PARALLEL IN TIME ALGORITHM WITH SPECTRAL-SUBDOMAIN ENHANCEMENT FOR VOLTERRA INTEGRAL EQUATIONS

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ABSTRACT. This paper proposes a parallel in time (called also time parareal) method to solve Volterra integral equations of the second kind. The parallel in time approach follows the same spirit as the domain decomposition that consists of breaking the domain of computation into subdomains and solving iteratively the sub-problems in a parallel way. To obtain high order of accuracy, a spectral collocation accuracy enhancement in subdomains will be employed. Our main contributions in this work are two folds: (i) the time parareal method is combined with the spectral method for each sub-problem, leading to an algorithm of very high accuracy; (ii) a rigorous convergence analysis of the overall method is provided. The numerical experiment confirms that the overall computational cost is considerably reduced while the desired spectral rate of convergence can be obtained.

1. INTRODUCTION

We consider the linear Volterra integral equations (VIEs) of the second kind

$$u(t) - \int_0^t K(t,s)u(s)\mathrm{d}s = g(t), \qquad \forall t \in I, \tag{1.1}$$

where $I = [0,T]$, g and K are sufficiently smooth in I and $I \times I$, respectively; and $K(t,t) \neq 0$ for all $t \in I$. Under these assumptions, smooth solution $u(t)$ to (1.1) exists and unique, see, e.g., [3].

The presence of the integral in (1.1) makes the problem *globally* time-dependent. This means that the solution at time t_k depends on the solutions at all previous time $t < t_k$. Consequently, it may require large storage if the solution time is large or high accuracy of approximations is needed. This may become more serious if partial integro-differential equations are considered. To handle this, we will propose a parallel in time method with spectral approach for subproblems. More precisely, we divide the time interval $[0,T]$ into N equi-spaced subintervals and then break the original problem into a series of independent problems on the small sub-intervals. These independent problems are solved by a fine approximation which can be implemented in a parallel way, together with some coarse grid approximations which have to be implemented in a sequential way.

The parallel in time algorithm for a model ordinary differential equation (ODE) was initially introduced by Lions, Maday, and Turinici ([14]) for solving evolution problems in parallel. It can be interpreted as a predictor-corrector scheme [1, 2], which involves a prediction step based on a coarse approximation and a correction step computed in parallel based on a fine approximation.

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Even though the time direction seems intrinsically sequential, the combination of a coarse and a fine solution procedure has proven to allow for more rapid (convergent) solutions if parallel architectures are available. The parallel in time algorithm has received considerable attention over the past years, especially in the community of the domain decomposition methods [6], fluid and structure problems [7], the Navier-Stokes equations [8], quantum control problems [15], and so on. For the parallel in time method based on the finite difference scheme, the convergence analysis for an ODE problem was given in [9].

In our algorithm, we also build in a recent spectral method approach to obtain exponential rate of convergence, see [5, 19, 20]. The main advantage of using high order methods for integral equations is their low storage requirement with the desired precision; this advantage makes high order methods attractive. Among the high order methods spectral methods are known very useful due to its exponential rate of convergence, which is also demonstrated for solving VIEs [18, 13]. However, a drawback of the spectral method is also well-known; i.e., the matrix associated with the spectral method is full and the computational cost grows more quickly than that of a low order method. Thus for long integration it is desirable to combine the spectral method with domain decomposition techniques to avoid using a single high-degree polynomials.

It is noted that there have been great interests in studying the parabolic integro-differential equations, in particular the study of discontinuous-Galerkin methods, see, e.g., [10, 17]. These studies are very relevant to the present study and it will be interesting to extend the present study to parabolic integro-differential equations. Another class of relevant problems is about space-Time fractional diffusion equation which have been extensively studied [12, 11]. It is also noted that the problems mentioned above also contain a weakly singular kernel in the memory term, which can provide extra difficulties in convergence analysis, see, e.g., the recent work on the spectral methods for VIEs with a weakly singular kernel [5, 13].

This paper is the first attempt to approximate solutions of the VIEs by the parallel in time method. In addition, we will also provide a rigorous analysis for the proposed algorithm. This paper is organized as follows. In section 2, we construct the parallel in time method based on the spectral collocation scheme for the underlying equation. Some lemmas useful for the convergence analysis are provided in section 3. The convergence analysis for the proposed method in L^∞ under some assumptions is given in section 4. Numerical experiments are carried out in section 5. In the final section we analyze the parallelism efficiency of the overall algorithm, together with a description of some implementation details.

2. The Parallel in Time Algorithm

The time interval $I = [0, T]$ is first partitioned into N subintervals, determined by the grid points $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$ with $t_n = n\Delta t, \Delta t = T/N$. We denote this partition by $I = \cup_{n=1}^N \Lambda_n, \Lambda_n = [t_{n-1}, t_n]$.

Let $\boldsymbol{u} = (u_1, u_2, \cdots u_N)^T$, where $u_n(t)$ is the solution of (1.1) on the n-th element, i.e.,

$$u_n(t) = u(t), \quad \forall t \in [t_{n-1}, t_n], \quad n = 1, \cdots, N.$$

It is readily seen that the following system of N separate initial value problems holds:

$$
(2.1)\quad \begin{cases}
u_1(t) - \displaystyle\int_{t_0}^{t} K(t,s)u_1(s)\mathrm{d}s = g(t), & t \in [t_0, t_1],\\
u_2(t) - \displaystyle\int_{t_1}^{t} K(t,s)u_2(s)\mathrm{d}s = g(t) + \displaystyle\int_{t_0}^{t_1} K(t,s)u_1(s)\mathrm{d}s, & t \in [t_1, t_2],\\
\qquad\vdots & \\
u_N(t) - \displaystyle\int_{t_{N-1}}^{t} K(t,s)u_N(s)\mathrm{d}s & \\
\quad = g(t) + \displaystyle\int_{t_{N-2}}^{t_{N-1}} K(t,s)u_{N-1}(s)\mathrm{d}s \cdots + \displaystyle\int_{t_0}^{t_1} K(t,s)u_1(s)\mathrm{d}s, & t \in [t_{N-1}, t_N].
\end{cases}
$$

To simplify the notation, we introduce the operator $\mathcal{S}_n$, such that $u_{n+1}(t) = \mathcal{S}_n(t, u_n, u_{n-1}, \cdots, u_1)$. Then problem (2.1) can be rewritten as

$$
(2.2)\quad \begin{cases}
u_1(t) = \mathcal{S}_1(t), & t \in [t_0, t_1],\\
u_2(t) = \mathcal{S}_2(t, u_1), & t \in [t_1, t_2],\\
\qquad\vdots & \\
u_N(t) = \mathcal{S}_N(t, u_{N-1}, u_{N-2}, \cdots, u_1), & t \in [t_{N-1}, t_N].
\end{cases}
$$

In most cases $\mathcal{S}_n$ is not realizable and needs to be approximated. Let $\mathcal{F}_n$ be an approximation to $\mathcal{S}_n$, which will be defined hereafter by a highly accurate operator on element Λ_n. More precisely, we will use a spectral collocation method to define this approximation. To this end, we define $\mathcal{P}_M(\Lambda_n)$ as the polynomial spaces of degree less than or equal to M. Let $\Lambda = [-1, 1]$. We denote by $L_M(x), -1 \le x \le 1$, the Legendre polynomial of degree M. Let x_i be the points of the Legendre-Gauss (LG) quadrature formula, defined by

$$
L_{M+1}(x_i) = 0,\ i = 0, \cdots, M,
$$

arranged by increasing order: $-1 < x_0 < x_1 < \cdots < x_M < 1$. The associated weights of the LG quadrature formula are denoted by ω_i, $0 \le i \le M$. Then it is well-known the following identity:

$$
\int_\Lambda \varphi(x)\mathrm{d}x = \sum_{i=0}^{M} \varphi(x_i)\omega_i,\ \forall \varphi \in \mathcal{P}_{2M+1}(\Lambda).
$$

The discrete L^2 inner product associated to the LG quadrature is denoted by:

$$
(2.3)\qquad (\varphi, \psi)_M := \sum_{i=0}^{M} \varphi(x_i)\psi(x_i)\omega_i.
$$

Furthermore, define the LG points on the element $\Lambda_n = [t_{n-1}, t_n]$, i.e.,

$$
\xi_n^i = \frac{t_n - t_{n-1}}{2} x_i + \frac{t_n + t_{n-1}}{2}, \quad 0 \le i \le M, \quad 1 \le n \le N,
$$

and the corresponding weights

$$
\rho_n^i = \frac{\Delta t}{2}\omega_i, \quad 0 \le i \le M, \quad 1 \le n \le N.
$$

Then it holds

$$\int_{\Lambda_n} \varphi(x)\mathrm{d}x = \sum_{i=0}^{M} \varphi(\xi_n^i)\rho_n^i, \quad \forall \varphi \in \mathcal{P}_{2M+1}(\Lambda_n), \ \ \forall n = 1, \cdots, N.$$

We consider the Legendre-collocation method for (2.2) as follows: find $U_n(t) \in \mathcal{P}_M(\Lambda_n), n = 1, \cdots, N$, such that, for all $0 \le i \le M$,

$$\left\{ \begin{aligned} & U_1(\xi_1^i) - \int_{t_0}^{\xi_1^i} K(\xi_1^i, s)U_1(s)\mathrm{d}s = g(\xi_1^i), \\ & U_2(\xi_2^i) - \int_{t_1}^{\xi_2^i} K(\xi_2^i, s)U_2(s)\mathrm{d}s = g(\xi_2^i) + \int_{t_0}^{t_1} K(\xi_2^i, s)U_1(s)\mathrm{d}s, \\ & \qquad \vdots \\ & U_N(\xi_N^i) - \int_{t_{N-1}}^{\xi_N^i} K(\xi_N^i, s)U_N(s)\mathrm{d}s \\ & \quad = g(\xi_N^i) + \int_{t_{N-2}}^{t_{N-1}} K(\xi_N^i, s)U_{N-1}(s)\mathrm{d}s + \cdots + \int_{t_0}^{t_1} K(\xi_N^i, s)U_1(s)\mathrm{d}s. \end{aligned} \right. \tag{2.4}$$

In the implementation, the integral terms on the left hand sides of (2.4) are evaluated by using the following Gauss quadrature:

$$\int_{t_{n-1}}^{\xi_n^i} K(\xi_n^i, s)U_n(s)\mathrm{d}s = \int_{-1}^{1} \bar{K}(\xi_n^i, s_n^i)U_n(s_n^i)\mathrm{d}x \approx \left(\bar{K}(\xi_n^i, s_n^i), U_n(s_n^i)\right)_M, \tag{2.5}$$

where

$$s_n^i = s_n^i(x) = \frac{\xi_n^i - t_{n-1}}{2}x + \frac{\xi_n^i + t_{n-1}}{2}, \quad -1 < x < 1, \tag{2.6a}$$

$$\bar{K}(\xi_n^i, s_n^i) = \left(\frac{\xi_n^i - t_{n-1}}{2}\right) K(\xi_n^i, s_n^i). \tag{2.6b}$$

The right-hand sides of (2.4) are computed in a similar way:

$$\int_{t_{l-1}}^{t_l} K(\xi_n^i, s)U_n(s)\mathrm{d}s = \frac{\Delta t}{2}\int_{-1}^{1} K(\xi_n^i, s_l)U_n(s_l)\mathrm{d}x \approx \frac{\Delta t}{2}\left(K(\xi_n^i, s_l), U_n(s_l)\right)_M, \tag{2.7}$$

where

$$s_l = s_l(x) = \frac{\Delta t}{2}x + \frac{t_l + t_{l-1}}{2}, \quad -1 < x < 1.$$

With the help of the above numerical integrations, (2.4) is further approximated by the following problem:

$$\left\{ \begin{aligned} & U_1(\xi_1^i) - \left(\bar{K}(\xi_1^i, s_1^i), U_1(s_1^i)\right)_M = g(\xi_1^i), \\ & U_2(\xi_2^i) - \left(\bar{K}(\xi_2^i, s_2^i), U_2(s_2^i)\right)_M = g(\xi_2^i) + \tfrac{\triangle t}{2}\left(K(\xi_2^i, s_1), U_1(s_1)\right)_M, \\ & \qquad \vdots \\ & U_N(\xi_N^i) - \left(\bar{K}(\xi_N^i, s_N^i), U_N(s_N^i)\right)_M \\ & \quad = g(\xi_N^i) + \tfrac{\triangle t}{2}\left(K(\xi_N^i, s_{N-1}), U_{N-1}(s_{N-1})\right)_M + \cdots + \tfrac{\triangle t}{2}\left(K(\xi_N^i, s_1), U_1(s_1)\right)_M. \end{aligned} \right. \tag{2.8}$$

As for the continuous solution, we use the operators $\mathcal{F}_n$ to denote the above approximations, i.e.,

$$\begin{cases} U_1(t) = \mathcal{F}_1(t), & t \in [t_0, t_1], \\ U_2(t) = \mathcal{F}_2(t, U_1), & t \in [t_1, t_2], \\ \quad \vdots & \\ U_N(t) = \mathcal{F}_N(t, U_{N-1}, U_{N-2}, \cdots, U_1), & t \in [t_{N-1}, t_N]. \end{cases} \tag{2.9}$$

Clearly, the above precess is sequential since $U_{n-1}, U_{n-2}, \cdots$, and U_1 are needed to compute U_n.

In addition to the fine approximations $\mathcal{F}_n$, the parallel in time method assumes we are given another coarse approximation operator, denoted by $\mathcal{G}_n$, which is cheaper (and consequently less accurate) than $\mathcal{F}_n$. It is defined by the same spectral collocation method but with the degree $\tilde{M}$, which is much less than M. Let $\{\eta_n^i\}_{i=0}^{\tilde{M}}$ be a set of LG-points on the element $[t_{n-1}, t_n]$ corresponding to the weight $\{\varrho_n^i\}_{i=0}^{\tilde{M}}$. Then the coarse approximation $\mathcal{G}_n(t, V_{n-1}, \cdots, V_1)$ consists in finding $V_n(t) \in \mathcal{P}_{\tilde{M}}(\Lambda_n)$, such that, for all $n = 1, \cdots, N$,

$$\begin{aligned} & V_n(\eta_n^i) - \left(\bar{K}(\eta_n^i, \tau_n^i), V_n(\tau_n^i)\right)_{\tilde{M}} \\ = \; & g(\eta_n^i) + \frac{\Delta t}{2}\left(K(\eta_n^i, s_{n-1}), V_{n-1}(s_{n-1})\right)_M + \cdots + \frac{\Delta t}{2}\left(K(\eta_n^i, s_1), V_1(s_1)\right)_M, \end{aligned} \tag{2.10}$$

where $\tau_{n,i}$ is defined by

$$\tau_n^i = \tau_n^i(x) = \frac{\eta_n^i - t_{n-1}}{2}x + \frac{\eta_n^i + t_{n-1}}{2}, \quad -1 < x < 1,$$

and $\bar{K}$ and s_l are respectively defined in (2.6) and (2.8). We will denote the coarse approximation (2.10) by

$$V_n(t) = \mathcal{G}_n(t, V_{n-1}, V_{n-2}, \cdots, V_1), \quad t \in [t_{n-1}, t_n], \;\; n = 1, \cdots, N. \tag{2.11}$$

Then the parallel in time algorithm proposes an approximation to each $U_n(t), n = 1, \cdots, N$, as the limit of the sequence $U_n^k(t)$ defined by

$$\begin{aligned} U_n^k \;=\; & \mathcal{G}_n\left(t, U_{n-1}^k, U_{n-2}^k, \cdots, U_1^k\right) \\ & + \mathcal{F}_n\left(t, U_{n-1}^{k-1}, U_{n-2}^{k-1}, \cdots, U_1^{k-1}\right) - \mathcal{G}_n\left(t, U_{n-1}^{k-1}, U_{n-2}^{k-1}, \cdots, U_1^{k-1}\right), \quad k \geq 1, \end{aligned} \tag{2.12}$$

with the initial value:

$$U_n^0 = \mathcal{G}_n\left(t, U_{n-1}^0, U_{n-2}^0, \cdots, U_1^0\right), \tag{2.13}$$

where subscript n refers to the element number, superscript k refers to the iteration number, and $U_n^k(t)$ represents an approximation of the solution $U_n(t)$ at time interval $[t_{n-1}, t_n]$.

It is an easy matter to realize first that the method is exact after enough iterations. Indeed, by induction we obtain that $U_n^n = U_n$ for any $n > 0$. However our numerical experiments show that the convergence U_n^k to U_n goes much faster. Secondly, the way of presenting the algorithm places it in the category of the predictor-corrector scheme, where the predictor is $\mathcal{G}_n(t, U_{n-1}^k, U_{n-2}^k, \cdots, U_1^k)$ while the corrector is

$$\mathcal{F}_n(t, U_{n-1}^{k-1}, U_{n-2}^{k-1}, \cdots, U_1^{k-1}) - \mathcal{G}_n(t, U_{n-1}^{k-1}, U_{n-2}^{k-1}, \cdots, U_1^{k-1}).$$

Notice that the parallel in time scheme can be easily implemented on a parallel architecture. In fact, suppose at step k, all $U_n^{k-1}, 1 \leq n \leq N$, are known, and we want to compute $U_n^k, 1 \leq n \leq N$.

Obviously, both fine and coarse solutions in the corrector, i.e., $\mathcal{F}_n(t, U_{n-1}^{k-1}, U_{n-2}^{k-1}, \cdots, U_1^{k-1})$ and $\mathcal{G}_n(t, U_{n-1}^{k-1}, U_{n-2}^{k-1}, \cdots, U_1^{k-1})$ can be calculated simultaneously with respect to each n since all its independent variables $U_n^{k-1}, 1 \leq n \leq N$, are available. On the other hand, although the computation of the coarse approximation $\mathcal{G}_n(t, U_{n-1}^{k}, U_{n-2}^{k}, \cdots, U_1^{k})$ has to be sequential, the cost of this computation is relatively inexpensive.

3. Useful Lemmas

The convergence analysis of the proposed algorithm is accomplished with aid of a series of lemmas. We first introduce some notations that will be used throughout the analysis. Let Λ be an arbitrary bounded interval, $L^2(\Lambda)$ be the space of measurable functions whose square is Lebesgue integrable in Λ. The inner product and norm of $L^2(\Lambda)$ are defined by

$$(u,v)_\Lambda = \int_\Lambda u(x)v(x)\mathrm{d}x \ , \quad \|v\|_\Lambda = \sqrt{(v,v)_\Lambda}, \quad \forall u,v \in L^2(\Lambda).$$

For non-negative integer m, define

$$H^m(\Lambda) := \{v;\ \|v\|_{m,\Lambda} < \infty\},$$

with

$$\|v\|_{m,\Lambda} = \left(\sum_{k=0}^{m} |v|_{k,\Lambda}^2\right)^{\frac{1}{2}}, \quad |v|_{k,\Lambda} = \|\partial_x^k v\|_\Lambda,$$

and

$$W^{m,\infty}(\Lambda) := \left\{v;\ \|v\|_{W^{m,\infty}(\Lambda)} < \infty\right\},$$

with

$$\|v\|_{W^{m,\infty}(\Lambda)} = \sum_{k=0}^{m} \|\partial_x^k v\|_{\infty,\Lambda}, \quad \|v\|_{\infty,\Lambda} = ess\sup_{x\in\Lambda} |v(x)|.$$

In the error analysis, it is sometimes convenient to introduce the seminorms as follows

$$|v|_{H^{m;M}(\Lambda)} = \left(\sum_{k=\min(m,M+1)}^{m} \|\partial_x^k v\|_\Lambda^2\right)^{\frac{1}{2}}.$$

Hereafter, in cases where no confusion would arise, the domain symbol Λ may be dropped from the notations. We denote by c generic positive constants independent of the discretization parameters, but may depend on the kernel function $K(\cdot,\cdot)$ or T.

In order to carry out the error analysis, we introduce two approximation operators. Firstly, we define the Lagrange interpolation operator $\mathcal{I}_\Lambda^M :\ \mathcal{C}(\Lambda) \to \mathcal{P}_M(\Lambda)$, by: $\forall v \in \mathcal{C}(\Lambda), \mathcal{I}_\Lambda^M v \in \mathcal{P}_M(\Lambda)$, such that

$$\mathcal{I}_\Lambda^M v(z_i) = v(z_i), \quad 0 \leq i \leq M,$$

where $\{z_i\}_{i=0}^M$ is the set of LG-points on the interval Λ. This polynomial can be expressed as

$$\mathcal{I}_\Lambda^M v(x) = \sum_{i=0}^{M} v(z_i) h_\Lambda^i(x),$$

where h_Λ^i is the Lagrange interpolation basis function associated with $\{z_i\}_{i=0}^M$. Particularly, we use $\mathcal{I}_n^M$ (resp., $h_n^i(x)$) to replace $\mathcal{I}_\Lambda^M$ (resp., $h_\Lambda^i(x)$) when $\Lambda = \Lambda_n$.

Then for all $v \in H^m(\Lambda), m \geqslant 1$, the following optimal error estimates hold (see, e.g., [4, 18]):

$$\|v - \mathcal{I}_\Lambda^M v\| \lesssim M^{-m} |v|_{H^{m;M}}, \tag{3.1}$$

$$\|v - \mathcal{I}_\Lambda^M v\|_{L^\infty} \lesssim M^{\frac{3}{4}-m} |v|_{H^{m;M}}. \tag{3.2}$$

For the discrete L^2-inner product defined in (2.3), it holds the following error estimate [4]: $\forall \phi \in \mathcal{P}_M(\Lambda)$,

$$|(v,\phi) - (v,\phi)_M| \lesssim M^{-m} |v|_{H^{m;M}} \|\phi\|, \quad v \in H^m(\Lambda), \quad m \geq 1. \tag{3.3}$$

From [16], we have the following result on the Lebesgue constant for the Lagrange interpolation polynomials associated with the LG-points.

Lemma 3.1. *Let $\{h_\Lambda^j(x)\}_{j=0}^M$ be the Lagrange interpolation polynomials associated with the $M+1$ LG-points on the interval Λ. Then*

$$\|I_\Lambda^M\|_\infty := \max_{x\in\Lambda} \sum_{j=0}^{M} |h_\Lambda^j(x)| \lesssim \sqrt{M}. \tag{3.4}$$

Lemma 3.2. *Let $m \geq 0$. If $v(x) \in W^{m,\infty}(\Lambda)$ satisfies*

$$u(t) = v(t) + \int_0^t K(t,s)u(s)\mathrm{d}s, \tag{3.5}$$

then

$$\|u\|_{W^{m,\infty}} \leq c \|v\|_{W^{m,\infty}}. \tag{3.6}$$

Proof. Applying the linear transformation $\tau = \frac{s}{t}$ to (3.5) gives

$$u(t) = v(t) + \int_0^1 tK(t,t\tau)u(t\tau)\mathrm{d}\tau. \tag{3.7}$$

Differentiating (3.7) m-times with respect to the variable t yields

$$\begin{aligned} u^{(m)}(t) &= v^{(m)}(t) + \int_0^1 \sum_{i=0}^{m} C_m^i \partial_t^{(m-i)}(K(t,t\tau)t)u^{(i)}(t\tau)\mathrm{d}\tau \\ &= v^{(m)}(t) + \sum_{i=0}^{m-1} C_m^i \int_0^1 \partial_t^{(m-i)}(K(t,t\tau)t)\partial_t^{(i)}u(t\tau)\mathrm{d}\tau + \int_0^t K(t,s)u^{(m)}(s)\frac{s^m}{t^m}\mathrm{d}s. \end{aligned}$$

It follows from the standard Gronwall inequality (see, e.g., [18]) that

$$\begin{aligned} \left\|u^{(m)}(t)\right\|_\infty &\leq c\left\|v^{(m)}(t)\right\|_\infty + c\sum_{i=0}^{m-1} C_m^i \left\|\int_0^1 \partial_t^{(m-i)}(K(t,t\tau)t)\partial_t^{(i)}u(t\tau)\mathrm{d}\tau\right\|_\infty \\ &\leq c\left\|v^{(m)}(t)\right\|_\infty + c\sum_{i=0}^{m-1} C_m^i \max_{0\leq\tau\leq 1}\left\|\partial_t^{(m-i)}(K(t,t\tau)t)\right\|_\infty \left\|u^{(i)}(t)\right\|_\infty \\ &\leq c\left\|v^{(m)}(t)\right\|_\infty + c(2^m-1)\sum_{i=0}^{m-1}\left\|u^{(i)}(t)\right\|_\infty, \end{aligned} \tag{3.8}$$

where c depends on

$$\max_{0\leq i\leq m-1}\max_{0\leq\tau\leq 1}\left\|\partial_t^{(m-i)}(K(t,t\tau)t)\right\|_\infty.$$

Consequently, we have

$$\|u\|_{W^{m,\infty}} \leq c\,\|v\|_{W^{m,\infty}} + c2^m\,\|u\|_{W^{m-1,\infty}}\,. \tag{3.9}$$

Let $a_m = \|u\|_{W^{m,\infty}}$ and $B = c\,\|v\|_{W^{m,\infty}}$. It follows from (3.9) that

$$\begin{aligned} a_m &\leq B + c2^m a_{m-1} \\ &\leq B + c2^m(B + c2^{m-1}a_{m-2}) \\ &\leq B + cB2^m + c^2 2^{m+(m-1)}a_{m-2} \\ &\cdots \\ &\leq B + cB2^m + \cdots + c^{m-1}B2^{m+(m-1)+\cdots+2} + c^m 2^{m+(m-1)+\cdots+1}a_0 \\ &\leq \big(1 + c2^m + c^2 2^{m+(m-1)} + \cdots + c^m 2^{m+(m-1)+\cdots+1}\big)B, \end{aligned}$$

which leads to the desired estimate (3.6). □

The following discrete Gronwall lemma can be found in [3].

Lemma 3.3. *Assume that ϕ_n, p_n and k_n are three sequences, k_n is non-negative, and they satisfy*

$$\begin{cases} \phi_0 \leq g_0 \\ \phi_n \leq g_0 + \sum\limits_{s=0}^{n-1} p_s + \sum\limits_{s=0}^{n-1} k_s\phi_s, \quad n \geq 1. \end{cases}$$

Then it holds

$$\begin{cases} \phi_1 \leq g_0(1+k_0) + p_0 \\ \phi_n \leq g_0 \prod\limits_{s=0}^{n-1}(1+k_s) + \sum\limits_{s=0}^{n-2} p_s \prod\limits_{\tau=s+1}^{n-1}(1+k_r) + p_{n-1}, \quad n \geq 2. \end{cases}$$

Moreover, if $g_0 \geq 0$ and $p_n \geq 0$ for all $n \geq 0$, then it holds

$$\phi_n \leq \left(g_0 + \sum_{s=0}^{n-1} p_s\right)\exp\left(\sum_{s=0}^{n-1} k_s\right), \quad n \geq 1.$$

4. Stability and Convergence Analysis

This section is devoted to the stability and convergence analysis of the proposed parallel in time scheme. To begin with, we will the following lemma which provides some property for the coarse approximation and find approximation operators.

Lemma 4.1. *For $1 \leq n \leq N$, let $\mathcal{F}_n$ be the fine approximation operator defined by the spectral collocation scheme (2.8), $\mathcal{G}_n$ defined by (2.10), $g_M = \mathcal{I}_n^M g$, and $\{\psi_i\}_{i=1}^{n-1}$ be a polynomial sequence, then, for sufficiently large M and $\tilde{M}$, we have*

$$\begin{aligned} &\|\mathcal{F}_n(t, \psi_{n-1}, \psi_{n-2}, \cdots, \psi_1)\|_\infty \\ &\qquad \leq c\Big(\|g_M\|_\infty + \Delta t\|\psi_{n-1}\|_\infty + \Delta t\|\psi_{n-2}\|_\infty + \cdots + \Delta t\|\psi_1\|_\infty\Big), \end{aligned} \tag{4.1}$$

$$\begin{aligned} &\|\mathcal{G}_n(t, \psi_{n-1}, \psi_{n-2}, \cdots, \psi_1)\|_\infty \\ &\qquad \leq c\Big(\|g_{\tilde{M}}\|_\infty + \Delta t\|\psi_{n-1}\|_\infty + \Delta t\|\psi_{n-2}\|_\infty + \cdots + \Delta t\|\psi_1\|_\infty\Big). \end{aligned} \tag{4.2}$$

Proof. Since the only difference between $\mathcal{F}_n$ and $\mathcal{G}_n$ is the polynomial degree, we only need to prove the inequality (4.1). Let $\psi_n := \mathcal{F}_n(t, \psi_{n-1}, \psi_{n-2}, \cdots, \psi_1)$, then by the spectral collocation scheme (2.8), we have

$$\begin{aligned}\psi_n(\xi_n^i) \;&=\; g(\xi_n^i) + \left(\bar{K}(\xi_n^i, s_n^i), \psi_n(s_n^i)\right)_M \\ &+\frac{\Delta t}{2}\left(K(\xi_n^i, s_{n-1}), \psi_{n-1}(s_{n-1})\right)_M + \cdots + \frac{\Delta t}{2}\left(K(\xi_n^i, s_1), \psi_1(s_1)\right)_M ,\end{aligned} \tag{4.3}$$

which can be further re-organized as

$$\begin{aligned}\psi_n(\xi_n^i) \;&=\; g(\xi_n^i) + \left(\bar{K}(\xi_n^i, s_n^i), \psi_n(s_n^i)\right) \\ &+\frac{\Delta t}{2}\left(K(\xi_n^i, s_n^i), \psi_n(s_n^i)\right)_M - \frac{\Delta t}{2}\left(K(\xi_n^i, s_n^i), \psi_n(s_n^i)\right) \\ &+\frac{\Delta t}{2}\left(K(\xi_n^i, s_{n-1}), \psi_{n-1}(s_{n-1})\right) \\ &+\frac{\Delta t}{2}\left(K(\xi_n^i, s_{n-1}), \psi_{n-1}(s_{n-1})\right)_M - \frac{\Delta t}{2}\left(K(\xi_n^i, s_{n-1}), \psi_{n-1}(s_{n-1})\right) \\ &+\cdots \\ &+\frac{\Delta t}{2}\left(K(\xi_n^i, s_1), \psi_1(s_1)\right) \\ &+\frac{\Delta t}{2}\left(K(\xi_n^i, s_1), \psi_1(s_1)\right)_M - \frac{\Delta t}{2}\left(K(\xi_n^i, s_1), \psi_1(s_1)\right).\end{aligned} \tag{4.4}$$

Multiplying both sides of equation (4.4) by h_n^i and summing up from $i=0$ to $i=M$, we obtain

$$\begin{aligned}\psi_n(t) \;&=\; \mathcal{I}_n^M g + \mathcal{I}_n^M \int_{t_{n-1}}^t K(t,s)\psi_n(s)\mathrm{d}s + \sum_{i=0}^M A_n^i h_n^i \\ &+\frac{\Delta t}{2}\mathcal{I}_n^M\left(K(t, s_{n-1}), \psi_{n-1}(s_{n-1})\right) + \sum_{i=0}^M A_{n-1}^i h_{n-1}^i \\ &+\cdots \\ &+\frac{\Delta t}{2}\mathcal{I}_n^M\left(K(t, s_1), \psi_1(s_1)\right) + \sum_{i=0}^M A_1^i h_1^i,\end{aligned} \tag{4.5}$$

where

$$\begin{aligned}A_n^i &= \left(\bar{K}(\xi_n^i, s_n^i), \psi_n(s_n^i)\right) - \left(\bar{K}(\xi_n^i, s_n^i), \psi_n(s_n^i)\right)_M , \\ A_l^i &= \frac{\Delta t}{2}\left(K(\xi_n^i, s_l), \psi_l(s_l)\right) - \frac{\Delta t}{2}\left(K(\xi_n^i, s_l), \psi_l(s_l)\right)_M , \quad 1 \le l \le n-1.\end{aligned}$$

Consequently, we have

$$\psi_n(t) = \int_{t_{n-1}}^t K(t,s)\psi_n(s)\mathrm{d}s + \mathcal{I}_n^M g + \sum_{l=1}^n J_l + \sum_{l=1}^n I_l + \sum_{l=1}^{n-1} R_l, \tag{4.6}$$

where

$$
\begin{aligned}
J_l &= \sum_{i=0}^{M} A_l^i h_l^i, \quad 1 \le l \le n, \\
I_n &= \mathcal{I}_n^M \left(\bar{K}(t, s_n^i), \psi_n(s_n^i)\right) - \left(\bar{K}(t, s_n^i), \psi_n(s_n^i)\right), \\
I_l &= \frac{\Delta t}{2} \mathcal{I}_n^M \left(K(t, s_l), \psi_l(s_l)\right) - \frac{\Delta t}{2}\left(K(t, s_l), \psi_l(s_l)\right), \quad 1 \le l \le n-1, \\
R_l &= \frac{\Delta t}{2}\left(K(t, s_l), \psi_l(s_l)\right), \quad 1 \le l \le n-1.
\end{aligned}
$$

It follows from the standard Gronwall inequality for (4.6) that

$$
\|\psi_n(t)\|_\infty \le c\left(\|g_M\|_\infty + \sum_{l=1}^{n}\|J_l\|_\infty + \sum_{l=1}^{n}\|I_l\|_\infty + \sum_{l=1}^{n-1}\|R_l\|_\infty\right). \tag{4.7}
$$

We now estimate the right hand side term by term. First, we estimate the terms J_l. Using the error estimate (3.3) for the LG-quadrature, we have

$$
\begin{aligned}
\max_{0\le i\le M} |A_n^i| &= \max_{0\le i\le M}\left|\left(\bar{K}(\xi_n^i, s_n^i), \psi_n(s_n^i)\right)_M - \int_{t_{n-1}}^{\xi_n^i} K(\xi_n^i, s)\psi_n(s)\mathrm{d}s\right| \\
&\le c\Delta t M^{-1}\max_{0\le i\le M}|K(\xi_n^i, s_n^i)|_{H^{1;M}}\max_{0\le i\le M}\|\psi_n(s_n^i)\| \\
&\le c_1\Delta t M^{-1}\|\psi_n\|_\infty,
\end{aligned} \tag{4.8}
$$

where c_1 depends on $|K(\xi_n^i, s_n^i(\cdot))|_{H^{1;M}}$. Similarly,

$$
\max_{0\le i\le M}|A_l^i| \le c_1\Delta t M^{-1}\|\psi_l\|_\infty, \quad 1 \le l \le n-1. \tag{4.9}
$$

Hence, combining (4.8) and (4.9) with Lemma 3.1 gives

$$
\begin{aligned}
\|J_n\|_\infty &= \left\|\sum_{i=0}^{M} A_n^i h_n^i\right\|_\infty \le \max_{0\le i\le M}|A_n^i|\max_{x\in\Lambda_n}\sum_{i=0}^{M}|h_n^i| \\
&\le c_1\Delta t M^{-\frac{1}{2}}\|\psi_n\|_\infty \le \frac{1}{3c}\|\psi_n\|_\infty
\end{aligned} \tag{4.10}
$$

for sufficiently large M. Following the same lines leads to

$$
\|J_l\|_\infty \le c_1\Delta t M^{-\frac{1}{2}}\|\psi_l\|_\infty, \quad 1 \le l \le n-1. \tag{4.11}
$$

We now estimate the second term I_l. For $l = n$, applying (3.2) leads to

$$
\begin{aligned}
\|I_n\|_\infty &= \left\|(\mathcal{I}_n^M - \mathcal{I})\int_{t_{n-1}}^{t} K(t, s)\psi_n(s)\mathrm{d}s\right\|_\infty \\
&\le c_1 M^{-\frac{1}{4}}\left|\int_{t_{n-1}}^{t} K(t, s)\psi_n(s)\mathrm{d}s\right|_{H^{1;M}} \le c_1 M^{-\frac{1}{4}}\|\psi_n\|_\infty \le \frac{1}{3c}\|\psi_n\|_\infty.
\end{aligned} \tag{4.12}
$$

For $1 \leq l \leq n-1$, in virtue of (3.2), we derive

$$\begin{aligned} \|I_l\|_\infty &= \left\| (\mathcal{I}_n^M - \mathcal{I}) \int_{t_{l-1}}^{t_l} K(t,s)\psi_l(s)\mathrm{d}s \right\|_\infty \\ &\leq c_1 M^{\frac{3}{4}-1} \left| \int_{t_{l-1}}^{t_l} K(t,s)\psi_l(s)\mathrm{d}s \right|_{H^{1;M}} \leq c_1 \Delta t M^{-\frac{1}{4}} \|\psi_l\|_\infty . \end{aligned} \tag{4.13}$$

It remains to estimate R_l. By a direct calculation, we have

$$\|R_l\|_\infty = \frac{\Delta t}{2} \|(K(t,s_l),\psi_l(s_l))\|_\infty \leq c\Delta t \|\psi_l\|_\infty, \quad 1 \leq l \leq n-1. \tag{4.14}$$

Finally, combining (4.7)-(4.14) gives

$$\begin{aligned} \|\psi_n\|_\infty &\leq c\|g_M\|_\infty + \frac{1}{3}\|\psi_n\|_\infty + c\Delta t M^{-\frac{1}{2}} \sum_{l=1}^{n-1} \|\psi_l\|_\infty \\ &\quad + \frac{1}{3}\|\psi_n\|_\infty + c\Delta t M^{-\frac{1}{4}} \sum_{l=1}^{n-1} \|\psi_l\|_\infty + c\Delta t \sum_{l=1}^{n-1} \|\psi_l\|_\infty . \end{aligned} \tag{4.15}$$

Now a simple rearrangement leads to (4.1). □

We are now ready to state and proof one of the main results of this paper.

Theorem 4.1. *(Stability) The iteration scheme* (2.12) *is stable in the sense that the solution* U_n^k *satisfies*

$$\|U_n^k\|_\infty \leq c\|g_M\|_\infty, \quad \forall k \geq 0, \;\; 1 \leq n \leq N. \tag{4.16}$$

Proof. For $k = 0$, in virtue of the scheme (2.13) and Lemma 4.1, we obtain

$$\begin{aligned} \|U_n^0\|_\infty &= \left\|\mathcal{G}_n\left(t, U_{n-1}^0, U_{n-2}^0, \cdots, U_1^0\right)\right\|_\infty \\ &\leq c_0\Big(\|g_M\|_\infty + \Delta t\|U_{n-1}^0\|_\infty + \Delta t\|U_{n-2}^0\|_\infty + \ldots + \Delta t\|U_1^0\|_\infty\Big). \end{aligned}$$

By applying the discrete Gronwall lemma 3.3, we get

$$\|U_n^0\|_\infty \leq c_0 e^{c_0(n-1)\Delta t}\|g_M\|_\infty \leq c_0 e^{c_0 T}\|g_M\|_\infty. \tag{4.17}$$

For $k \geq 1$, according to the iteration scheme (2.12), we have

$$\begin{aligned} \|U_n^k\|_\infty &\leq \left\|\mathcal{G}_n\left(t, U_{n-1}^k, U_{n-2}^k, \cdots, U_1^k\right)\right\|_\infty \\ &\quad + \left\|\mathcal{F}_n\left(t, U_{n-1}^{k-1}, U_{n-2}^{k-1}, \cdots, U_1^{k-1}\right)\right\|_\infty + \left\|\mathcal{G}_n\left(t, U_{n-1}^{k-1}, U_{n-2}^{k-1}, \cdots, U_1^{k-1}\right)\right\|_\infty . \end{aligned} \tag{4.18}$$

Applying Lemma 4.1 to the right hand side of (4.18) yields

$$\begin{aligned} \|U_n^k\|_\infty &\leq c\left(\|g_M\|_\infty + \Delta t\|U_{n-1}^k\|_\infty + \Delta t\|U_{n-2}^k\|_\infty + \cdots + \Delta t\|U_1^k\|_\infty\right) \\ &\quad + c\left(\|g_{\tilde{M}}\|_\infty + \|g_M\|_\infty + \Delta t\|U_{n-1}^{k-1}\|_\infty + \Delta t\|U_{n-2}^{k-1}\|_\infty + \cdots + \Delta t\|U_1^{k-1}\|_\infty\right) \\ &\leq c_0\left(\Delta t\|U_{n-1}^k\|_\infty + \Delta t\|U_{n-2}^k\|_\infty + \cdots + \Delta t\|U_1^k\|_\infty\right) \\ &\quad + c_0\left(\|g_M\|_\infty + \Delta t\|U_{n-1}^{k-1}\|_\infty + \Delta t\|U_{n-2}^{k-1}\|_\infty + \cdots + \Delta t\|U_1^{k-1}\|_\infty\right). \end{aligned} \tag{4.19}$$

In the following, we will derive the following inequality

$$\|U_n^k\|_\infty \le c_0 \sum_{l=1}^{k} \frac{e^{c_0(l+1)(n-1)\Delta t}}{l!}\|g_M\|_\infty, \quad \forall k \ge 0, 1 \le n \le N, \tag{4.20}$$

where c_0 is a constant independent of k. We do this by induction. First, for $k = 1$, from (4.19), we have

$$\begin{aligned}\|U_n^1\|_\infty \quad \le \quad & c_0\Big(\Delta t\|U_{n-1}^1\|_\infty + \Delta t\|U_{n-2}^1\|_\infty + \cdots + \Delta t\|U_1^1\|_\infty\Big)\\ & + c_0\Big(\|g_M\|_\infty + \Delta t\|U_{n-1}^0\|_\infty + \Delta t\|U_{n-2}^0\|_\infty + \cdots + \Delta t\|U_1^0\|_\infty\Big).\end{aligned}$$

By applying the discrete Gronwall lemma, Lemma 3.3, to the sequence $\{U_j^1\}_{j=1}^n$, we obtain, for all $n = 1, \cdots, N$,

$$\|U_n^1\|_\infty \le c_0 e^{c_0(n-1)\Delta t}\left(\|g_M\|_\infty + \Delta t\|U_{n-1}^0\|_\infty + \Delta t\|U_{n-2}^0\|_\infty + \cdots + \Delta t\|U_1^0\|_\infty\right). \tag{4.21}$$

Combining (4.21) with estimate (4.17), we arrive at

$$\begin{aligned}\|U_n^1\|_\infty \quad & \le \quad c_0 e^{c_0(n-1)\Delta t}\left(\|g_M\|_\infty + c_0\Delta t\sum_{i=1}^{n-1} e^{c_0(i-1)\Delta t}\|g_M\|_\infty\right)\\ & = \quad c_0 e^{c_0(n-1)\Delta t}\|g_M\|_\infty\left(1 + c_0\Delta t\frac{e^{c_0(n-1)\Delta t} - 1}{e^{c_0\Delta t} - 1}\right)\\ & \le \quad c_0 e^{c_0(n-1)\Delta t}\|g_M\|_\infty\left(1 + e^{c_0(n-1)\Delta t} - 1\right) \le c_0 e^{2c_0(n-1)\Delta t}\|g_M\|_\infty.\end{aligned}$$

This means (4.20) holds for $k = 1$. Now we need to prove that if (4.20) holds for a given k, it holds also for $k+1$. By using (4.19), (4.20), and discrete Gronwall lemma, Lemma 3.3, we get

$$\begin{aligned}\|U_n^{k+1}\|_\infty \quad & \le \quad c_0 e^{c_0(n-1)\Delta t}\left(\|g_M\|_\infty + \Delta t\|U_{n-1}^k\|_\infty + \Delta t\|U_{n-2}^k\|_\infty + \cdots + \Delta t\|U_1^k\|_\infty\right)\\ & \le \quad c_0\|g_M\|_\infty e^{c_0(n-1)\Delta t}\left(1 + c_0\Delta t\sum_{l=1}^{k}\frac{e^{c_0(l+1)(n-2)\Delta t}}{l!} + \cdots + c_0\Delta t\sum_{l=1}^{k}\frac{1}{l!}\right)\\ & = \quad c_0\|g_M\|_\infty e^{c_0(n-1)\Delta t}\left(1 + c_0\Delta t\sum_{l=1}^{k}\frac{1}{l!}\frac{e^{c_0(l+1)(n-1)\Delta t} - 1}{e^{c_0(l+1)\Delta t} - 1}\right)\\ & \le \quad c_0\|g_M\|_\infty e^{c_0(n-1)\Delta t}\left(1 + \sum_{l=1}^{k}\frac{1}{(l+1)!}(e^{c_0(l+1)(n-1)\Delta t} - 1)\right)\\ & \le \quad c_0\|g_M\|_\infty e^{c_0(n-1)\Delta t}\left(1 + \sum_{l=1}^{k}\frac{1}{(l+1)!}e^{c_0(l+1)(n-1)\Delta t}\right)\\ & \le \quad c_0\|g_M\|_\infty\sum_{l=1}^{k+1}\frac{e^{c_0(l+1)(n-1)\Delta t}}{l!}.\end{aligned}$$

This implies that (4.20) also holds for k being replaced by $k+1$. Finally, by noticing the fact that

$$\sum_{l=1}^{\infty}\frac{e^{c_0(l+1)(n-1)\Delta t}}{l!} \le e^{c_0(n-1)\Delta t}e^{e^{c_0(n-1)\Delta t}} \le e^{c_0 T}e^{e^{c_0 T}}, \quad 1 \le n \le N,$$

the desired result (4.16) follows from (4.20). □

In the following, we will analyze the convergence of the iteration scheme (2.12).

Lemma 4.2. *Let u_n be the solution of* (2.1) *and U_n be the solution of* (2.9) *with sufficiently large M. If $u \in H^m(I), m \geq 1$, we have*

$$\|u_n - U_n\|_\infty \leq c\left(M^{\frac{3}{4}-m}|u|_{H^{m;M}} + M^{\frac{1}{2}-m}\|u\|_\infty\right), \quad 1 \leq n \leq N.$$

Proof. From the definitions (2.1) and (2.8), we have

$$u_n(\xi_n^i) = g(\xi_n^i) + \left(\bar{K}(\xi_n^i, s_n^i), u_n(s_n^i)\right) + \frac{\Delta t}{2}\left(K(\xi_n^i, s_{n-1}), u_{n-1}(s_{n-1})\right) + \cdots + \frac{\Delta t}{2}\left(K(\xi_n^i, s_1), u_1(s_1)\right), \tag{4.22}$$

$$U_n(\xi_n^i) = g(\xi_n^i) + \left(\bar{K}(\xi_n^i, s_n^i), U_n(s_n^i)\right)_M + \frac{\Delta t}{2}\left(K(\xi_n^i, s_{n-1}), U_{n-1}(s_{n-1})\right)_M + \cdots + \frac{\Delta t}{2}\left(K(\xi_n^i, s_1), U_1(s_1)\right)_M. \tag{4.23}$$

Let $e_n = u_n - U_n$. By subtracting (4.23) from (4.22), we deduce

$$\begin{aligned} u_n(\xi_n^i) - U_n(\xi_n^i) \;=\;& \left(\bar{K}(\xi_n^i, s_n^i), u_n(s_n^i)\right) - \left(\bar{K}(\xi_n^i, s_n^i), U_n(s_n^i)\right)_M \\ &+ \frac{\Delta t}{2}\left(K(\xi_n^i, s_{n-1}), u_{n-1}(s_{n-1})\right) - \frac{\Delta t}{2}\left(K(\xi_n^i, s_{n-1}), U_{n-1}(s_{n-1})\right)_M \\ &+ \cdots \\ &+ \frac{\Delta t}{2}\left(K(\xi_n^i, s_1), u_1(s_1)\right) - \frac{\Delta t}{2}\left(K(\xi_n^i, s_1), U_1(s_1)\right)_M. \end{aligned}$$

It can be further re-organized as

$$\begin{aligned} u_n(\xi_n^i) - U_n(\xi_n^i) \;=\;& \left(\bar{K}(\xi_n^i, s_n^i), e_n(s_n^i)\right) \\ &+ \left(\bar{K}(\xi_n^i, s_n^i), U_n(s_n^i)\right) - \left(\bar{K}(\xi_n^i, s_n^i), U_n(s_n^i)\right)_M \\ &+ \frac{\Delta t}{2}\left(K(\xi_n^i, s_{n-1}), e_{n-1}(s_{n-1})\right) \\ &+ \frac{\Delta t}{2}\left(K(\xi_n^i, s_{n-1}), U_{n-1}(s_{n-1})\right) - \frac{\Delta t}{2}\left(K(\xi_n^i, s_{n-1}), U_{n-1}(s_{n-1})\right)_M \\ &+ \cdots + \frac{\Delta t}{2}\left(K(\xi_n^i, s_1), e_1(s_1)\right) \\ &+ \frac{\Delta t}{2}\left(K(\xi_n^i, s_1), U_1(s_1)\right) - \frac{\Delta t}{2}\left(K(\xi_n^i, s_1), U_1(s_1)\right)_M .i \end{aligned}$$

Multiplying both sides of the above equation by h_n^i and summing up from $i=0$ to $i=M$, we have

$$\begin{aligned}\mathcal{I}_n^M u_n - U_n &= \mathcal{I}_n^M\left(\bar{K}(\xi_n^i, s_n^i), e_n(s_n^i)\right) + \sum_{i=0}^M A_n^i h_n^i \\ &\quad + \frac{\Delta t}{2}\mathcal{I}_n^M\left(K(\xi_n^i, s_{n-1}), e_{n-1}(s_{n-1})\right) + \sum_{i=0}^M A_{n-1}^i h_{n-1}^i \\ &\quad + \cdots + \frac{\Delta t}{2}\mathcal{I}_n^M\left(K(\xi_n^i, s_1), e_1(s_1)\right) + \sum_{i=0}^M A_1^i h_1^i,\end{aligned}$$

where

$$\begin{aligned}A_n^i &= \left(\bar{K}(\xi_n^i, s_n^i), U_n(s_n^i)\right) - \left(\bar{K}(\xi_n^i, s_n^i), U_n(s_n^i)\right)_M, \\ A_l^i &= \frac{\Delta t}{2}\left(K(\xi_n^i, s_l), U_l(s_l)\right) - \frac{\Delta t}{2}\left(K(\xi_n^i, s_l), U_l(s_l)\right)_M, \quad 1 \le l \le n-1.\end{aligned}$$

Thus

$$u_n - U_n = \int_{t_{n-1}}^{\xi_n^i} K(\xi_n^i, s)e_n(s)\mathrm{d}s + u_n - \mathcal{I}_n^M u_n + \sum_{l=1}^n J_l + \sum_{l=1}^n I_l + \sum_{l=1}^{n-1} R_l,$$

where

$$\begin{aligned}&J_l = \sum_{i=0}^M A_l^i h_l^i, \quad 1 \le l \le n; \quad R_l = \frac{\Delta t}{2}\left(K(\xi_n^i, s_l), e_l\right), \quad 1 \le l \le n-1, \\ &I_n = \mathcal{I}_n^M\left(\bar{K}(\xi_n^i, s_n^i), e_n(s_n^i)\right) - \left(\bar{K}(\xi_n^i, s_n^i), e_n(s_n^i)\right), \\ &I_l = \frac{\Delta t}{2}\mathcal{I}_n^M\left(K(\xi_n^i, s_l), e_l(s_l)\right) - \frac{\Delta t}{2}\left(K(\xi_n^i, s_l), e_l(s_l)\right), \quad 1 \le l \le n-1.\end{aligned}$$

By applying the Gronwall lemma, we obtain

$$\|u_n - U_n\|_\infty \le c\left(\left\|\mathcal{I}_n^M u_n - u_n\right\|_\infty + \sum_{l=1}^n \|J_l\|_\infty + \sum_{l=1}^n \|I_l\|_\infty + \sum_{l=1}^{n-1}\|R_l\|_\infty\right). \tag{4.24}$$

We now estimate the right hand side of (4.24) term by term. First, noting the inequality (3.2), we have

$$\left\|\mathcal{I}_n^M u_n - u_n\right\|_\infty \le cM^{\frac{3}{4}-m}|u_n|_{H^{m;M}}. \tag{4.25}$$

Then by using a similar technique as in the proof of Lemma 4.1, we can estimate $\|I_l\|_\infty$ and $\|J_l\|_\infty$ as follows. For $l=n$ and sufficiently large M,

$$\begin{aligned}\|J_n\|_\infty = \left\|\sum_{i=0}^M A_n^i h_n^i\right\|_\infty &\le c_1 \Delta t M^{\frac{1}{2}-m}\left(\|e_n\|_\infty + \|u_n\|_\infty\right) \\ &\le \frac{1}{3c}\|e_n\|_\infty + c\Delta t M^{\frac{1}{2}-m}\|u_n\|_\infty,\end{aligned} \tag{4.26}$$

where, as in Lemma 4.1, c_1, and therefore c, depend on $|K(\xi_n^i, s_n^i(\cdot))|_{H^{m;M}}$. Similarly, we have

$$\|I_n\|_\infty = \left\|\mathcal{I}_n^M\left(\bar{K}(\xi_n^i, s_n^i), e_n(s_n^i)\right) - \left(\bar{K}(\xi_n^i, s_n^i), e_n(s_n^i)\right)\right\|_\infty \le \frac{1}{3c}\|e_n\|_\infty. \tag{4.27}$$

For $1 \leq l \leq n-1$,

$$\|J_l\|_\infty = \left\| \sum_{i=0}^{M} A_l^i h_l^i \right\|_\infty \leq c\Delta t M^{\frac{1}{2}-m} \left(\|e_l\|_\infty + \|u_l\|_\infty \right), \tag{4.28a}$$

$$\|I_l\|_\infty = \frac{\Delta t}{2} \left\| \mathcal{I}_n^M \left(K(\xi_n^i, s_l), e_l(s_l) \right) - \left(K(\xi_n^i, s_l), e_l(s_l) \right) \right\|_\infty \leq c\Delta t \|e_l\|_\infty, \tag{4.28b}$$

$$\|R_l\|_\infty = \frac{\Delta t}{2} \left\| (K(t, s_l), e_l(s_l)) \right\|_\infty \leq c\Delta t \|e_l\|_\infty. \tag{4.28c}$$

We deduce from combining (4.24)-(4.28) that

$$\begin{aligned} \|e_n\|_\infty \leq\ & c\Big(M^{\frac{3}{4}-m} |u_n|_{H^{m;M}} + \frac{2}{3c} \|e_n\|_\infty + \Delta t M^{\frac{1}{2}-m} \|u_n\|_\infty \Big) \\ & + c\Delta t M^{\frac{1}{2}-m} \Big(\|u_{n-1}\|_\infty + \cdots + \|u_1\|_\infty \Big) \\ & + c\Delta t \Big(\|e_{n-1}\|_\infty + \cdots + \|e_1\|_\infty \Big). \end{aligned}$$

Using the discrete Gronwall lemma 3.3, we arrive at the following inequality

$$\|e_n\|_\infty \leq c \left(M^{\frac{3}{4}-m} |u|_{H^{m;M}} + n\Delta t M^{\frac{1}{2}-m} \|u\|_\infty \right) e^{c(n-1)\Delta t}. \tag{4.29}$$

Finally the lemma is proved by observing that $n\Delta t$ and $e^{c(n-1)\Delta t}$ can be bounded by a constant depending on T. □

By following the same lines as in the proof of Lemma 4.1, we can prove the following continuity of the approximation operators $\mathcal{F}_n$ and $\mathcal{G}_n$.

Lemma 4.3. *For $1 \leq n \leq N$, both operators $\mathcal{F}_n$ and $\mathcal{G}_n$ are continuous, i.e., for any two polynomial sequences $\{\psi_i\}_{i=1}^{n-1}$ and $\{\varphi_i\}_{i=1}^{n-1}$, we have*

$$\begin{aligned} & \|\mathcal{F}_n(t, \psi_{n-1}, \psi_{n-2}, \cdots, \psi_1) - \mathcal{F}_n(t, \varphi_{n-1}, \varphi_{n-2}, \cdots, \varphi_1)\|_\infty \\ & \quad \leq c\Delta t \left(\|\psi_{n-1} - \varphi_{n-1}\|_\infty + \|\psi_{n-2} - \varphi_{n-2}\|_\infty + \cdots + \|\psi_1 - \varphi_1\|_\infty \right), \end{aligned} \tag{4.30}$$

$$\begin{aligned} & \|\mathcal{G}_n(t, \psi_{n-1}, \psi_{n-2}, \cdots, \psi_1) - \mathcal{G}_n(t, \varphi_{n-1}, \varphi_{n-2}, \cdots, \varphi_1)\|_\infty \\ & \quad \leq c\Delta t \left(\|\psi_{n-1} - \varphi_{n-1}\|_\infty + \|\psi_{n-2} - \varphi_{n-2}\|_\infty + \cdots + \|\psi_1 - \varphi_1\|_\infty \right). \end{aligned} \tag{4.31}$$

We now define an operator $\tilde{\mathcal{S}}_n$. For a sequence $\{\psi_i\}_{i=1}^{n-1}$, we define $\tilde{\mathcal{S}}_n(t, \psi_{n-1}, \cdots, \psi_1)$ as the function ψ_n, which is the solution of the following problem:

$$\begin{aligned} & \psi_n(t) - \int_{t_{n-1}}^{t} K(t,s) \psi_n(s) \mathrm{d}s \\ = \ & g(t) + \frac{\Delta t}{2} \left(K(t, s_{n-1}), \psi_{n-1}(s_{n-1}) \right)_M + \cdots + \frac{\Delta t}{2} \left(K(t, s_1), \psi_1(s_1) \right)_M . \end{aligned} \tag{4.32}$$

Lemma 4.4. *For $1 \leq n \leq N$, let $\delta\mathcal{F}_n$ (resp. $\delta\mathcal{G}_n$) be the difference $\delta\mathcal{F}_n = \tilde{\mathcal{S}}_n - \mathcal{F}_n$ (resp. $\delta\mathcal{G}_n = \tilde{\mathcal{S}}_n - \mathcal{G}_n$). Then they are continuous in the sense that they satisfy, for any two sequences*

$\{\psi_i\}_{i=1}^{n-1}$ *and* $\{\varphi_i\}_{i=1}^{n-1}$,

$$\begin{aligned}(4.33)\qquad &|\delta\mathcal{F}_n(t,\psi_{n-1},\psi_{n-2},\cdots,\psi_1)-\delta\mathcal{F}_n(t,\varphi_{n-1},\varphi_{n-2},\cdots,\varphi_1)|\\ &\quad\le c\Delta t M^{\frac{3}{4}-m}\left(\|\psi_{n-1}-\varphi_{n-1}\|_\infty+\|\psi_{n-2}-\varphi_{n-2}\|_\infty+\cdots+\|\psi_1-\varphi_1\|_\infty\right),\end{aligned}$$

$$\begin{aligned}(4.34)\qquad &|\delta\mathcal{G}_n(t,\psi_{n-1},\psi_{n-2},\cdots,\psi_1)-\delta\mathcal{G}_n(t,\varphi_{n-1},\varphi_{n-2},\cdots,\varphi_1)|\\ &\quad\le c\Delta t \tilde{M}^{\frac{3}{4}-m}\left(\|\psi_{n-1}-\varphi_{n-1}\|_\infty+\|\psi_{n-2}-\varphi_{n-2}\|_\infty+\cdots+\|\psi_1-\varphi_1\|_\infty\right),\end{aligned}$$

where c *depends on* $\max_{s\in I}\|K(\cdot,s)\|_{W^{m,\infty}}$, $m\ge 1$.

Proof. As Lemma 4.1, we only need to prove the first inequality. Let

$$\begin{aligned} q_n(t)&=\mathcal{F}_n(t,\psi_{n-1},\psi_{n-2},\cdots,\psi_1),\quad p_n(t)=\mathcal{F}_n(t,\varphi_{n-1},\varphi_{n-2},\cdots,\varphi_1),\\ v_n(t)&=\tilde{\mathcal{S}}_n(t,\psi_{n-1},\psi_{n-2},\cdots,\psi_1),\quad w_n(t)=\tilde{\mathcal{S}}_n(t,\varphi_{n-1},\varphi_{n-2},\cdots,\varphi_1).\end{aligned}$$

Then

$$\delta\mathcal{F}_n(t,\psi_{n-1},\psi_{n-2},\cdots,\psi_1)-\delta\mathcal{F}_n(t,\varphi_{n-1},\varphi_{n-2},\cdots,\varphi_1)=(v_n-q_n)-(w_n-p_n).$$

Let $e_l=\psi_l-\varphi_l$. From the definitions (4.32) and (2.8), we have

$$\begin{aligned}(4.35)\qquad (v_n-w_n)(\xi_n^i)&=\left(\bar{K}(\xi_n^i,s_n^i),(v_n-w_n)(s_n^i)\right)+\frac{\Delta t}{2}\left(K(\xi_n^i,s_{n-1}),e_{n-1}(s_{n-1})\right)_M\\ &\quad+\cdots+\frac{\Delta t}{2}\left(K(\xi_n^i,s_1),e_1(s_1)\right)_M,\end{aligned}$$

$$\begin{aligned}(4.36)\qquad (q_n-p_n)(\xi_n^i)&=\left(\bar{K}(\xi_n^i,s_n^i),(q_n-p_n)(s_n^i)\right)_M+\frac{\Delta t}{2}\left(K(\xi_n^i,s_{n-1}),e_{n-1}(s_{n-1})\right)_M\\ &\quad+\cdots+\frac{\Delta t}{2}\left(K(\xi_n^i,s_1),e_1(s_1)\right)_M.\end{aligned}$$

By subtracting (4.36) from (4.35), we deduce

$$\begin{aligned}(4.37)\qquad &(v_n-w_n)(\xi_n^i)-(q_n-p_n)(\xi_n^i)\\ =\;&\left(\bar{K}(\xi_n^i,s_n^i),(v_n-w_n)(s_n^i)\right)-\left(\bar{K}(\xi_n^i,s_n^i),(q_n-p_n)(s_n^i)\right)_M.\end{aligned}$$

Let $\Delta_n=(v_n-q_n)-(w_n-p_n)$. By multiplying both sides of equation (4.37) by h_n^i and summing up from $i=0$ to $i=M$, we obtain

$$\mathcal{I}_n^M(v_n-w_n)-(q_n-p_n)=\mathcal{I}_n^M\left(\bar{K}(t,s_n^i),\Delta_n(s_n^i)\right)+\sum_{i=0}^{M}A_n^i h_n^i,$$

where

$$A_n^i=\left(\bar{K}(\xi_n^i,s_n^i),(q_n-p_n)(s_n^i)\right)-\left(\bar{K}(\xi_n^i,s_n^i),(q_n-p_n)(s_n^i)\right)_M.$$

Consequently,

$$(4.38)\qquad \Delta_n(t)=\int_{t_{n-1}}^{t}K(t,s)\Delta_n(s)\mathrm{d}s+\mathcal{I}_n^M(v_n-w_n)-(v_n-w_n)+J_n+I_n,$$

where

$$J_n=\sum_{i=0}^{M}A_n^i h_n^i,\quad I_n=\mathcal{I}_n^M\left(\bar{K}(t,s_n^i),\Delta_n(s_n^i)\right)-\left(\bar{K}(t,s_n^i),\Delta_n(s_n^i)\right).$$

Applying the standard Gronwall inequality to (4.38) gives

$$\|\Delta_n\|_\infty \leq c\Big(\big\|\mathcal{I}_n^M(v_n - w_n) - (v_n - w_n)\big\|_\infty + \|J_n\|_\infty + \|I_n\|_\infty\Big). \tag{4.39}$$

Applying inequality (3.2), we have the following estimate:

$$\begin{aligned} & \big\|\mathcal{I}_n^M(v_n - w_n) - (v_n - w_n)\big\|_\infty \\ \leq\ & cM^{\frac{3}{4}-m}|v_n - w_n|_{H^{m;M}} \leq cM^{\frac{3}{4}-m}\|v_n - w_n\|_{W^{m,\infty}}.i \end{aligned} \tag{4.40}$$

Noting that

$$\begin{aligned} v_n - w_n \ =\ & \big(\bar{K}(t,s_n), (v_n - w_n)(s_n)\big) + \frac{\Delta t}{2}\left(K(t,s_{n-1}), e_{n-1}(s_{n-1})\right)_M \\ & + \cdots + \frac{\Delta t}{2}\left(K(t,s_1), e_1(s_1)\right)_M, \end{aligned}$$

and employing Lemma 3.2, we have

$$\begin{aligned} & \|v_n - w_n\|_{W^{m,\infty}} \\ \leq\ & c\Big(\|\left(K(t,s_{n-1}), e_{n-1}(s_{n-1})\right)_M\|_{W^{m,\infty}} + \cdots + \|(K(t,s_1), e_1(s_1))_M\|_{W^{m,\infty}}\Big) \\ \leq\ & c\Delta t \max_{s\in I}\|K(\cdot,s)\|_{W^{m,\infty}}\Big(\|e_{n-1}\|_\infty + \cdots + \|e_1\|_\infty\Big) \leq c\Delta t\Big(\|e_{n-1}\|_\infty + \cdots + \|e_1\|_\infty\Big). \end{aligned} \tag{4.41}$$

Using Lemma 4.3 gives

$$\begin{aligned} \|J_n\|_\infty \ =\ & \left\|\sum_{i=0}^{M} A_n^i h_n^i\right\|_\infty \leq c\Delta t M^{\frac{1}{2}-m}\|q_n - p_n\|_\infty \\ \leq\ & c\Delta t^2 M^{\frac{1}{2}-m}\Big(\|e_{n-1}\|_\infty + \cdots + \|e_1\|_\infty\Big). \end{aligned} \tag{4.42}$$

Applying the technique used in Lemma 4.1, we obtain

$$\|I_n\|_\infty = \big\|\mathcal{I}_n^M\left(\bar{K}(t,s_n^i), \Delta_n(s_n^i)\right) - \left(\bar{K}(t,s_n^i), \Delta_n(s_n^i)\right)\big\|_\infty \leq \frac{1}{3c}\|\Delta_n\|_\infty. \tag{4.43}$$

Combining (4.39)-(4.43), we conclude

$$\|\Delta_n\|_\infty \leq c\Delta t M^{\frac{3}{4}-m}\Big(\|e_{n-1}\|_\infty + \cdots + \|e_1\|_\infty\Big).$$

The proof is then complete. □

Theorem 4.2. *(Convergence) For $1 \leq n \leq N$, let u_n be the solution of* (2.1) *and U_n^k be the solution of iteration scheme* (2.12). *If $\max_{s\in I}\|K(\cdot,s)\|_{W^{m,\infty}} \leq c$, $u \in H^m(I), m \geq 1$, then*

$$\|u_n - U_n^k\|_\infty \leq c\Big(M^{\frac{3}{4}-m} + \tilde{M}^{(\frac{3}{4}-m)(k+1)}\Big)\Big(|u|_{H^{m;M}} + \|u\|_\infty\Big), \tag{4.44}$$

where c depends on T and $\max_{s\in I}\|K(\cdot,s)\|_{W^{m,\infty}}$.

Proof. From (2.9) and (2.12), we deduce

$$
\begin{aligned}
U_n^k - U_n &= U_n^k - \mathcal{F}_n\left(t, U_{n-1}, U_{n-2}, \cdots, U_1\right) \\
&= \mathcal{G}_n\left(t, U_{n-1}^k, U_{n-2}^k, \cdots, U_1^k\right) - \mathcal{G}_n\left(t, U_{n-1}^{k-1}, U_{n-2}^{k-1}, \cdots, U_1^{k-1}\right) \\
&\quad + \mathcal{F}_n\left(t, U_{n-1}^{k-1}, U_{n-2}^{k-1}, \cdots, U_1^{k-1}\right) - \mathcal{F}_n\left(t, U_{n-1}, U_{n-2}, \cdots, U_1\right) \\
&= \mathcal{G}_n\left(t, U_{n-1}^k, U_{n-2}^k, \cdots, U_1^k\right) - \mathcal{G}_n\left(t, U_{n-1}, U_{n-2}, \cdots, U_1\right) \\
&\quad - \mathcal{G}_n\left(t, U_{n-1}^{k-1}, U_{n-2}^{k-1}, \cdots, U_1^{k-1}\right) + \mathcal{G}_n\left(t, U_{n-1}, U_{n-2}, \cdots, U_1\right) \\
&\quad + \mathcal{F}_n\left(t, U_{n-1}^{k-1}, U_{n-2}^{k-1}, \cdots, U_1^{k-1}\right) - \mathcal{F}_n\left(t, U_{n-1}, U_{n-2}, \cdots, U_1\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\left|U_n^k - U_n\right| \leq & \left|\mathcal{G}_n\left(t, U_{n-1}^k, U_{n-2}^k, \cdots, U_1^k\right) - \mathcal{G}_n\left(t, U_{n-1}, U_{n-2}, \cdots, U_1\right)\right| \\
& + \left|\delta\mathcal{G}_n\left(t, U_{n-1}^{k-1}, U_{n-2}^{k-1}, \cdots, U_1^{k-1}\right) - \delta\mathcal{G}_n(t, U_{n-1}, U_{n-2}, \cdots, U_1)\right| \\
& + \left|\delta\mathcal{F}_n\left(t, U_{n-1}^{k-1}, U_{n-2}^{k-1}, \cdots, U_1^{k-1}\right) - \delta\mathcal{F}_n(t, U_{n-1}, U_{n-2}, \cdots, U_1)\right|,
\end{aligned}
$$

where $\delta\mathcal{F}_n = \tilde{\mathcal{S}}_n - \mathcal{F}_n$ and $\delta\mathcal{G}_n = \tilde{\mathcal{S}}_n - \mathcal{G}_n$ with $\tilde{\mathcal{S}}_n$ defined by (4.32). By using Lemmas 4.3 and 4.4, we get

$$
\begin{aligned}
& \|U_n^k - U_n\|_\infty \\
\leq\ & c\Delta t\left(\|U_{n-1}^k - U_{n-1}\|_\infty + \|U_{n-2}^k - U_{n-2}\|_\infty + \cdots + \|U_1^k - U_1\|_\infty\right) \\
& + c\varepsilon\Delta t\left(\|U_{n-1}^{k-1} - U_{n-1}\|_\infty + \|U_{n-2}^{k-1} - U_{n-2}\|_\infty + \cdots + \|U_1^{k-1} - U_1\|_\infty\right),
\end{aligned}
$$

where $\varepsilon = \tilde{M}^{\frac{3}{4}-m}$, and we have used the fact that $M^{\frac{3}{4}-m} \leq \tilde{M}^{\frac{3}{4}-m}$.

Now we want to derive the following inequality

$$
\|U_n^k - U_n\|_\infty \leq \frac{c}{k!} e^{c(k+1)(n-1)\Delta t} \varepsilon^{k+1} |u|_{H^{m;M}}, \ \forall k = 0, 1, \cdots, \tag{4.45}
$$

where c is a constant independent of k. The deduction is done by induction. For $k = 1$, using Gronwall inequality and (4.29), we have

$$
\begin{aligned}
& \|U_n^1 - U_n\|_\infty \qquad (4.46) \\
\leq\ & c\Delta t(\|U_{n-1}^1 - U_{n-1}\|_\infty + \|U_{n-2}^1 - U_{n-2}\|_\infty + \cdots + \|U_1^1 - U_1\|_\infty) \\
& + c\Delta t\varepsilon(\|U_{n-1}^0 - U_{n-1}\|_\infty + \|U_{n-2}^0 - U_{n-2}\|_\infty + \cdots + \|U_1^0 - U_1\|_\infty) \\
\leq\ & c\Delta t\varepsilon e^{c(n-1)\Delta t}(\|U_{n-1}^0 - U_{n-1}\|_\infty + \|U_{n-2}^0 - U_{n-2}\|_\infty + \cdots + \|U_1^0 - U_1\|_\infty) \\
\leq\ & c\Delta t\varepsilon e^{c(n-1)\Delta t}(\|U_{n-1}^0 - u_{n-1}\|_\infty + \|U_{n-2}^0 - u_{n-2}\|_\infty + \cdots + \|U_1^0 - u_1\|_\infty) \\
& + c\Delta t\varepsilon e^{c(n-1)\Delta t}(\|u_{n-1} - U_{n-1}\|_\infty + \|u_{n-2} - U_{n-2}\|_\infty + \cdots + \|u_1 - U_1\|_\infty) \\
\leq\ & c^2\Delta t\varepsilon^2 e^{c(n-1)\Delta t}|u|_{H^{m;M}} \sum_{i=1}^{n-1} e^{c(i-1)\Delta t} \\
\leq\ & c^2\Delta t e^{c(n-1)\Delta t}\frac{e^{c(n-1)\Delta t} - 1}{e^{c\Delta t} - 1}\varepsilon^2|u|_{H^{m;M}} \leq c e^{2c(n-1)\Delta t}\varepsilon^2|u|_{H^{m;M}}.
\end{aligned}
$$

This means (4.45) holds for $k = 1$. Now we assume (4.45) holds for a given k, we want to prove that it also holds for $k+1$. In fact,

$$
\begin{aligned}
(4.47) \qquad & \|U_n^{k+1} - U_n\|_\infty \\
\le\ & c\Delta t(\|U_{n-1}^{k+1} - U_{n-1}\|_\infty + \|U_{n-2}^{k+1} - U_{n-2}\|_\infty + \cdots + \|U_1^{k+1} - U_1\|_\infty) \\
& + c\Delta t\varepsilon(\|U_{n-1}^{k} - U_{n-1}\|_\infty + \|U_{n-2}^{k} - U_{n-2}\|_\infty + \cdots + \|U_1^{k} - U_1\|_\infty) \\
\le\ & c\Delta t\varepsilon e^{c(n-1)\Delta t}(\|U_{n-1}^{k} - U_{n-1}\|_\infty + \|U_{n-2}^{k} - U_{n-2}\|_\infty + \cdots + \|U_1^{k} - U_1\|_\infty) \\
\le\ & \frac{c^2}{k!}\Delta t\varepsilon e^{c(n-1)\Delta t}\sum_{i=1}^{n-1}\varepsilon^{k+1}e^{c(k+1)(i-1)\Delta t}|u|_{H^{m;M}} \\
\le\ & \frac{c^2}{k!}\Delta t e^{c(n-1)\Delta t}\frac{e^{c(k+1)(n-1)\Delta t}-1}{e^{c(k+1)\Delta t}-1}\varepsilon^{k+2}|u|_{H^{m;M}} \\
\le\ & \frac{c}{(k+1)!}e^{c(k+2)(n-1)\Delta t}\varepsilon^{k+2}|u|_{H^{m;M}}.
\end{aligned}
$$

Thus we have proved (4.45) for all $k \ge 1$. Then as in the proof of Theorem 4.1, the boundedness of $\frac{1}{k!}e^{c(k+1)(n-1)\Delta t}$ implies

$$(4.48) \qquad \|U_n^k - U_n\|_\infty \le c\varepsilon^{k+1}|u|_{H^{m;M}} \le c\tilde{M}^{(\frac{3}{4}-m)(k+1)}|u|_{H^{m;M}},\ \forall k = 0, 1, \cdots .$$

Finally, by combining the above estimate with Lemma 4.2, we obtain

$$
\begin{aligned}
\|u_n - U_n^k\|_\infty \le\ & \|u_n - U_n\|_\infty + \|U_n^k - U_n\|_\infty \\
\le\ & c\left(\left(M^{\frac{3}{4}-m} + \tilde{M}^{(\frac{3}{4}-m)(k+1)}\right)|u|_{H^{m;M}} + n\Delta t M^{\frac{1}{2}-m}\|u\|_\infty\right), \quad 1 \le n \le N.
\end{aligned}
$$

Note that $n\Delta t \le T$, the above estimate means (4.44). This completes the proof. □

5. Numerical Results

We first give some implementation details of the parallel in time scheme (2.12). The overall algorithm is described below:

Algorithm (A1)

- Initialization ($k = 0$): $\{U_1^0(t), U_2^0(t), \cdots, U_N^0(t)\}$, given by solving $\mathcal{G}_n\left(t, U_{n-1}^0, U_{n-2}^0, \cdots, U_1^0\right)$ *successively* for $n = 1, 2, \cdots, N$.
- At step k: Suppose known $\{U_1^{k-1}(t), U_2^{k-1}(t), \cdots, U_N^{k-1}(t)\}$. For $n = 1\cdots, n$
  (1) solve the fine problem $\mathcal{F}_n\left(t, U_{n-1}^{k-1}, U_{n-2}^{k-1}, \cdots, U_1^{k-1}\right)$ *simultaneously*;
  (2) solve the coarse problem $\mathcal{G}_n\left(t, U_{n-1}^{k}, U_{n-2}^{k}, \cdots, U_1^{k}\right)$ *successively*;
  (3) $\mathcal{G}_n\left(t, U_{n-1}^{k-1}, U_{n-2}^{k-1}, \cdots, U_1^{k-1}\right)$, are available from the previous step.
- Update: Using (2.12) to update $\{U_1^k(t), U_2^k(t), \cdots, U_N^k(t)\}$.

Obviously, the most expensive calculation of this algorithm is associated to solving the fine problem $\mathcal{F}_n\left(t, U_{n-1}^{k-1}, U_{n-2}^{k-1}, \cdots, U_1^{k-1}\right)$. The parallel realization of this step is the key of the efficiency of the algorithm. Next we describe the matrix form of the linear system associated to $\mathcal{F}_n\left(t, U_{n-1}^{k-1}, U_{n-2}^{k-1}, \cdots, U_1^{k-1}\right)$, defined over the subinterval Λ_n.

Let $p_n^k = \mathcal{F}_n\left(t, U_{n-1}^{k-1}, U_{n-2}^{k-1}, \cdots, U_1^{k-1}\right)$. Then by definition (2.8), $p_n^k \in \mathcal{P}_M(\Lambda_n)$ satisfies, $\forall i = 0, 1, \cdots, M$,

$$
\begin{aligned}
(5.1) \qquad & p_n^k(\xi_n^i) - \left(\bar{K}(\xi_n^i, s_n^i), p_n^k(s_n^i)\right)_M \\
= \; & g(\xi_n^i) + \frac{\Delta t}{2}\left(K(\xi_n^i, s_{n-1}), U_{n-1}^{k-1}(s_{n-1})\right)_M + \cdots + \frac{\Delta t}{2}\left(K(\xi_n^i, s_1), U_1^{k-1}(s_1)\right)_M .
\end{aligned}
$$

Here we use the Lagrangian polynomials as a basis of the approximation space $\mathcal{P}_M(\Lambda_n)$. Let $\{h_n^i : i = 0, 1, \cdots, M\}$ be the Lagrangian polynomials associated with LG points $\{\xi_n^j : j = 0, 1, \cdots, M\}$. That is, $h_n^i \in \mathcal{P}_M(\Lambda_n)$, such that $h_n^i(\xi_n^j) = \delta_{ij}$, where δ denotes the Kronecker delta. It is seen that the set $\{h_n^i, i = 0, 1, \cdots, M\}$ forms a basis of $\mathcal{P}_M(\Lambda_n)$, i.e.,

$$
\mathcal{P}_M(\Lambda_n) = span\Big\{h_n^i, \; i = 0, 1, \cdots, M\Big\}.
$$

Thus u_N has the following expression:

$$
(5.2) \qquad p_n^k(t) = \sum_{j=0}^{M} p_n^k(\xi_n^j) h_n^j(t), \quad n = 1, \cdots, N.
$$

Plugging the above expression into (5.1) leads to, for all $i = 0, 1, \cdots, M$,

$$
\begin{aligned}
(5.3) \qquad & p_n^k(\xi_n^i) - \sum_{j=0}^{M} p_n^k(\xi_n^j)\left(\bar{K}(\xi_n^i, s_n^i), h_n^j(s_n^i)\right)_M \\
= \; & g(\xi_n^i) + \frac{\Delta t}{2}\left(K(\xi_n^i, s_{n-1}), U_{n-1}^{k-1}(s_{n-1})\right)_M + \cdots + \frac{\Delta t}{2}\left(K(\xi_n^i, s_1), U_1^{k-1}(s_1)\right)_M ,
\end{aligned}
$$

which can be rewritten under the matrix form:

$$
(5.4) \qquad (\mathbf{I} - \mathbf{A})\mathbf{p}_n^k = \mathbf{f}_n^k, \quad n = 1, \cdots, N.
$$

In (5.4), $\mathbf{I}$ denotes the identity matrix, $\mathbf{A} = (a_{ij})_{M \times M}$ with

$$
a_{ij} = \left(\bar{K}(\xi_n^i, s_n^i), h_n^j(s_n^i)\right)_M .
$$

$\mathbf{p}_n^k$ is the unknown vector of the nodal values of p_n^k at the LG points $\{\xi_n^j\}_{j=0}^M$, and $\mathbf{f}_n^k$ is the known vector, whose components are given by the right hand side of (5.3).

In our calculations that follow, the linear system (5.4) is solved by the Gauss-Seidel iterative method with the initial guess U_n^{k-1}, obtained at the previous step in the algorithm (**A1**).

The coarse problems $\mathcal{G}_n\left(t, U_{n-1}^k, U_{n-2}^k, \cdots, U_1^k\right)$ can be solved in a similar way. Note that solving the coarse problems is strictly sequential with respect to n, but relatively inexpensive.

Below we will present some numerical results obtained by the proposed parallel in time scheme.

Example 5.1. *Linear Volterra integral equation with a regular kernel:*

$$
(5.5) \qquad u(t) + \int_0^t \sin(\pi(t - \tau))u(\tau)\mathrm{d}\tau = g(t), \qquad 0 \le t \le 100.
$$

We take

$$
g(t) = (1 - 1/2\pi)\sin(\pi t) - \cos(\pi t)/2\pi
$$

such that the exact solution $u(t) = \sin(\pi t)$.

The main purpose is to investigate the convergence behavior of numerical solutions with respect to the polynomial degrees M and iteration number k. In Figure 1, we plot the L^∞-errors in semi-log scale as a function of M, with $\tilde{M}$ fixed to 13 and N fixed to 20. That is, the domain is partitioned into 20 subintervals and the coarse algorithm is solved with 13 collocation points in each sub-interval. In order to separate different error sources, the solution is iterated with sufficiently large number of k so that the error $\tilde{M}^{(\frac{3}{4}-m)(k+1)}$ (see (4.44)) is negligible as compared with the error of the fine resolution. As expected, the errors show an exponential decay, since in this semi-log representation one observes that the error variations are linear versus the degrees of polynomial M.

Next we investigate the convergence behavior with respect to the iteration number k, which is more interesting to us. For a similar reason mentioned above, we now fix a large enough $M = 25$, and let k vary for different values of $\tilde{M}$. In Figure 2, we plot the error decay with increasing iteration number k for several values of $\tilde{M}$. It is observed that the error curves are all straight lines in this semi-log representation, which means the convergence of order k with respect to the error associated to the coarse resolution. It is observed that for $\tilde{M} = 13$ only 6 iterations are required to reach the machine accuracy.

Finally the error behavior with respect to the degree of freedom $\tilde{M}$ is presented in Figure 3, where we test for two values $k = 2$ and $k = 3$ with $M = 25$. Once again the errors show an exponential decay as predicted by the estimate in (4.44).

Question: it seems $k = 2$ gives better results than $k = 3$?

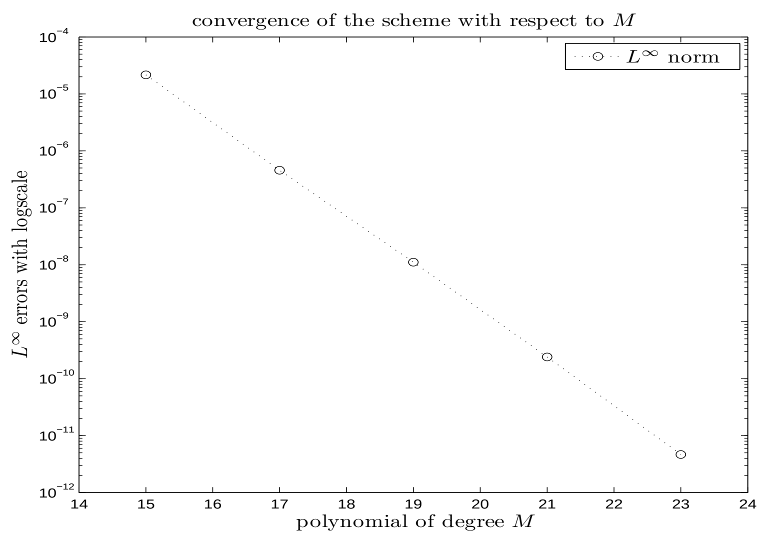

FIGURE 1. L^∞-errors versus the degree of freedom for fine approximation M with $\tilde{M} = 13, N = 20$.

6. Parallelism Efficiency

Although not yet tested in a parallel machine, the parallelism efficiency of the proposed scheme can be investigated through a cost estimate. To simplify the cost estimation, we suppose that the inter-processor communication cost in the implementation of the parallel in time scheme is

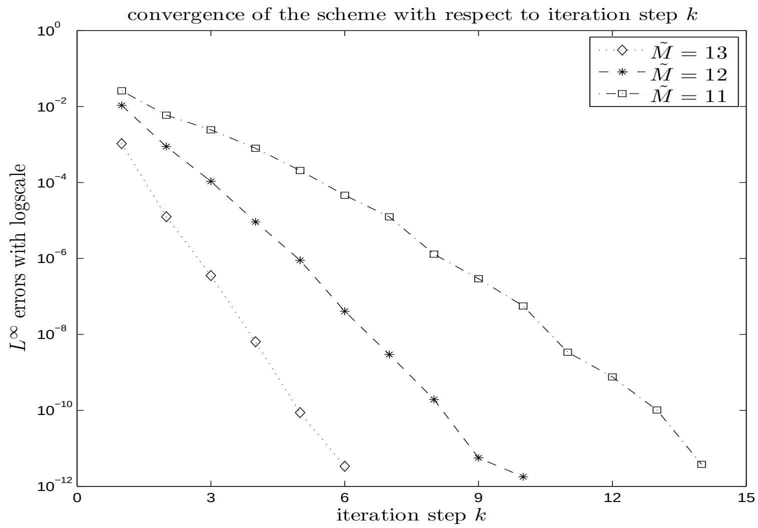

FIGURE 2. L^∞-errors versus iteration numbers k with $M = 25, N = 20$, $\tilde{M} = 11, 12, 13$.

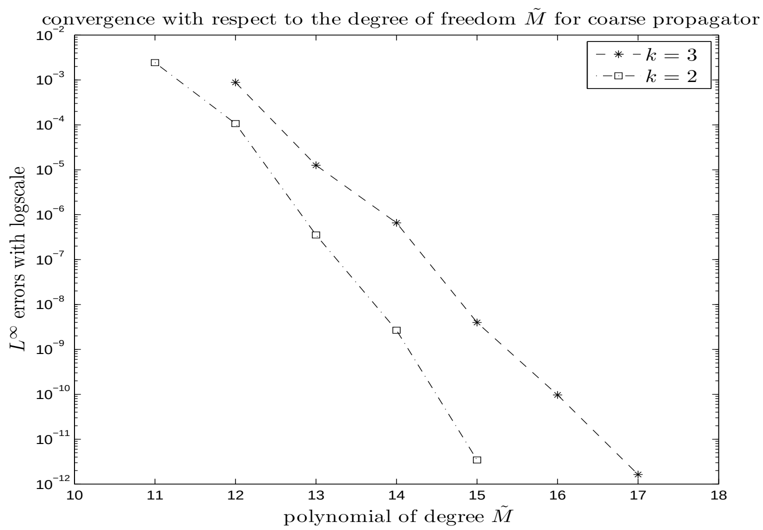

FIGURE 3. L^∞-errors versus the degree of freedom in coarse approximation $\tilde{M}$ with $M = 25$,$N = 20$ and $k = 2, 3$.

negligible as compared to the overall cost. The parallelism efficiency is demonstrated by a cost comparison between the parallel in time scheme (2.12) and the classical sequential scheme based on the corresponding fine mesh.

Firstly, the classical sequential scheme based on the fine mesh consists of solving the problems $\mathcal{F}_n(t, U_{n-1}, U_{n-2}, \cdots, U_1)$ consecutively for $n = 1, \cdots, N$. The computational complexity is equal to the sum of all the elementary operations in $\Lambda_n, n = 1, \cdots, N$. Denote the total computational cost by $\mathcal{C}_{\mathcal{F}}$ and the cost for n-th subproblem by $\mathcal{C}_{\mathcal{F}}^n$. Then

$$\mathcal{C}_{\mathcal{F}} = \sum_{n=1}^{N} \mathcal{C}_{\mathcal{F}}^n.$$

Neglecting the cost of evaluating the integral terms on the right hand sides, the spectral discretization $\mathcal{F}_n$ produces an elementary cost $\mathcal{C}_{\mathcal{F}}^n$ approximately equal to $\mathcal{O}(M^2 M)$, where $\mathcal{O}(M^2)$ is the number of operations needed for the matrix vector multiplication and $\mathcal{O}(M)$ is the estimated iteration number required to achieve the convergence of the iterative method. As a result, the total computational complexity of the sequential fine solutions is

$$\mathcal{C}_{\mathcal{F}} = \mathcal{O}(NM^3). \tag{6.6}$$

If we implement the scheme (2.12) in a parallel architecture with enough processors, then the total computational time corresponds to the cost to solve a sequential set of N coarse subproblems and a fine subproblem in a single processor. The cost of solving the sequential set of N coarse subproblems is estimated to be $\sum_{n=1}^{N} \mathcal{C}_{\mathcal{G}}^n$, where $\mathcal{C}_{\mathcal{G}}^n = \mathcal{O}(\tilde{M}^2 + M\tilde{M})$ is the cost to solve the n-th coarse subproblem. Note that here we only count the number of operations needed for the matrix vector multiplication in the coarse mesh, which is equal to $\mathcal{O}(\tilde{M}^2)$, because the Gauss-Seidel iterative method has been employed to solve the final linear system with the previous solution as the initial guess, and it is found that the convergence was achieved within a few iterations. For a same reason, the cost of solving a single fine subproblem is approximately $\mathcal{O}(M^2)$. Note also, in the implementation of the parallel in time scheme there is a need to interpolate the solution between the fine mesh and coarse mesh, the cost of which is $\mathcal{O}(NM\tilde{M})$. Therefore if K is the number of iterations required to achieve the desired convergence of the parallel in time algorithm, then the total computational complexity is

$$K\Big[\mathcal{O}(N\tilde{M}^2 + NM\tilde{M}) + \mathcal{O}(M^2) + \mathcal{O}(NM\tilde{M})\Big]. \tag{6.7}$$

Comparing (6.6) with (6.7), we obtain a speed up (i.e., the percentage used with respect the sequential scheme) close to

$$\mathcal{O}\left(K\frac{\tilde{M}^2}{M^3} + K\frac{\tilde{M}}{M^2} + \frac{K}{NM}\right).$$

It can be verified that in general case the speed up us better than $\mathcal{O}(K/M)$. In some particular cases, for example if the number of degrees of freedom for the coarse solver is far less than that of the fine solver, i.e., $\tilde{M} \ll M$ or $\tilde{M}N = M$, then the speed up by using the parallel in time algorithm would be close to $\mathcal{O}(\frac{K}{NM})$.

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