Chapter P: Preliminaries

Winter 2016

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The preliminary chapter reviews the most important things that you should know before beginning calculus.

Depending on your pre-calculus background, you may or may not be familiar with the topics in the preliminaries chapter.

1. If you are, you may want to skim over these materials to refresh your understanding of the terms used;

2. If not, you should study this chapter in detail.
§P.1 Real Numbers and the Real Line

- **Real numbers** are numbers that can be expressed as decimals, for example,

\[
\begin{align*}
5 &= 5.00000... \\
\frac{4}{3} &= 1.33333...
\end{align*}
\]

\[
\begin{align*}
\sqrt{2} &= 1.41421... \\
\pi &= 3.14159... \\
e &= 2.71828...
\end{align*}
\]

- The real numbers can be represented geometrically as points on the **real line**, denoted by \( \mathbb{R} \).
Some important special subsets of real numbers are:

(i) **natural numbers** (or **positive integers**): the numbers 1, 2, 3, 4, 5, ....

(ii) **integers**: the numbers 0, ±1, ±2, ±3, ±4, ±5, ....

(iii) **rational numbers**: the numbers that can be expressed in the form of a fraction $m/n$, where $m$ and $n$ are integers and $n \neq 0$.

(iv) **irrational numbers**: the real numbers that are not rational.

**Example**: $\sqrt{2}$ is an irrational number. *Why?*
Intervals

Let $a$ and $b$ be two real numbers and $a < b$. Then

(i) $(a, b)$: the **open interval** from $a$ to $b$, consisting of all real numbers $x$ satisfying $a < x < b$.

(ii) $[a, b]$: the **closed interval** from $a$ to $b$, consisting of all real numbers $x$ satisfying $a \leq x \leq b$.

(iii) $[a, b)$: the **half-open interval** from $a$ to $b$, consisting of all real numbers $x$ satisfying $a \leq x < b$.

(iv) $(a, b]$: the **half-open interval** from $a$ to $b$, consisting of all real numbers $x$ satisfying $a < x \leq b$.

**Remark:** Note that the whole real line is also an interval, denoted by $\mathbb{R} = (-\infty, \infty)$. The symbol $\infty$ is called “infinity”. 
The **absolute value**, or **magnitude**, of a number $x$, denoted by $|x|$, is defined by the formula

$$
|x| = \begin{cases} 
    x, & \text{if } x \geq 0 \\
    -x, & \text{if } x < 0.
\end{cases}
$$

Properties of absolute values include:

1. $|-a| = |a|$.
2. $|ab| = |a| \cdot |b|$ and $|a/b| = |a|/|b|$.
3. $|a \pm b| \leq |a| + |b|$.
4. If $D$ is a positive number, then
   $$
   |x - a| < D \iff a - D < x < a + D. \\
   |x - a| > D \iff \text{either } x < a - D \text{ or } x > a + D.
   $$
§P.2 Cartesian Coordinates in the Plane

- The positions of all points in a plane can be measured with respect to two perpendicular real lines (referred to as \textit{x-axis} and \textit{y-axis}) in the plane intersecting at the 0-point of each.

- The point of intersection of the x-axis and the y-axis is called the \textit{origin} and is often denoted by the letter $O$.

- For a point $P(a, b)$, we call $a$ the \textit{x-coordinate} of $P$, and $b$ the \textit{y-coordinate} of $P$. The \textbf{ordered pair} $(a, b)$ is called the \textbf{Cartesian coordinates} of the point $P$.

- The coordinate axes divide the plane into four regions called \textbf{quadrants}, numbered from I to IV.
Figure P.8  The coordinate axes and the point \( P \) with coordinates \( (a, b) \)
Figure P.10  The four quadrants
Figure P.12  The distance from $P$ to $Q$ is

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
Figure P.14  The parabola $y = x^2$
(a) $x^2 + y^2 = 4$, and (b) $x^2 + y^2 \leq 4$
Given two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the plane, there is a unique **straight line** passing through them both. We call the line $L = P_1P_2$.

For any nonvertical line $L$, we define the **slope** of the line is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \tan(\phi),$$

where the angle $\phi$ is the **inclination** of the line $L$. We let $0^\circ \leq \phi < 180^\circ$ (or $0 \leq \phi < \pi$).
Figure P.16  Line $L$ has inclination $\phi$
Equations of Straight Lines

- The **point-slope equation** of the line that passes through the point \((x_1, y_1)\) and has slope \(m\) is

\[
y = m(x - x_1) + y_1.
\]
Equations of Straight Lines

- The **point-slope equation** of the line that passes through the point \((x_1, y_1)\) and has slope \(m\) is
  \[ y = m(x - x_1) + y_1. \]

- The **point-point equation** of the line that passes through the point \((x_1, y_1)\) and \((x_2, y_2)\) is
  \[ y = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1. \]
Equations of Straight Lines

- The **point-slope equation** of the line that passes through the point \((x_1, y_1)\) and has slope \(m\) is
  \[ y = m(x - x_1) + y_1. \]

- The **point-point equation** of the line that passes through the point \((x_1, y_1)\) and \((x_2, y_2)\) is
  \[ y = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1. \]

- The **horizontal line** passing through the point \((x_1, y_1)\) is
  \[ y = y_1. \]

The **vertical line** passing through the point \((x_1, y_1)\) is
\[ x = x_1. \]
Figure P.19  Line $L$ has $x$-intercept $a$ and $y$-intercept $b$
For a nonhorizontal and nonvertical line, let $a$ be the x-intercept and $b$ be the y-intercept. Then

- The **slope–y-intercept equation** of the line with slope $m$ and y-intercept $b$ is
  
  $$y = mx + b.$$
For a nonhorizontal and nonvertical line, let $a$ be the x-intercept and $b$ be the y-intercept. Then

- The **slope–y-intercept equation** of the line with slope $m$ and y-intercept $b$ is
  \[ y = mx + b. \]

- The **slope–x-intercept equation** of the line with slope $m$ and x-intercept $a$ is
  \[ y = m(x - a). \]
Example 1:

Find the slope and the two intercepts of the line with equation $8x + 5y = 20$.

**Solution**: Solving the equation for $y$, we have

$$y = -\frac{8}{5}x + 4.$$ 

This shows that the slope of the line is $m = -8/5$, and the $y$-intercept is $b = 4$.

Second, to find the $x$-intercept, we represent the equation as

$$y = -\frac{8}{5}(x - \frac{5}{2}).$$

Hence, the $x$-intercept is $a = 5/2$. 
The term **function** was first used by Leibniz in 1673 to denote the dependence of one quantity on another.

For instance, the area $A$ of a circle depends on the radius $r$ according to the formula

$$A = \pi r^2.$$

We can say that the area (as the dependent variable) is a function of the radius (as the independent variable).

As a generic function, we often refer to $y$ as the **dependent variable** and $x$ as the **independent variable**. Then to denote that $y$ is a function of $x$, we write

$$y = f(x).$$
A function $f$ on a set $D$ into a set $S$ is a rule that assigns a unique element $f(x)$ in $S$ to each element $x$ in $D$. We write the function as

$$y = f(x).$$

Remark:

(1) $D = \mathcal{D}(f)$ is the **domain** of the function $f$: *the set of all possible input values for the independent variable.*

(2) The **range** of $f$, $\mathcal{R}(f)$, is the subset of $S$: *the set of all possible output values for the dependent variable.*
A function can be treated as a kind of machine.

Figure P.35  A function machine
Graphs of Functions

For any given pair of \((x, f(x))\), we can find a unique point in the \(x\)-\(y\) plane to represent it. That is, the function \(y = f(x)\) can be represented by a curve in Cartesian coordinates system.
Example 2

Find the domain and range of function:

\[ y = f(x) = 2x^2 + 1, \]

where \( x \in [1, 3]. \)

Solution:

(1) The domain of \( f \) is given as

\[ D(f) = [1, 3]. \]

(2) The range of \( f \) is

\[ R(f) = [3, 19]. \]
Example 3

Find the domain and the range of function:

\[ y = f(x) = \sqrt{9 - x^2} + 1. \]

Solution:

(1) The domain of \( f \) is

\[ \mathcal{D}(f) = [-3, 3]. \]

(2) The range of \( f \) is

\[ \mathcal{R}(f) = [1, 4]. \]
Example 4

Find the range of function \( y = \frac{x}{x^2 + 1} \), where \( x \in \mathbb{R} \).

**Solution:**

For any \( x \in \mathbb{R} \), we can find \( y \) by the relation of

\[
    y = \frac{x}{x^2 + 1},
\]

or equivalently, by \( yx^2 - x + y = 0 \). Now since \( x \) is a real number, we have \( \Delta = 1 - 4y^2 \geq 0 \). This leads to the range of \( y \) as

\[
    \mathcal{R}(y) = [-0.5, 0.5].
\]
Example 5

Find the range of function $y = \frac{x}{x^4 + 1}$, where $x \in \mathbb{R}$.

Solution:

To find the range of $y$, we need to learn Calculus. More specifically, the differentiation in Chapter 2 will help us to solve this problem. Before starting differentiation, however, we need to first introduce the limits and continuity of a function in Chapter 1.
Even and Odd Functions

Definition

Suppose that $-x$ belongs to the domain of $f$ whenever $x$ does.

1. We say that $f$ is an **even function** if

\[ f(-x) = f(x) \quad \text{for every } x \text{ in the domain of } f. \]

2. We say that $f$ is an **odd function** if

\[ f(-x) = -f(x) \quad \text{for every } x \text{ in the domain of } f. \]

Remark:

1. The graph of an even function is *symmetric about the y-axis*.
2. The graph of an odd function is *symmetric about the origin*.
§P.5 Combining Functions to Make New Functions

Functions can be combined in a variety of ways to produce new functions.

- Like numbers, functions can be added, subtracted, multiplied, and divided (when the denominator is nonzero) to produce new functions.

- There is another method, called composition, by which two functions can be combined to form a new function.

- We can also define new functions by using different formulas on different parts of its domain, referred to as “piecewise defined functions”.
Sums, Differences, Products, Quotients, and Multiples

**Definition**

Let $f$ and $g$ be two functions and $c$ is a real number. Then for every $x$ that belongs to the domains of both $f$ and $g$, we define functions $f + g$, $f - g$, $f \cdot g$, $f/g$, and $cf$ by the formulas:

\[
(f + g)(x) = f(x) + g(x), \\
(f - g)(x) = f(x) - g(x), \\
(fg)(x) = f(x)g(x), \\
\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad \text{where } g(x) \neq 0, \\
(cf)(x) = c \cdot f(x).
\]
Example 6

The functions $f$ and $g$ are defined by the formulas

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = \sqrt{2 - x}.$$  

Find the new functions $f + g$, $f - g$, $fg$, $f/g$, $g/f$ and $f^4$ at $x$. Also specify the domain of each function.

Solution:
Example 6

The functions $f$ and $g$ are defined by the formulas

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = \sqrt{2 - x}.$$  

Find the new functions $f + g$, $f - g$, $fg$, $f/g$, $g/f$ and $f^4$ at $x$. Also specify the domain of each function.

Solution:

1. $(f + g)(x) = \sqrt{x} + \sqrt{2 - x}, \quad D(f + g) = [0, 2]$.
2. $(f - g)(x) = \sqrt{x} - \sqrt{2 - x}, \quad D(f - g) = [0, 2]$.
3. $(fg)(x) = \sqrt{x(2 - x)}, \quad D(fg) = [0, 2]$.
4. $(f/g)(x) = \sqrt{x/(2 - x)}, \quad D(f/g) = [0, 2)$.
5. $(g/f)(x) = \sqrt{(2 - x)/x}, \quad D(g/f) = (0, 2]$.
6. $(f^4)(x) = x^2, \quad D(f^4) = [0, \infty)$.
**Composite Functions**

**Definition**

Let $f$ and $g$ be two functions. The *composite* function of $f \circ g$ is defined by

$$f \circ g(x) = f(g(x)).$$

The domain of $f \circ g$ consists of those numbers $x$ in the domain of $g$ for which $g(x)$ is the domain of $f$.

**Remark:**

(1) If the range of $g$ is contained in the domain of $f$, then the domain of $f \circ g$ is just the domain of $g$.

(2) $g$ is called the *inner* function and $f$ the *outer* function. To calculate $f(g(x))$, we first calculate $g$ and then calculate $f$.

(3) The functions $f \circ g$ and $g \circ f$ are usually quite different.
The output of $g$ becomes the input of $f$.

Figure P.60  $f \circ g(x) = f(g(x))$
Example 4

Let \( f(x) = x^2 + 1 \) and \( g(x) = \sqrt{x - 2} \).

(a) Find the composite function \( f \circ g \) and identify its domain;

(b) Find the composite function \( g \circ f \) and identify its domain.

Solution:
Example 4

Let \( f(x) = x^2 + 1 \) and \( g(x) = \sqrt{x - 2} \).

(a) Find the composite function \( f \circ g \) and identify its domain;
(b) Find the composite function \( g \circ f \) and identify its domain.

Solution:

(a) For the first composition, we have

\[(f \circ g)(x) = f(\sqrt{x - 2}) = (\sqrt{x - 2})^2 + 1 = x - 1.\]

The domain of \( f \circ g \) is \( \mathcal{D}(f \circ g) = [2, \infty) \).
Example 4

Let \( f(x) = x^2 + 1 \) and \( g(x) = \sqrt{x - 2} \).

(a) Find the composite function \( f \circ g \) and identify its domain;

(b) Find the composite function \( g \circ f \) and identify its domain.

Solution:

(a) For the first composition, we have

\[
(f \circ g)(x) = f(\sqrt{x - 2}) = (\sqrt{x - 2})^2 + 1 = x - 1.
\]

The domain of \( f \circ g \) is \( \mathcal{D}(f \circ g) = [2, \infty) \).

(b) For the second composition, we have

\[
(g \circ f)(x) = g(x^2 + 1) = \sqrt{x^2 - 1}.
\]

The domain of \( g \circ f \) is \( \mathcal{D}(g \circ f) = (-\infty, -1] \cup [1, \infty) \).
Piecewise Defined Functions

(1) One example is the absolute value function:

\[
|x| = \begin{cases} 
  x, & \text{if } x \geq 0 \\
  -x, & \text{if } x < 0
\end{cases}
\]

(2) Another example is the sign function:

\[
\text{sgn}(x) = \frac{x}{|x|} = \begin{cases} 
  1, & \text{if } x > 0 \\
  -1, & \text{if } x < 0 \\
  \text{undefined}, & \text{if } x = 0
\end{cases}
\]
§P.6 Polynomials and Rational Functions

**Definition**

A **polynomial** is a function $P$ whose value at $x$ is

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_n, a_{n-1}, \ldots, a_1$ and $a_0$, called the **coefficients** of the polynomial, are constants and, if $n > 0$, then $a_n \neq 0$.

**Remark:** The number $n$, the degree of the highest power of $x$ in the polynomial, is called the **degree** of the polynomial.
**Definition**

If \( P(x) \) and \( Q(x) \) are two polynomials and \( Q(x) \) is not the zero polynomial (i.e., \( Q(x) \equiv 0 \)), then the function

\[
R(x) = \frac{P(x)}{Q(x)}
\]

is called a **rational function**.

**Remark**: By the domain convention, the domain of \( R(x) \) consists of all real numbers \( x \) except those for which \( Q(x) = 0 \).
The Quadratic Formula

**Theorem**

The two solutions of the quadratic equation

\[ Ax^2 + Bx + C = 0 \]

where \( A, B, \) and \( C \) are constants and \( A \neq 0 \), are given by

\[ x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}. \]

**Remark:** The quantity \( \Delta = B^2 - 4AC \) is called the discriminant of the quadratic equation or polynomial.
Basic Elementary Functions

The following functions are commonly used in Calculus and are the so-called basic elementary functions:

1) **Power functions**: $x^a$  
   (a is a real number)

2) **Exponential functions**: $a^x$  
   (a is a positive real number, one important value of $a$ is $e = 2.71828$)

3) **Logarithmic functions**: $\log_a(x)$  
   (a is a positive real number, with $\log_e(x)$ denoted by $\ln(x)$)

4) **Trigonometric functions**: $\sin(x), \cos(x), \tan(x), \cot(x), \sec(x), \csc(x)$

5) **Inverse Trigonometric functions**: $\arcsin(x), \arccos(x), \arctan(x), \arccot(x), \text{arcsec}(x), \text{arccsc}(x)$
Elementary Functions

Definition

An **elementary function** is a function of one variable built from a finite number of basic elementary functions and constants through ‘composition’ and ‘combinations’ using the four elementary operations (+, −, ×, /), where

\[
(f + g)(x) = f(x) + g(x),
\]
\[
(f - g)(x) = f(x) - g(x),
\]
\[
(f \cdot g)(x) = f(x) \cdot g(x),
\]
\[
(f/g)(x) = f(x)/g(x) \quad \text{where } g(x) \neq 0.
\]
Examples of elementary functions include:

- \( f(x) = \ln(\sin(2x^2 + 1)) + \sqrt{e^x + 1} \)
- The polynomial: \( P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \)
- The rational function: \( P(x)/Q(x) \)
- The absolute value function: \( f(x) = |x| \)
Not all functions are elementary. Examples of non-elementary functions include:

- **The Dirichlet function:**
  \[
  f(x) = \begin{cases} 
  0 & \text{if } x \text{ is a rational number,} \\
  1 & \text{if } x \text{ is an irrational number.}
  \end{cases}
  \]

- **The error function:**
  \[
  f(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.
  \]
§P.7 The Trigonometric Functions

\[ \sin \theta = \frac{b}{c} = \frac{\text{side opposite } \theta}{\text{hypotenuse}} \]
\[ \cos \theta = \frac{a}{c} = \frac{\text{side adjacent } \theta}{\text{hypotenuse}} \]
\[ \tan \theta = \frac{b}{a} = \frac{\text{side opposite } \theta}{\text{side adjacent } \theta} = \frac{\sin \theta}{\cos \theta} \]
\[ \cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{1}{\tan \theta} \]
\[ \sec \theta = \frac{1}{\cos \theta} \]
\[ \csc \theta = \frac{1}{\sin \theta} \]
Let $C$ be the circle with center at the origin $O$ and radius 1. Recall that its equation is $x^2 + y^2 = 1$.

We use the arc length $t$ as a measure of the size of the angle $AOP_t$. Specifically, the **radian measure** of angle $AOP_t$ is

$$\angle AOP_t = t \text{ radians}.$$ 

Note that the angle can also be measured in **degrees**. Since $P_\pi$ is the point $(-1, 0)$, halfway around the circle $C$ from the point $A$, we have

$$\angle AOP_\pi = \pi \quad \text{and also} \quad \angle AOP_\pi = 180^\circ.$$ 

Hence, $\pi = 180^\circ$. Some frequently used degrees are, e.g., $\pi/2 = 90^\circ$, $\pi/3 = 60^\circ$, $\pi/4 = 45^\circ$, and $\pi/6 = 30^\circ$. 
$P_t = (\cos t, \sin t)$

Arc length $t$

$x^2 + y^2 = 1$

$O$, $A = (1, 0)$

$C$
Some Useful Identities

(1) The range of Sine and Cosine:

\[-1 \leq \sin(t) \leq 1 \quad \text{and} \quad -1 \leq \cos(t) \leq 1.\]

(2) The Pythagorean identity:

\[\sin^2(t) + \cos^2(t) = 1.\]

(3) Periodicity:

\[\sin(t + 2\pi) = \sin(t) \quad \text{and} \quad \cos(t + 2\pi) = \cos(t).\]
(4) Sine is an odd function, and Cosine is an even function:

\[
sin(-t) = -\sin(t) \quad \text{and} \quad \cos(-t) = \cos(t).
\]

(5) Complementary angle identities:

\[
\sin\left(\frac{\pi}{2} - t\right) = \cos(t) \quad \text{and} \quad \cos\left(\frac{\pi}{2} - t\right) = \sin(t).
\]

(6) Supplementary angle identities:

\[
\sin(\pi - t) = \sin(t) \quad \text{and} \quad \cos(\pi - t) = -\cos(t).
\]
The graph of $\sin(x)$
The graph of $\cos(x)$
Additional Formulas for Sine and Cosine

\[
\begin{align*}
\sin(s + t) &= \sin(s) \cos(t) + \cos(s) \sin(t), \\
\sin(s - t) &= \sin(s) \cos(t) - \cos(s) \sin(t), \\
\cos(s + t) &= \cos(s) \cos(t) - \sin(s) \sin(t), \\
\cos(s - t) &= \cos(s) \cos(t) + \sin(s) \sin(t).
\end{align*}
\]
**Proof:** The last formula, \( \cos(s - t) = \cos(s) \cos(t) + \sin(s) \sin(t) \), can be shown by noting that \( P_s P_t = P_{s-t} A \). Then by this formula, we can readily derive the remaining three formulas. *[The proof is not required.]*
Letting $s = t$, by the first and third additional formulas we have

$$\sin(2t) = 2 \sin(t) \cos(t),$$
$$\cos(2t) = \cos^2(t) - \sin^2(t).$$

Further, we can show that

$$\sin^2(t) = \frac{1 - \cos(2t)}{2},$$
$$\cos^2(t) = \frac{1 + \cos(2t)}{2}.$$