Multigrid Method

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1. Introduction

Multigrid techniques give algorithms that solve sparse linear systems

\[ Ax = b \]

of \( N \) unknowns with \( O(N) \) computational complexity for large classes of problems.

The systems arise from approximation of partial differential equations. The condition number is usually very large when the mesh size becomes small.
Background:

- Idea (Finite difference methods), 60’s: [Fedorenko, 1964], [Bachvalov, 1966]
- Applications, 70’s: [Brandt, 1973], etc.
- Theoretical Analysis, 80’s: [Bank and Dupont, 1981], [Braess and Hackbusch, 1983], [Bramble, Pasciak and Xu, 1987], etc.

Features:

For many iterative methods one can prove that the iterates \( \{x^i\} \) satisfy:
\[
\|x^{i+1} - x\| \leq \delta \|x^i - x\|
\]
for \( \delta \in (0, 1) \) in certain norm \( \| \cdot \| \).

- Jacobi, Gauss-Seidel: \( \delta = 1 - O(h^2) \).
- MG: \( \delta < 1 \) is independent of the mesh size \( h \).
Observation

Two solution techniques

**Iterative method**: Jacobi, Gauss-Seidel
Easy to perform (matrix $\times$ vector), but slow convergence with operations $O(N^2)$

Reason: the higher frequencies of the residual can be suppressed quickly, but lower frequencies are reduced very slowly.

After few iterations, the residual becomes smooth, but not small.

**Direct method**: Gauss elimination
Good for small-size problem
Rather difficult to perform for large-size problem, the accuracy is not high (round error accumulation)
MGM:
Combination of the above two techniques
Using different techniques on different levels of grid

1. Fine grid (smoothing)
   Iterative method–to smooth the residual

2. Coarse grid (correction)
   Direct method–to solve the residual equation

3. Add the correction to the initial residual–a more precise residual


Total operations: $O(N)$
Two types of MGM:

1. **W cycle**: two corrections per cycle (1980’)
   - Difficult to carry out, easy to analyze

2. **V cycle**: one correction per cycle (1990’)
   - Easy to carry out, difficult to analyze

**New scheme**

Cascadic scheme

No correction at all, only iterations on each level.
Very easy to carry out, difficult to analyze
Model problem:

The second order elliptic problem

\[
\begin{cases}
- \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} (a_{ij} \frac{\partial u}{\partial x_j}) = f \quad \text{in} \quad \Omega, \\
u = 0 \quad \text{on} \quad \partial \Omega,
\end{cases}
\]

The differential operator is uniformly elliptic:

\[c\xi^t \xi \leq \sum_{i,j=1}^{d} a_{ij} \xi_i \xi_j \leq C \xi^t \xi \quad \forall (x, y) \in \Omega, \xi \in R^d.\]

The variational form: find \( u \in H^1_0(\Omega) \) such that

\[a(u, v) = (f, v) \quad \forall v \in H^1_0(\Omega),\]

where the bilinear form is

\[a(u, v) = \int_{\Omega} \sum_{i,j=1}^{d} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} dx \quad \forall u, v \in H^1(\Omega).\]
Let $\Gamma_l (l \geq 0)$ denote triangulation of $\Omega$, which is generated uniformly from the initial triangulation $\Gamma_0$. The mesh size is $h_l = h_0 2^{-l}$. For instance, in 2D case, $\Gamma_l$ is obtained by linking the midpoints of three edges of triangle or the midpoints of counteredges of rectangle on $\Gamma_{l-1}$. Let $V_l \subset L^2(\Omega)$ denote the finite element space on $\Gamma_l$. On each level, we define

$$a_l(u, v) = \sum_{K \in \Gamma_l} \int_K \sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} dx$$

and

$$|v|_{i,l} = \sum_{K \in \Gamma_l} |v|_{i,K}^2, \quad i = 0, 1,$$

here $\|v\|_0 = |v|_{0,l}$.
It is easy to see that the following properties are satisfied for $V_l$:

1. $a_l(u, v) \leq C|u|_{1,l}|v|_{1,l} \quad \forall u, v \in V_l.$
2. $a_l(v, v) \geq C|v|_{1,l}^2 \quad \forall v \in V_l.$

We assume that $|v|_{1,l}$ is a norm over $V_l$.

The finite element approximation on $V_l$: Find $u_l \in V_l$ such that

$$a_l(u_l, v) = (f, v) \quad \forall v \in V_l.$$

It is known that the above equation has a unique solution $u_l \in V_l$. Define the operator $A_l : V_l \to V_l$ as follows

$$(A_l u, v) = a_l(u, v) \quad \forall u, v \in V_l.$$

Then we get a linear system

$$A_l u_l = f_l,$$

here $f_l \in V_l$ and $(f_l, v) = (f, v) \quad v \in V_l.$
2. Cascadic Multigrid Method

- Bornemann and Deuflhard (1996)
- Shaidurov (1996)
- Shi and Xu (1998)
- Braess and Dahmen (1999)
- Stevesen (2002)

Advantages:

- Simplicity
- No coarse grid corrections
- One way multigrid
Cascadic Multigrid Algorithm

(1) Set $u_0^0 = u_0^* = u_0$ and let

$$u_l^0 = I_l u_{l-1}^*.$$  

(2) For $l = 1, \ldots, L$, do iterations:

$$u_l^{m_l} = C_l^{m_l} u_l^0.$$  

(3) Set $u_l^* = u_l^{m_l}$. 

Choose the iterative operator $C_l : V_l \to V_l$ on the level $l$ and assume that there exists a linear operator $T_l : V_l \to V_l$ such that $u_l - C_l^{m_l} u_l^0 = T_l^{m_l} (u_l - u_l^0)$. 

$C_l :$ Gauss-Seidel, Jacobi, CG.
Full Multigrid Algorithm

(1) Set $u_0^0 = u_0^* = u_0$ and let
\[ u_0^l = I_l u_{l-1}^*. \]

(2) For $l = 1, \ldots, L$, do iterations:
\[ u_{l}^{m_i} = MG(l, u_{l}^{0}, r). \]

(3) Set $u_l^* = u_l^{m_i},$

where $MG(l, u_{l}^{0}, r)$ denotes to do $r$-th multigrid iterations.

Difference:

No coarse grid corrections in CMG.
Optimal Cascadic Multigrid Method:

If we obtain both the Accuracy

$$\|u_L - u^*_L\|_L \approx |u - u_L|_L$$

which means that the iterative error is comparable to the approximation error of the finite element method

and the multigrid Complexity

$$\text{amount of work} = O(n_L), \quad n_L = \text{dim}V_L.$$
General Framework: (Shi and Xu, 1998)

Three assumptions:

Define the intergrid transfer operator $I_l : V_{l-1} \rightarrow V_l$ which is assumed to satisfy the condition:

( H1)

\begin{align*}
(1). & |v - I_l v|_{t-1,l} \leq C h_l |v|_{t,l-1} \quad \forall v \in V_{l-1}, \\
(2). & |u_l - I_l u_{l-1}|_{t-1,l} \leq C h_l^2 \|f\|_0,
\end{align*}

where $u_l$ is the finite element solution on $V_l$. 
Choose the iterative operator $C_l : V_l \to V_l$ on the level $l$ and assume that there exists a linear operator $T_l : V_l \to V_l$ such that $u_l - C_l^{m_l} u_l^0 = T_l^{m_l}(u_l - u_l^0)$

( H2)

\[ \|T_l^{m_l} v\|_l \leq C \frac{h_l^{-1}}{m_l^{\gamma}} |v|_{l-1,l} \quad \forall v \in V_l, \]

\[ \|T_l^{m_l} v\|_l \leq \|v\|_l \quad \forall v \in V_l, \]

where $m_l$ is the number of iteration steps on the level $l$, and $\gamma$ is a positive number depending on the given iteration.
Introduce a projection operator $P_l : V_{l-1} + V_l \to V_l$ defined by

$$a_l(P_l u, v) = a_l(u, v) \quad \forall v \in V_l.$$  

From the definition, it is easily seen that

$$\|P_l v\|_l \leq \|v\|_{l-1} \quad \forall v \in V_{l-1}.$$  

For the operator $P_l$, we assume that

(H3) \quad \|v - P_l v\|_{t-1,l} \leq Ch_l \|v\|_{t,l-1} \quad \forall v \in V_{l-1}.$$

Note that the project operator $P_l$ is used only in the convergence analysis, it is not needed in real computation.
Note that the mesh size on the level \( l \) is
\[ h_l = h_L 2^{L-l}. \]

Let \( m_l \) (\( 0 \leq l \leq L \)) be the smallest integer satisfying
\[ m_l \geq \beta^{L-l} m_L \tag{1} \]
for some fixed \( \beta > 1 \), where \( m_L \) is the number of iterations on the finest level \( L \).

**Under the assumptions (H1), (H2) and (H3), if \( m_l \), the number of iterations on level \( l \) is given by (1), then the accuracy of the cascadic multigrid is**
\[ \| u_L - u_L^* \|_L \leq \begin{cases} C \frac{1}{1 - \left( \frac{2}{\beta} \right)^{\gamma}} m_L^{\gamma} \| f \|_0 & \text{for } \beta > 2^{\frac{1}{\gamma}}, \\ CL m_L^{\gamma} \| f \|_0 & \text{for } \beta = 2^{\frac{1}{\gamma}}. \end{cases} \]
Complexity of CMG:

The computational cost of the cascadic multigrid is proportional to

\[
\sum_{l=0}^{L} m_{l}n_{l} \leq \begin{cases} 
C \frac{1}{1 - \frac{\beta}{2^d}} m_{L}n_{L} & \text{for } \beta < 2^d, \\
CLm_{L}n_{L} & \text{for } \beta = 2^d,
\end{cases}
\]

where \(d\) is the dimension of the domain \(\Omega\).
Main Results:

Suppose (H1), (H2) and (H3) hold.

(1) If $\gamma = \frac{1}{2}$, $d = 3$, then the cascadic multigrid is optimal.

(2) If $\gamma = 1$, $d = 2, 3$, then the cascadic multigrid is optimal.

(3) If $\gamma = \frac{1}{2}$, $d = 2$, and the number of iterations on the level $L$ is

$$m_L = \lceil m_* L^2 \rceil,$$

then the error in the energy norm is

$$\|u_L - u^*_L\|_L \leq C \frac{h_L}{m_*^{\frac{1}{2}}} \|f\|_0,$$

and the complexity is

$$\sum_{l=0}^{L} m_l n_l \leq c m_* n_L (1 + \log n_L)^3.$$ 

It means that the cascadic multigrid is nearly optimal in this case.
Applications:

1. Lagrange conforming elements

In this case, $V_{l-1} \subset V_l$. Therefore, both the transfer operator $I_l$ and the projection operator $P_l$ are the identity $I$, so that (H1)-1 and (H3) are trivial. By finite element error estimates,

$$\|u_l - I_l u_{l-1}\|_0 = \|u_l - u_{l-1}\|_0 \leq \|u_l - u\|_0 + \|u - u_{l-1}\|_0 \leq Ch_l^2 \|f\|_0,$$

so (H1)-2 also holds.

(H2) has been proved by Bank and Dupont (1981)
2. Nonconforming elements

Let $V_l$ be the $P_1$ nonconforming finite element space. Define the intergrid transfer operator $I_l : V_{l-1} \rightarrow V_l$ as follows: let $m$ be the midpoint of a side of triangles in $\Gamma_l$.
(1) If $m$ lies in the interior of a triangle in $\Gamma_{l-1}$, then

$$(I_l v)(m) := v(m).$$

(2) If $m$ lies on the common edge of two adjacent triangles $T_1$ and $T_2$ in $\Gamma_{l-1}$, then

$$(I_l v)(m) := \frac{1}{2}(v|_{T_1}(m) + v|_{T_2}(m)).$$

(H1) is valid for the P1 nonconforming element.

Using a duality argument, we can show that (H3) is valid for the P1 nonconforming element.
(H2) is obvious.
Other nonconforming elements: the Wilson element, the Carey element.
3. The plate bending problem

Let $\Omega$ be a convex polygon in $\mathbb{R}^2$, the variational form of the plate bending problem is to find $u \in H^2_0(\Omega)$ such that

$$a(u, v) = (f, v) \quad \forall v \in H^2_0(\Omega),$$

$$a(u, v) = \int_{\Omega} \{ \triangle u \triangle v + (1 - \sigma)(2\frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 v}{\partial x_1 \partial x_2} - \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 v}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_2^2} \frac{\partial^2 v}{\partial x_1^2}) \} dx,$$

$$(f, v) = \int_{\Omega} f v dx,$$

where $f \in L^2(\Omega)$, and $0 < \sigma < \frac{1}{2}$ is the Possion ratio. In this case $V = H^2_0(\Omega)$. 
Morley element

The Morley finite element space is nonconforming and nonnested. We define the intergrid transfer operator as follows:

1. $I_l v = 0$ at the vertex of $\Gamma_l$ along $\partial \Omega$ and $\frac{\partial I_l v}{\partial n} = 0$ at the midpoint of each edge of $\Gamma_l$ along $\partial \Omega$.

2. Let $p$ be a vertex of $\Gamma_l$ inside $\Omega$. If $p$ is also a vertex of $\Gamma_{l-1}$, then $(I_l v)(p) = v(p)$. If $p$ is the midpoint of a common edge of two triangles $K_1$ and $K_2$ in $\Gamma_{l-1}$, then

$$
(I_l v)(p) = \frac{1}{2}(v|_{K_1}(p) + v|_{K_2}(p)).
$$
(3) Let $m$ be a midpoint of the edge $e$ of $\Gamma_l$ inside $\Omega$ and $n$ denote the unit normal of $e$. If $m$ is inside of a triangle in $\Gamma_{l-1}$, then

$$\frac{\partial I_l v}{\partial n}(m) = \frac{\partial v}{\partial n}(m).$$

If $m$ is on the common edge of two triangles $K_1$ and $K_2$ in $\Gamma_{l-1}$, then

$$\frac{\partial I_l v}{\partial n}(m) = \frac{1}{2} \left[ \frac{\partial v|_{K_1}}{\partial n}(m) + \frac{\partial v|_{K_2}}{\partial n}(m) \right].$$

We can prove (Shi and Xu 1998)

(H1)-(H3) hold for the Morley element.
3. Economical Cascadic Multigrid Method

Advantages:

- Less operations on the each level.
- Computational costs can be greatly reduced.
Example:

CMG:

\[ m_l = [m_L \beta^{L-l}] \quad l = 1, \ldots, L, \]

ECMG:

a new criterion for choosing smoothing steps on each level. When \( d = 2 \), that is:

(i) If \( l > L_0 \), then

\[ m_l = [m_L \beta^{L-l}] \]

(ii) If \( l \leq L_0 \), then

\[ m_l = [m_*^{\frac{1}{2}} (L - (2 - \varepsilon_0)l) h_l^{-2}] \]

here \( L_0 \) is a positive integer which will be determined later, and \( 0 < \varepsilon_0 \leq 1 \) is a fixed positive number, \( m_L = m_0(L - L_0)^2 \). Note that in the standard cascadic algorithm \( m_L = m_0L^2 \).
Consider the second order elliptic problem and use Gauss-Seidel iteration as a smoother. For simplicity, we only consider 2D case. Then $L_0 = L/2 = 4$, $\varepsilon_0 = \frac{1}{2}$, $h_0 = \frac{1}{4}$, and $m_0 = 2$, $m_* = 1$. The following table shows the iteration steps on each level for the two methods.

<table>
<thead>
<tr>
<th>Level</th>
<th>8</th>
<th>7</th>
<th>6</th>
<th>5</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>CMG</td>
<td>128</td>
<td>512</td>
<td>2048</td>
<td>8192</td>
<td>32768</td>
<td>131072</td>
<td>524288</td>
<td>2097152</td>
</tr>
<tr>
<td>ECMG</td>
<td>32</td>
<td>128</td>
<td>512</td>
<td>2048</td>
<td>1024</td>
<td>896</td>
<td>320</td>
<td>88</td>
</tr>
</tbody>
</table>

We will prove that the new cascadic multigrid algorithm is still optimal in both accuracy and complexity as the standard CMG.
Basic smoothers:

For the basic iterative smoothers, we have

\[ \|T_l^{m_l}v\|_l \leq \epsilon(m_l, h_l)\|v\|_0 \quad \forall v \in V_l, \]

where

\[ \epsilon(m_l, h_l) = \begin{cases} 
    C \frac{h_l^{-1}}{m_l^{\frac{1}{2}}}, & \text{for } m_l < \kappa_l, \\
    C 2^{-m_l \frac{1}{\kappa_l}}, & \text{for } m_l \geq \kappa_l,
\end{cases} \]

where \( \kappa_l \) denotes the condition number of the matrix \( A_l \).
New criteria for choosing iteration parameters

Determine the largest positive integer $L_0$, which satisfies the following inequality

$$\beta^{L-L_0}m_L \geq \kappa L_0.$$ 

For simplicity, we assume that $\kappa = h_i^{-2}$, then by a simple manipulation,

$$L_0 \leq \frac{L \log \beta + \log m_L + 2 \log h_0}{\log \beta + 2 \log 2}.$$ 

Define the level parameter $L_0$ as the largest positive integer, which satisfies the following inequality:

$$L_0 \leq \min\left\{ \frac{L \log \beta + \log m_L + 2 \log h_0}{\log \beta + 2 \log 2}, \frac{dL}{2 + d} \right\}. \quad (2)$$
New criteria:

1. When \( d = 2 \),

   (i) If \( l > L_0 \), then

   \[
   m_l = [m_L \beta^{L-l}].
   \]

   (ii) If \( l \leq L_0 \), then

   \[
   m_l = [m_1 \left( L - (2 - \varepsilon_0)l \right) \kappa_l].
   \]

In practical implementation, because \( \kappa_l \approx h_l^{-2} \), the above terms can be replaced by:

\[
 m_l = [m_1^{\frac{1}{2}} (L - (2 - \varepsilon_0)l) h_l^{-2}].
\]
2. When $d = 3$,

(i) If $l > L_0$, then

$$m_l = [m_L \beta^{L-l}].$$

(ii) If $l \leq L_0$, there are two cases:

(a) If $(2 - \varepsilon_0) L_0 \leq L$, then

$$m_l = [m_1^2 (L - (2 - \varepsilon_0) l) \kappa_l].$$

(b) If $(2 - \varepsilon_0) L_0 > L$, then there exists a largest positive integer $L' < L_0$ such that $(2 - \varepsilon_0) L' \leq L$. For all $l \leq L'$, we choose $m_l$ as follows:

$$m_l = [m_*^2 (L - (2 - \varepsilon_0) l) \kappa_l],$$

for all $L' < l \leq L_0$, we choose $m_l$ as follows:

$$m_l = [m_*^2 \kappa_l].$$
Economical Cascadic Multigrid

(1) Set $u_0^0 = u_0^* = u_0$ and let
$$u_l^0 = I_l u_{l-1}^*.$$

(2) For $l = 1, \ldots, L$, do
$$u_l^{m_l} = C_l^{m_l} u_l^0.$$

(3) Set $u_l^* = u_l^{m_l}$,
where the $m_l$ is determined by the new criteria.
Main Results:

Suppose (H1) and (H3) hold.

(1) If \( d = 3 \), then the cascadic multigrid is optimal, i.e., the error in the energy norm is

\[
\| u_L - u^*_L \|_L \leq C h_L \| f \|_0 (\frac{h_0^{1/2}}{m_L^{1/2}} + \frac{h_0^{1/2}}{m_*^{1/2}})
\]

and the complexity is

\[
\sum_{l=0}^{L} m_l n_l \leq C \left( h_0^{-2} m_* + \frac{1}{1 - \frac{\beta}{2^d}} \right) m_L n_L.
\]
(2) If $d = 2$, and the number of iterations on the level $L$ is

$$m_L = \lfloor m_0(L - L_0)^2 \rfloor,$$

then the error in the energy norm is

$$\| u_L - u^*_L \|_L \leq C h_L \| f \|_0 (\frac{h_0}{m_*^\frac{3}{2}} + \frac{h_0}{m_0^\frac{3}{2}})$$

and the complexity of computation is

$$\sum_{l=0}^{L} m_l n_l \leq C (m_* h_0^{-2} + m_0 (L - L_0)^3) n_L.$$

It means that the cascadic multigrid is nearly optimal in this case.
**CG method as a smoother:**

**Main results:**

Suppose (H1) and (H2) hold. For $d = 2, 3$, the cascadic multigrid with CG smoother is optimal, i.e., the error in the energy norm is

$$
\| u_L - u_L^* \|_L \leq C h_L \| f \|_0 \left( \frac{h_0}{m_L} + \frac{h_0}{m_*} \right)
$$

and the complexity of computation is

$$
\sum_{l=0}^{L} m_l n_l \leq C (h_0^{-2} + \frac{1}{1 - \frac{\beta}{2^d}}) n_L.
$$

It means that the cascadic multigrid is optimal in this case.
Applications:

- Conforming and Nonconforming elements for the second order problem.
- Nonconforming plate elements, Morley element, Adini element.
Numerical experiments:

We use ECMG and standard CMG algorithms to solve the following problem:

\[
\begin{align*}
-\Delta u &= f, \quad \text{in } \Omega = (-1, 1) \times (-1, 1), \\
u &= 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( f \) is chosen such that the exact solution of the problem is \( u(x, y) = (1 - x^2)(1 - y^2) \).
ECMG and CMG with GS smoother (P1 conforming element)

<table>
<thead>
<tr>
<th>Mesh</th>
<th>CMG</th>
<th></th>
<th>ECMG</th>
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</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Energy error</td>
<td>CPU</td>
<td>Energy error</td>
<td>CPU</td>
</tr>
<tr>
<td>$512 \times 512$</td>
<td>$7.68851949e-03$</td>
<td>38(s)</td>
<td>$7.76912806e-03$</td>
<td>20(s)</td>
</tr>
<tr>
<td>$1024 \times 1024$</td>
<td>$3.82012827e-03$</td>
<td>323(s)</td>
<td>$3.83635112e-03$</td>
<td>101(s)</td>
</tr>
<tr>
<td>$2048 \times 2048$</td>
<td>$1.90676237e-03$</td>
<td>1420(s)</td>
<td>$1.91763892e-03$</td>
<td>446(s)</td>
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</tbody>
</table>
ECMG and CMG with CG smoother (P1 conforming element)

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<thead>
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<th>Mesh</th>
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<th>ECMG</th>
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<tbody>
<tr>
<td></td>
<td>Energy error</td>
<td>Energy error</td>
</tr>
<tr>
<td>512 × 512</td>
<td>7.83833771e-03</td>
<td>7.87421203e-03</td>
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<tr>
<td>1024 × 1024</td>
<td>3.95663358e-03</td>
<td>3.96542864e-03</td>
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<tr>
<td>2048 × 2048</td>
<td>1.99369401e-03</td>
<td>1.99632148e-03</td>
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## ECMG and CMG with GS smoother (P1 nonconforming element)

<table>
<thead>
<tr>
<th>Mesh</th>
<th>CMG</th>
<th>ECMG</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Energy error</td>
<td>CPU</td>
</tr>
<tr>
<td>256 × 256</td>
<td>1.18506418e-02</td>
<td>40(s)</td>
</tr>
<tr>
<td>512 × 512</td>
<td>5.91793596e-03</td>
<td>294(s)</td>
</tr>
<tr>
<td>1024 × 1024</td>
<td>2.95883706e-03</td>
<td>1382(s)</td>
</tr>
</tbody>
</table>
ECMG and CMG with CG smoother (P1 nonconforming element)

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<thead>
<tr>
<th>Mesh</th>
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<th>ECMG</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Energy error</td>
<td>CPU</td>
</tr>
<tr>
<td>256 × 256</td>
<td>1.18435038e-02</td>
<td>18(s)</td>
</tr>
<tr>
<td>512 × 512</td>
<td>5.91886995e-03</td>
<td>72(s)</td>
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<tr>
<td>1024 × 1024</td>
<td>2.95842421e-03</td>
<td>447(s)</td>
</tr>
</tbody>
</table>
4. Summary

- Simplicity
- No coarse grid corrections, one way multigrid
- ECMG: less operations on the each level.
- ECMG: computational costs can be greatly reduced.
5. Appendix

Efficiency—Computer vs Algorithm

- **Computer**
  - 1950 → 2000 from $10^3 (Kilo) \rightarrow 10^{12} (Tera)$, \textbf{increase} $10^9 (Giga)$

- **Algorithm** (solution of linear system)
  - 1950 (Gauss elimination) → 2000 (multigrid method)
  - Convergence rate: from $O(N^3) \rightarrow O(N)$, \textbf{increase} $O(N^2)$

- **3D problem:** $N = (10^2)^3 (Mega)$
  - Operations: $10^{18} \rightarrow 10^6$, \textbf{reduce} $10^{12} (Tera)$ !!
THANK YOU!

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