Adaptive Fourier-Galerkin Methods

C. Canuto
Dipartimento di Matematica, Politecnico di Torino, Italy
R.H. Nochetto
Department of Mathematics and Institute for Physical Science and Technology, University of Maryland, U.S.A.
M. Verani
MOX, Dipartimento di Matematica, Politecnico di Milano, Italy

The design of adaptive spectral-element (or h-p) discretization algorithms for elliptic self-adjoint problems relies on the dynamic interplay between two fundamental stages: the refinement of the geometric decomposition into elements and the enrichment of the local basis within an element. While the former stage is by now well understood in terms of practical realization and optimality properties, less theoretical attention has been devoted to the latter one.

With the aim of shedding light on this topic, we focus on what happens in a single element of the decomposition. In order to reduce at a minimum the technical burdens, we actually assume periodic boundary conditions on the d-dimensional box $\Omega = (0, 2\pi)^d$, in order to exploit the orthogonality properties of the Fourier basis. Thus, we consider a fully adaptive Fourier method, in which at each iteration of the adaptive algorithm the Galerkin solution is spanned by a dynamically selected, finite subset of the whole set of Fourier basis functions. The active set is determined by looking at a fixed fraction of largest (scaled) Fourier coefficients of the residual, according to the bulk-chasing (or Dörfler marking) philosophy. The algorithm is proven to be convergent; in addition, exploiting the concentration properties of the inverse of the elliptic operator represented in the Fourier basis, one can show that the error reduction factor per iteration tends to 0 as the bulk-chasing fraction tends to 1.

After convergence has been established, one is faced with the issue of optimality. This leads to the comparison of the adaptive Galerkin solution spanned by, say, N Fourier modes, with the best N-term approximation of the exact solution. Consequently, we are led to introduce suitable sparsity classes of periodic H^1-functions for which the best N-term approximation error fulfils a prescribed decay as N tends to infinity. These classes can also be characterized in terms of behavior of the rearranged sequence of the (normalized) Fourier coefficients of the functions.

If the best N-term approximation error of the exact solution decays at an algebraic rate (this occurs if the solution belongs to a certain "oblique" scale of Besov spaces of periodic functions, corresponding to a finite regularity), then the arguments developed by Cohen, Dahmen and DeVore and by Stevenson et al in the framework of wavelet bases apply to our situation as well, and one can establish the optimality of the approximation without coarsening. The crucial ingredients in the analysis are the minimality property of the active set of degrees of freedom determined by bulk-chasing, and a geometric-series

argument (essentially, the estimated number of degrees of freedom added at each iteration is comparable to the total number of degrees of freedom added in all previous iterations).

On the other hand, the case of a solution having (local) infinite-order or analytical regularity is quite significant in dealing with spectral (spectral-element) methods. For the Navier-Stokes equations, regularity results in Gevrey classes have been first established by Foias and Temam; in these classes, the best N-term approximation error of a function decays precisely at a sub-exponential or exponential rate. Then, the analysis of an adaptive strategy becomes much more delicate, as the previous arguments fail to apply. For instance, even for a linear operator with analytic ceofficients, it is shown that the Gevrey sparsity class of the residual is different from (actually, worse than) the one of the exact solution. Therefore, we devise and analyse a new version of the adptive algorithm considered before: since the choice of the next set of active degrees of freedom is made on the basis of the sparsity of the residual, we incorporate a coarsening step in order to get close to the sparsity pattern of the solution.