Chapter III – Combinatorics

Permutations and Combinations

The Multiplication Principle

Suppose n choices must be made, with m_1 ways to make choice 1, and for each of these ways, m_2 ways to make choice 2, and so on, with m_n ways to make choice n. Then there are $m_1 \cdot m_2 \cdot \cdots \cdot m_n$ different ways to make the entire sequence of choices.

Factorial Notation

For any natural number n,

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 .$$

Also, $0! = 1$.

Permutations

A *permutation* of r (where $r \geq 1$) elements from a set of n elements is any specific ordering or arrangement, without repetition, of the r elements. Each rearrangement of the r elements is a different permutation. The number of permutations of n things taken r at a time (with $r \leq n$) is written

$$P(n,r) \text{ or } P_r^n .$$

If $P(n,r)$ (where $r \leq n$) is the number of permutations of n elements taken r at a time, then

$$P(n,r) = \frac{n!}{(n-r)!} .$$

Remark: $P_n^n = n!$

Examples

I) Three married couples have bought six seats in a row for a performance of a musical comedy.

(a) In how many ways can they be seated?

$6! = 720$

(b) In how many ways can they be seated if each couple is to sit together with the husband to the left of his wife?

$3! = 6$

(c) In how many ways can they be seated if each couple is to sit together?

$3! \times 2! \times 2! \times 2! = 48$

(d) In how many ways can they be seated if all the men are to sit together and all the women are to sit together?

$2! \times 3! \times 3! = 72$

II) In how many ways can 8 people *A*, *B*, *C*, *D*, *E*, *F*, *G* and *H* be seated in a row if

(a) there are no restrictions on seating arrangement;

$8! = 40320$

(b) persons A and B must not sit next to each other;

$8! - 7!2! = 30240$

III) In how many ways can six coupons for free lunches at different restaurants be distributed among 10 students

(a) if none is to receive more than one coupon;

$P_6^{10} = 151200$

(b) if there is no restriction on the number of coupons that each student can receive?

$10^6 = 1000000$

Remark: The number of *r*-permutations of a set of *n* objects with repetition allowed is n^r.

Combinations

A *combination* of *r* (where $r \geq 1$) elements from a set of *n* elements is a subset of *r* elements without regard to order.

If $C(n,r)$ (or C_r^n) denotes the number of combinations of *n* elements taken *r* at a time, where $r \leq n$, then

$$C_r^n = \frac{n!}{(n-r)!r!}.$$

Remark: $C_n^n = 1$ and $C_r^n = C_{n-r}^n$.

http://www.omegamath.com/Data/d2.2.html

Example

For betting on the Mark Six draw,

(a) how many single entries can be split from an 8-number multiple entry?

$C_6^8 = 28$

(b) how many single entries can be split from a 3-banker-and-7-leg-number entry?

$C_3^7 = 35$

Remark: There are C_r^{n+r-1} combinations of *r* elements from a set of *n* elements when repetition of elements is allowed.

Example

Suppose there are 1 red ball, 1 blue ball and 1 green ball in a box. Five students are invited to come out one by one to draw a ball from the box and put it back. How many combinations of colors are possible? (Note: “GRBBR” and “RBRGB” are regarded as the same combination.)

$C_5^{3+5-1} = C_5^7 = 21$

Permutations with Indistinguishable Objects

Theorem The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2,…, and n_k indistinguishable objects of type k, is $\dfrac{n!}{n_1!n_2!\cdots n_k!}$.

Examples

I) How many strings can be made by reordering the letters of the word "daricks"?

$$P_7^7 = 7! = 5040$$

II) How many strings can be made by reordering the letters of the word "darickschan"?

$$\frac{11!}{1!\times 2!\times 1!\times 1!\times 2!\times 1!\times 1!\times 1!\times 1!} = \frac{11!}{2!2!} = 9979200$$

III) How many strings can be made by reordering the letters of the word "darickswaihongchan"?

$$\frac{18!}{3!\times 2!\times 2!\times 2!\times 2!} = 66691392768000$$

Theorem The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, $i = 1,2,\ldots,k$, equals $\dfrac{n!}{n_1!n_2!\cdots n_k!}$

Examples

I) In a class of 20 students, 5 of them will get Grade A, 10 of them Grade B, 3 of them Grade C, and 2 will be fail. How many grade distributions are possible among 20 students?

$$\frac{20!}{5!10!3!2!} = 465585120$$

II) How many ways can we distribute a standard deck of 52 playing cards into 4 sets of 13 cards each?

$$\frac{52!}{13!13!13!13!} = 53644737765488792839237440000 \quad \text{(very large!!!)}$$

The Pigeonhole Principle

Suppose that a flock of pigeons flies into a set of pigeonholes to roost. The *pigeonhole principle* states that if there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it.

Theorem [The Pigeonhole Principle] If $k+1$ or more objects are placed into k boxes, then there is at least on box containing two or more of the objects.

Proof Suppose contrary that there is at most 1 object in each box. The total number of objects in the k boxes should be less than or equal to k.

Example

If every student in a class will receive a grade from A to E, and there are 6 students in the class, then there should be at least 2 students will receive the same grade.

Theorem [The Generalized Pigeonhole Principle] If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Proof Suppose contrary that all boxes are containing at most $\lceil N/k \rceil - 1$ objects. Then, the total number of objects is at most $\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) \times k < \left(\left(\frac{N}{k} + 1\right) - 1\right) \times k < N$.

Example

I) How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?
$\left\lceil \frac{N}{4} \right\rceil \geq 3 \Rightarrow \frac{N}{4} > 2 \Rightarrow N > 8$; therefore the least N is 9.

II) How many must be selected to guarantee that at least three hearts are selected?
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III) During a month of 30 days a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Let a_i be the number of games played from day 1 to day *i*. Then $a_1, a_2, \ldots, a_{30}$ is a strictly increasing sequence and so is $a_1 + 14, a_2 + 14, \ldots, a_{30} + 14$. Clearly, $1 \le a_i \le 45$ and $15 \le a_i + 14 \le 59$. However, there are totally 60 numbers in the two sequences. Then two of them must be equal, and they are not in the same sequence. Thus, $a_i = a_j + 14$ or $a_i - a_j = 14$ for some $i > j$. This means that exactly 14 games were played from day $j+1$ to day *i*.

Recurrence Relations

A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers *n* with $n \ge n_0$, where n_0 is a nonnegative integer. A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.

Examples

I) 0, 1, 3, 7, 15, 31, …[from the Tower of Hanoi]
$a_n = 2a_{n-1} + 1$ for $n \ge 1$, with $a_0 = 0$
Solution: $\{2^n - 1\}$

II) 0, 1, 1, 2, 3, 5, 8, 13, …[Fibonacci sequence]
$a_n = a_{n-1} + a_{n-2}$ for $n \ge 2$ with $a_0 = 0$ and $a_1 = 1$
Solution: $\left\{ \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n \right\}$

Linear Homogeneous Recurrence Relations

A *linear homogeneous recurrence relation of degree k* with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \ne 0$.

Example

The recurrence relation of the terms in the Fibonacci sequence is a linear homogeneous recurrence relation of degree 2.

Characteristic Equation

The *characteristic equation* of a linear homogeneous recurrence relation of degree *k*, $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$, is defined as

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0 .$$

The roots of this equation are called *characteristics roots* of the associated recurrence relation.

Theorem r_0 is a root of the characteristic equation of a linear homogeneous recurrence relation if and only if $a_n = r_0^n$ is a solution of the recurrence relation.

Proof Assume $a_n = r_0^n$ is a solution of the recurrence relation. $r_0^n = c_1 r_0^{n-1} + c_2 r_0^{n-2} + \cdots + c_k r_0^{n-k}$ or $\left(r_0^k - c_1 r_0^{k-1} - c_2 r_0^{k-2} - \cdots - c_{k-1} r - c_k\right) r_0^{n-k} = 0$. Then $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$ as $r_0 \neq 0$. Conversely, if r_0 is a roots of the characteristic equation, i.e. $r_0^k - c_1 r_0^{k-1} + c_2 r_0^{k-2} - \cdots - c_k = 0$, then $r_0^n = c_1 r_0^{n-1} + c_2 r_0^{n-2} + \cdots + c_k r_0^{n-k}$. Hence $a_n = r_0^n$ is a solution of the recurrence relation.

Remark: The linear combinations of the *n*th power of the roots of the characteristics equation of a linear homogeneous recurrence relation is also a solution of the recurrence relation.

I.e. if $r_1, r_2, \ldots, r_k$ are *k* distinct roots of $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$, then $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$ is also a solution $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constant.

Proof Let $r_1, r_2, \ldots, r_k$ are *k* distinct roots of $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$. From the theorem above, $a_n = r_i^n$, $i = 1,2,\ldots,k$, are solutions of the recurrence relation (i.e. $r_i^n = c_1 r_i^{n-1} + c_2 r_i^{n-2} + \cdots + c_k r_i^{n-k}$). Then, $\alpha_i r_i^n = c_1 \alpha_i r_i^{n-1} + c_2 \alpha_i r_i^{n-2} + \cdots + c_k \alpha_i r_i^{n-k}$, for any constant α_i, and $\sum_{i=1}^{k} \alpha_i r_i^n = c_1 \sum_{i=1}^{k} \alpha_i r_i^{n-1} + c_2 \sum_{i=1}^{k} \alpha_i r_i^{n-2} + \cdots + c_k \sum_{i=1}^{k} \alpha_i r_i^{n-k}$.

Hence, $a_n = \sum_{i=1}^{k} \alpha_i r_i^n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$ is a solution of the recurrence relation.

Example

[Fibonacci sequence] $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$ with $a_0 = 0$ and $a_1 = 1$

The characteristic equation $r^2 - r - 1 = 0$ has two roots $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$. From the theorem above, $a_n = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^n$ is a solution of $a_n = a_{n-1} + a_{n-2}$. Since $a_0 = 0$ and $a_1 = 1$, we have $\begin{cases} 0 = \alpha_1 + \alpha_2 \\ 1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right) \end{cases}$ or $\begin{cases} \alpha_1 = \frac{1}{\sqrt{5}} \\ \alpha_2 = \frac{-1}{\sqrt{5}} \end{cases}$. Hence, $\left\{\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$ is a solution.