THE SHARPNESS OF KUZNETSOV’S O(√Δx) L1-ERROR ESTIMATE FOR MONOTONE DIFFERENCE SCHEMES

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Abstract. We derive a lower error bound for monotone difference schemes to the solution of the linear advection equation with BV initial data. A rigorous analysis shows that for any monotone difference scheme the lower L1-error bound is O(√Δx), where Δx is the spatial stepsize.

1. Introduction

Conservative monotone difference schemes, which include the Lax-Friedrichs scheme, Godunov’s scheme, and the Engquist-Osher scheme [3], play an important role in both theoretical analysis and practical computation for hyperbolic conservation laws. From the viewpoint of numerical computation, accuracy and error bounds are of particular interest. Harten, Hyman, and Lax [4] pointed out that the monotone difference schemes are of at most first-order accuracy and Kuznetsov [6] showed that their (upper) L1-error bound is O(√Δx) as Δx goes to zero, where Δx is the spatial stepsize.

In this paper we demonstrate that all monotone schemes applied to linear first-order conservation laws in one dimension have a best possible √Δx rate of convergence when applied to discontinuous data.

A (p + q + l)-point conservative finite difference scheme

\[ v_j^{n+1} = H(v_j^n, v_{j-p}^n, \ldots, v_{j+q}^n) \]

\[ = v_j^n - \lambda [f(v_{j-p+1}^n, \ldots, v_{j+q}^n) - f(v_{j-p}^n, \ldots, v_{j+q-1}^n)] \]

is said to be monotone if H is a monotone nondecreasing function of each of its arguments, and is said to be consistent with a scalar conservation law

\[ \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad x \in \mathbb{R}, \quad t > 0, \]

\[ u|_{t=0} = u_0(x), \]

if the numerical flux \( \bar{f} \) satisfies

\[ \bar{f}(w, \ldots, w) = f(w), \]

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where $\lambda = \Delta t / \Delta x = \text{const}$, $p$ and $q$ are given nonnegative integers, and
\begin{equation}
(1.4) \quad v_j^0 = T_{\Delta x}(u_0)(x_j) = \frac{1}{\Delta x} \int_{x_j - \Delta x/2}^{x_j + \Delta x/2} u_0(x) \, dx, \quad x_j = j\Delta x.
\end{equation}

Stability, convergence, and error estimates for monotone difference schemes can be found in [2], [6], and [8].

It is easy to see that if (1.2) is the linear advection equation
\begin{equation}
(1.5a) \quad \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0 \quad (a = \text{const}),
\end{equation}
\begin{equation}
(1.5b) \quad u|_{t=0} = u_0(x),
\end{equation}
then a linear $(p + q + 1)$-point monotone difference scheme is of the form
\begin{equation}
(1.6) \quad v_j^{n+1} = \sum_{s=-p}^{q} a_s v_{j+s}^n,
\end{equation}
where
\begin{equation}
(1.7) \quad a_s \geq 0 \quad \text{for} \quad s = -p, \ldots, q.
\end{equation}

The consistency condition (1.3) implies that
\begin{equation}
(1.8) \quad \sum_{s=-p}^{q} a_s = 1
\end{equation}
and
\begin{equation}
(1.9) \quad \sum_{s=-p}^{q} s a_s = -\lambda a.
\end{equation}

Denote
\begin{equation}
\mathcal{S}_1 = \{s \mid a_s > 0\} \quad \text{and} \quad \mathcal{S}_1 \setminus s_0 = \{s \mid s \in \mathcal{S}_1 \text{ and } s \neq s_0\},
\end{equation}
where $s_0$ is an index which satisfies $a_{s_0} = \max_{s \in \mathcal{S}_1} a_s$. For the analysis of (1.6), we introduce
\begin{equation}
(1.10) \quad v_{\Delta x}(x, t) = v_j^n \quad \text{for} \quad (x, t) \in [x_{j-1/2}, x_{j+1/2}) \times [t_n, t_{n+1}),
\end{equation}
where $x_{j+1/2} = (j + 1/2)\Delta x$, $t_n = n\Delta t$, $n \in \mathbb{N}$ and $j \in \mathbb{Z}$.

In this paper we will prove the following theorem.

**Theorem.** Any monotone difference scheme (1.6), which is consistent with (1.5), has the following $L^1$-error bounds: for any $M > 0$ and $t > 0$
\begin{equation}
(1.11) \quad c(p, q)M \sum_{s \neq s_0} \sqrt{s} a_s \sqrt{\frac{t}{\lambda}} \sqrt{\Delta x} \leq \sup_{\|u_0\|_{BV} \leq M} \|v_{\Delta x}(\cdot, t) - u(\cdot, t)\|_{L^1(\mathbb{R})}
\end{equation}
\begin{equation}
\leq M \left[ 2 \sum_{s} s^2 a_s - \lambda^2 a^2 \sqrt{\frac{t}{\lambda}} \sqrt{\Delta x + \Delta x} \right],
\end{equation}
provided that $\Delta x$ is small enough. Here, $c(p, q) > 0$ is a constant depending only on $p$ and $q$, $u(x, t)$ is the solution of (1.5) and
\begin{equation}
(1.12) \quad \|u_0\|_{BV} = \sup_{\|u\| \neq 0} \frac{1}{\|u\|} \|u_0(\cdot + h) - u_0(\cdot)\|_{L^1(\mathbb{R})}.
\end{equation}
Remark 1. The lower bound of (1.11) indicates that, except in a trivial case \(a_s = 0\) for \(s \neq s_0\), a pure translation), any monotone difference scheme applied to a linear advection equation has an \(L^1\) convergence order of at most one-half in the class BV of solutions.

Remark 2. Several authors have studied error estimates for difference schemes to first-order hyperbolic equations by using Fourier methods (see [1,5] and references therein). However, to the best of our knowledge, none of these includes a lower error bound for monotone difference schemes in the presence of discontinuous initial data.

2. Some lemmas

A key step in proving the lower error bound of (1.11) is to get a precise lower bound of a sum of terms with multi-indices running over a set \(J_n\) (see the right-hand side of (3.7)). Lemma 2 and Lemma 4 below provide a precise lower cardinality of \(J_n\) and precise lower bounds of the summand terms, respectively. Consequently, they yield the desired lower bound. Lemma 1 gives a multinomial equality, while Lemma 3 is a generalized de Moivre theorem [7], which gives an asymptotic formula for the multinomial probabilities.

Lemma 1. If \(a = (a_{-p}, \ldots, a_q) \in \mathbb{R}^{p+q+1}\) satisfies (1.7) and (1.8), then

\[
\sum_{|\alpha|=n} C_n(\alpha) a^\alpha = 1,
\]

where \(\alpha = (\alpha_{-p}, \ldots, \alpha_q) \in \mathbb{N}^{p+q+1}\), \(|\alpha| = \alpha_{-p} + \cdots + \alpha_q\), \(a^\alpha = a_{-p}^{\alpha_{-p}} \cdots a_q^{\alpha_q}\) and \(C_n(\alpha)\) is the multinomial coefficient defined by

\[
C_n(\alpha) = \frac{n!}{\alpha_{-p}! \cdots \alpha_q!}.
\]

Proof. The above equality can be easily derived, and the proof is omitted. \(\square\)

Denote

\[
I_n = \{\alpha \mid |\alpha| = n, \quad \alpha \in \mathbb{N}^{p+q+1}\},
\]

\[
J_n = \{\xi \mid \xi \in \mathbb{R}^{p+q+1}, \quad \xi_s = a_s n + y_s \sqrt{n} \quad (s \in \mathcal{S}_1) \quad \text{and} \quad \xi_s = 0 \quad (s \in \mathcal{S}_0); \quad y_s \in [y_s, y_s + \Delta y_s] \quad (s \in \mathcal{S}_1 \setminus \mathcal{S}_0) \quad \text{and} \quad \sum_{s \in \mathcal{S}_1 \setminus \mathcal{S}_0} y_s = 0\}
\]

and

\[
J_n = \{\alpha \mid \alpha \in \mathcal{S}_1 \cap \mathbb{N}^{p+q+1}\},
\]

where \(a = (a_{-p}, \ldots, a_q)\) satisfies (1.7) and (1.8).

Lemma 2. For sufficiently large \(n\),

\[
|J_n| \geq |\mathcal{S}_1|/2 \left( \prod_{s \in \mathcal{S}_1 \setminus \mathcal{S}_0} |\Delta y_s| \right) n^{(|\mathcal{S}_1|-1)/2},
\]
where \(|J_n|\) and \(|S_1|\) are the cardinalities of \(J_n\) and \(S_1\), respectively.

**Proof.** Since the set \(J_n\) consists of all lattice points of \(J_n\), we have

\[
\lim_{n \to \infty} \frac{|J_n|}{\text{meas}(J_n)} = 1,
\]

and thus for sufficiently large \(n\),

\[
|J_n| \geq \frac{1}{2} \text{meas}(J_n).
\]

But from calculus we find that

\[
\text{meas}(J_n) = |S_1| \left( \prod_{y \in S_1 \setminus s_0} |\Delta y_s| \right) n^{(|S_1| - 1)/2}.
\]

**Lemma 3.** If \(\alpha \in J_n\), then

\[
\left(2^m\right)^{(m+1)/2} \left(1 + R(m)\right)
\]

uniformly for \(y_s \in [y_s, y_s + \Delta y_s] \quad (s \in S \setminus s_0)\), i.e., as \(n \to \infty\),

\[
\sup_{y_s \in [y_s, y_s + \Delta y_s]} \left| \frac{C_n(\alpha) a^\alpha}{\left(2^m\right)^{(m+1)/2} \left(1 + R(m)\right)} \exp\left(-\sum_{s \in S} \frac{y_s^2}{2a_s}\right) \right| \to 0.
\]

**Proof.** The proof depends on Stirling's formula

\[
m! = \sqrt{2\pi m} e^{-m} m^m (1 + R(m)),
\]

where \(R(m) \to 0\) as \(m \to \infty\). Since \(\alpha_s = a_s n + y_s \sqrt{n} \to \infty\) as \(n \to \infty\) for \(s \in S\), we have, by using the definition (2.2) and Stirling's formula,

\[
C_n(\alpha) a^\alpha = \frac{\sqrt{2\pi n} e^{-n} n^n a^\alpha (1 + R(n))}{(2\pi)^{|S|/2} \prod_{s \in S} \sqrt{a_s} e^{-\alpha_s} \prod_{s \in S} \left(1 + R(\alpha_s)\right)}
\]

\[
= \frac{1}{1 + R(n)} \times \frac{1}{\prod_{s \in S \setminus s_0} \left(1 + y_s/(a_s \sqrt{n})\right)^{a_n y_s \sqrt{n} \prod_{s \in S \setminus s_0} \left(1 + R(\alpha_s)\right)},
\]

where the last equality follows from the fact that

\[
\alpha^\alpha = a^\alpha \prod_{s \in S} n^{a_s} \prod_{s \in S \setminus s_0} \left(1 + y_s/(a_s \sqrt{n})\right)^{a_s} = a^\alpha n^n \prod_{s \in S} \left(1 + y_s/(a_s \sqrt{n})\right)^{a_n y_s \sqrt{n}}.
\]

Here we have used (1.8) and \(\sum_{s \in S} y_s = 0\). In order to prove (2.6), we need to verify the following formulas:

\[
\sqrt{a_s + \frac{y_s}{\sqrt{n}}} \sim \sqrt{a_s}
\]

and

\[
F(n) = \prod_{s \in S} \left(1 + \frac{y_s}{a_s \sqrt{n}}\right)^{a_n y_s \sqrt{n}} \sim \exp\left(\sum_{s \in S} \frac{y_s^2}{2a_s}\right).
\]
The first, (2.7), is obvious. Now we prove (2.8). Since
\[
\ln F(n) = \sum_{s \in \mathcal{I}} (a_s n + y_s \sqrt{n}) \ln \left(1 + \frac{y_s}{a_s \sqrt{n}}\right),
\]
we find, by applying Taylor's formula to \(\ln(1 + y_s/(a_s \sqrt{n}))\), that
\[
\ln F(n) = \sum_{s \in \mathcal{I}} \left( a_s n + y_s \sqrt{n} \right) \left[ \frac{y_s}{a_s \sqrt{n}} - \frac{1}{2} \left( \frac{y_s}{a_s \sqrt{n}} \right)^2 + \frac{1}{3} \left( \frac{y_s}{a_s \sqrt{n}} \right)^3 \right] = \sum_{s \in \mathcal{I}} \left[ y_s \sqrt{n} + \frac{y_s^2}{2a_s} + \frac{1}{a_s \sqrt{n}} \left( -\frac{y_s^3}{2a_s} + \frac{\tilde{y}_s^3}{3a_s^2} + \frac{y_s \tilde{y}_s^3}{3\sqrt{n}a_s^3} \right) \right],
\]
where \(\tilde{y}_s \in (0, y_s)\) and \(y_s \in [\tilde{y}_s, y_s + \Delta y_s]\). Now using \(\sum_{s \in \mathcal{I}} y_s = 0\), we obtain that for sufficiently large \(n\)
\[
\ln F(n) = \sum_{s \in \mathcal{I}} \left( \frac{y_s^2}{2a_s} + O\left(\frac{1}{\sqrt{n}}\right)\right).
\]
This verifies (2.8) and hence (2.6) is proved. \(\square\)

Lemma 4. If parameters \(y_s\) and \(\Delta y_s\), defined in \(J_n\), are given by
\[
(2.9) \quad y_s = \begin{cases} \frac{a_s}{s - s_0}, & s - s_0 > 0, \\ \frac{2\sqrt{a_s}}{s - s_0}, & s - s_0 < 0, \end{cases} \quad \Delta y_s = \frac{\sqrt{a_s}}{|s - s_0|} \quad (s \in \mathcal{I} \setminus \{s_0\})
\]
and \(a\) satisfies (1.7)--(1.9), then for sufficiently large \(n\),
\[
(2.10) \quad \min_{\alpha \in J_n} C_n(\alpha) a^{\alpha} \geq \frac{0.5}{(2\pi n)^{(|\mathcal{I}| - 1)/2} \prod_{s \in \mathcal{I}} \sqrt{a_s}} \exp\{-2(|\mathcal{I}| - 1)|\mathcal{I}|\},
\]
\[
(2.11) \quad \min_{\alpha \in J_n} \left| \sum_{s = -p}^{q} s \alpha_s + \lambda an \right| \geq \sqrt{n} \sum_{s \neq s_0} \sqrt{a_s}.
\]
Proof. By using Lemma 3, we have for sufficiently large \(n\),
\[
(2.12) \quad C_n(\alpha) a^{\alpha} \geq \frac{0.5}{(2\pi n)^{(|\mathcal{I}| - 1)/2} \prod_{s \in \mathcal{I}} \sqrt{a_s}} \exp\left(-\sum_{s \in \mathcal{I}} \frac{y_s^2}{2a_s}\right).
\]
Since \(y_s \in [\tilde{y}_s, \tilde{y}_s + \Delta y_s]\) for \(s \neq s_0\), we have, on account of (2.8),
\[
(2.13) \quad \sum_{s \in \mathcal{I} \setminus s_0} \frac{y_s^2}{2a_s} \leq \frac{y_{s_0}^2}{2a_{s_0}} + \sum_{s \in \mathcal{I} \setminus s_0} \frac{2}{(s - s_0)^2} \leq \frac{y_{s_0}^2}{2a_{s_0}} + 2(|\mathcal{I}| - 1).
\]
On the other hand, from \(\sum_{s \in \mathcal{I} \setminus s_0} y_s = 0\) and \(a_{s_0} = \max_s a_s\), we see that
\[
y_{s_0}^2 \leq \left( \sum_{s \in \mathcal{I} \setminus s_0} y_s \right)^2 \leq \left( \sum_{s \in \mathcal{I} \setminus s_0} \frac{2\sqrt{a_s}}{|s - s_0|} \right)^2 \leq 4a_{s_0}(|\mathcal{I}| - 1)^2,
\]
or
\[
\frac{v_{s_0}^2}{2a_{s_0}} \leq 2(|\mathcal{S}| - 1)^2.
\]
Substituting this into (2.13) gives
\[
\sum_{s \in \mathcal{S}} \frac{v_s^2}{2a_s} \leq 2(|\mathcal{S}| - 1)|\mathcal{S}|,
\]
and combining this with (2.12) yields (2.10).

We now turn to (2.11). By using (2.4b), (1.9), and (2.9), we have that
\[
\sum_{s} s\alpha_s + \lambda n = \sqrt{n} \left| \sum_{s \in \mathcal{S}} s y_s \right| = \sqrt{n} \left| \sum_{s \in \mathcal{S} \setminus s_0} (s - s_0) y_s \right| \\
\geq \sqrt{n} \sum_{s \neq s_0} \sqrt{a_s}, \quad \forall \alpha \in J_n \subset \mathbb{R}.
\]
This concludes the proof of (2.11). □

3. Proof of the main theorem

Proof of Theorem. By using (1.6) repeatedly for \( n := 0, \ldots, n - 1 \), we can express \( v_j^n \) in terms of the initial data \( v_j^0 \) as
\[
v_j^n = \sum_{\alpha \in I_n} C_n(\alpha) a^\alpha v_j^0 \sum_{s, s} s\alpha_s,
\]
where \( \sum_{s} \) is the sum over \( s \) from \(-p\) to \( q\). It is also known that the solution of (1.5) is of the form
\[
u(x, t) = u_0(x - at).
\]
Thus, we have
\[
v_j^n - u(x, t_n) = \sum_{\alpha \in I_n} C_n(\alpha) a^\alpha (v_j^0 \sum_{s, s} s\alpha_s - u_0(x - at_n)),
\]
where we have used the equality (2.1). The upper error bound of (1.11) is a special case of the error estimate for scalar conservation laws [6], but the coefficient given here is more accurate. We will not present the proof here.

In order to prove the lower error bound of (1.11), we only need to verify that the first inequality holds for the Riemann initial data
\[
u_0(x) = \begin{cases} M/2 & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -M/2 & \text{for } x < 0. \end{cases}
\]
From (1.4) and (3.4), we see that
\[
v_j^0 \sum_{s, s} s\alpha_s = u_0 \left( x_j + \Delta x \sum_s s\alpha_s \right),
\]
and hence
\[
v_j^0 + \sum_{s=0}^{s_0} a_j^s - u_0(x_j - \Delta t) = \begin{cases} 
-M, & -\sum_s s_0 > j > \lambda n, \\
-M/2, & -\sum_s s_0 = j > \lambda n \text{ or } -\sum_s s_0 > j = \lambda n, \\
0, & j > (\lambda n) \vee \left(-\sum_s s_0\right) \text{ or } j < (\lambda n) \wedge \left(-\sum_s s_0\right), \\
M/2, & -\sum_s s_0 = j < \lambda n \text{ or } -\sum_s s_0 < j = \lambda n, \\
M, & -\sum_s s_0 < j < \lambda n,
\end{cases}
\]

where \( c \lor d = \max\{c, d\} \) and \( c \land d = \min\{c, d\} \). Substituting this into (3.3) yields for \( j \neq \lambda n \)

\[
(3.5)
\]

\[
v^n_j - u(x_j, t^n) = \sum_{\alpha \in L_\alpha} C_n(\alpha) \mathbf{a}^\alpha (v_j^0 + \sum_{s=0}^{s_0} a_j^s - u_0(x_j - \Delta t))
\]

\[
= \begin{cases} 
-M \sum_{\{\alpha \in L_\alpha\} \cap \{j < -\sum_s s_0\}} C_n(\alpha) \mathbf{a}^\alpha \\
-M/2 \sum_{\{\alpha \in L_\alpha\} \cap \{j = -\sum_s s_0\}} C_n(\alpha) \mathbf{a}^\alpha \text{ for } j > \lambda n, \\
M \sum_{\{\alpha \in L_\alpha\} \cap \{j > -\sum_s s_0\}} C_n(\alpha) \mathbf{a}^\alpha \\
+M/2 \sum_{\{\alpha \in L_\alpha\} \cap \{j = -\sum_s s_0\}} C_n(\alpha) \mathbf{a}^\alpha \text{ for } j < \lambda n.
\end{cases}
\]

For simplicity, we assume \( t = t^n \) for some \( n (= t/\Delta t) \). Since \( u(x, t^n) \) is a two-piecewise constant function and \( |v^n_j| \leq M/2 \), we have

\[
(3.6a)
\]

\[
\|v_\Delta^\alpha(\cdot, t) - u(\cdot, t^n)\|_{L^1(\mathbb{R})} = \|v_\Delta^\alpha(\cdot, t^n) - u(\cdot, t^n)\|_{L^1(\mathbb{R})} \geq \Delta x \sum_j |v^n_j - u(x_j, t^n)| - M \Delta x
\]

We divide the sum in the last term of (3.6a) into two parts,

\[
(3.6b)
\]

\[
\sum_{j \neq \lambda n} |v^n_j - u(x_j, t^n)| = I_+ + I_-,
\]

where

\[
I_+ = \sum_{j > \lambda n} |v^n_j - u(x_j, t^n)|
\]

and

\[
I_- = \sum_{j < \lambda n} |v^n_j - u(x_j, t^n)|.
\]
Substituting (3.5) into $I_+$ gives

$$I_+ = M \sum_{j > \lambda n} \left\{ \sum_{\{a \in I_n \cap \{j < -s_0\}} C_n(\alpha) a^a \right\} + M/2 \sum_{j > \lambda n} \left\{ \sum_{\{a \in I_n \cap \{j < -s_0\}} C_n(\alpha) a^a \right\} \geq M/2 \sum_{j > \lambda n} \left\{ \sum_{\{a \in I_n \cap \{j < -s_0\}} C_n(\alpha) a^a \right\}$$

$$= M/2 \sum_{\{a \in I_n \cap \{j < -s_0\}} \left( -\sum_{s} s\alpha_s - \lfloor \lambda n \rfloor \right) C_n(\alpha) a^a \geq M/2 \sum_{\{a \in I_n \cap \{j < -s_0\}} \left( -\sum_{s} s\alpha_s - \lambda n \right) C_n(\alpha) a^a ,$$

where $[\eta]$ means the largest integer less than or equal to $\eta$. Similarly, we have

$$I_- \geq M/2 \sum_{\{a \in I_n \cap \{j > -s_0\}} \left( \sum_{s} s\alpha_s + \lambda n \right) C_n(\alpha) a^a .$$

Adding $I_+$ and $I_-$ yields

$$I_+ + I_- \geq M/2 \sum_{\{a \in I_n \}} \left| \sum_{s} s\alpha_s + \lambda n \right| C_n(\alpha) a^a = M/2 \sum_{a \in I_n} \left| \sum_{s} s\alpha_s + \lambda n \right| C_n(\alpha) a^a .$$

By the definitions (2.3) and (2.4b) we know that $J_n \subset I_n$ and, furthermore, assume that the parameters in $J_n$ are given by (2.9), so that (2.10) and (2.11) hold. Then, on account of (2.5), we obtain

(3.7)

$$I_+ + I_- \geq M/2 \sum_{a \in J_n} \left\{ \left| \sum_{s} s\alpha_s + \lambda n \right| C_n(\alpha) a^a \right\}$$

$$\geq \frac{M}{2} |J_n| \min_{a \in J_n} \left| \sum_{s} s\alpha_s + \lambda n \right| \min_{a \in J_n} C_n(\alpha) a^a$$

$$\geq \frac{M}{2} \sqrt{|J_n|} \left( \prod_{s \in J_n \setminus 0} \frac{\sqrt{a_s}}{|s - s_0|} \right) n^{(|J_n| - 1)/2} \sqrt{n} \sum_{s \neq 0} \sqrt{a_s}$$

$$\times \frac{0.5}{(2\pi n)(|J_n| - 1)/2 \prod_{s \in J_n} \sqrt{a_s}} \exp\{-2(|J_n| - 1)|J_n|\}$$

$$= \frac{\sqrt{|J_n|}}{8} \left( \prod_{s \in J_n \setminus 0} \frac{1}{|s - s_0|} \right) \exp\{-2(|J_n| - 1)|J_n|\} (2\pi n)(|J_n| - 1)/2 \sqrt{a_0} M \left( \sum_{s \neq 0} \sqrt{a_s} \right) \sqrt{n}.$$
It follows from \(|s - s_0| \leq (p + q)\), \(1 \leq |\mathcal{S}| \leq (p + q + 1)\), and \(a_{s_0} \leq 1\), that

\[
I_+ + I_- \geq \frac{1}{8(p + q)^{(p+q)}(2\pi)^{(p+q)/2}} \exp\{-2(p + q)(p + q + 1)\} M \left( \sum_{s \neq s_0} \sqrt{a_s} \right) \sqrt{n}
\]

\[
= 2c(p, q) M \left( \sum_{s \neq s_0} \sqrt{a_s} \right) \sqrt{n}.
\]

We can see that \(c(p, q) > 0\) is a constant which depends only on \(p\) and \(q\). Since the Riemann initial data \((3.4)\) satisfies \(|u_0(\cdot)|_{BV(\mathbb{R})} = M\), combining \((3.6)\) and \((3.8)\) yields the desired lower error bound, provided

\[
\Delta x \leq \left[ c(p, q) \sum_{s \neq s_0} \sqrt{a_s} \right]^2 t/\lambda.
\]

This completes the proof of the main theorem. \(\square\)

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