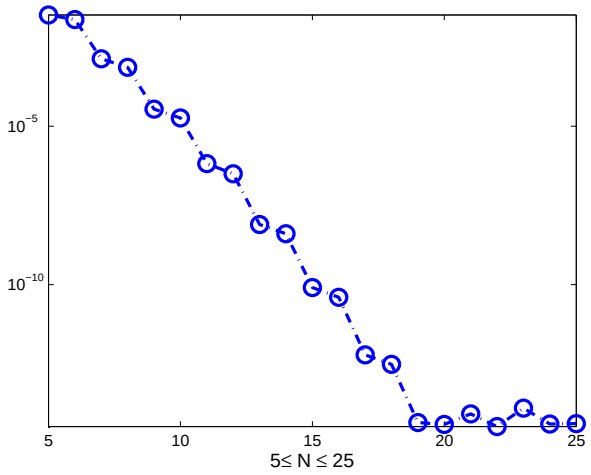

Figure 2: Example 4.2: maximum error for the 1-D nonlinear Volterra equation.

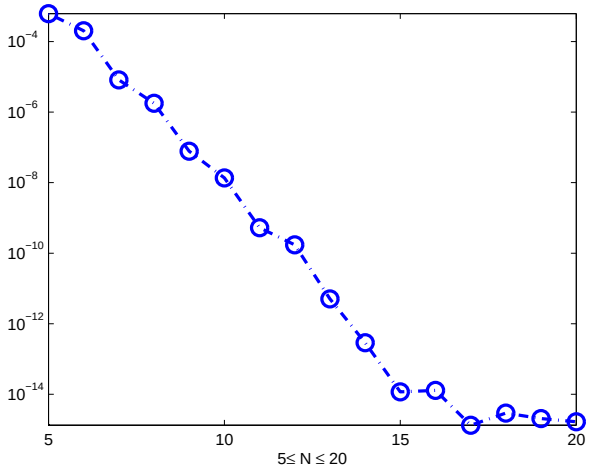

Figure 3: Example 4.3: maximum error for the 2-D linear Volterra equation.

the estimate in Lemma 3.3 can be improved. One possible improvement would be to prove

$$\sum_{j=0}^{N} \|F_j\|_{L^1(I)} = \mathcal{O}(1). \tag{5.1}$$

If this is true, then we can have

$$\|J_1\|_{L^1(I)} = \mathcal{O}(N^{-m}), \tag{5.2}$$

in the proof of Theorem 1. And as a result, the optimal order of convergence $\mathcal{O}(N^{-m})$ can be obtained.

Acknowledgments. The research of this project was partially supported by CERG Grants of Hong Kong Research Grant Council, FRG grants of Hong Kong Baptist University. Part of the research was carried out while the second author was visiting Hong Kong Baptist University supported by an CERG Grant of Hong Kong RGC.

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