Implicit differentiation is useful when a function $y = f(x)$ is known to satisfy a given equation, but it is difficult or impossible to isolate $y$ from it.

**Problem 1:** Calculate $dy/dx$ at $x = 3/5$, where $y$ is a function of $x$ defined implicitly by

$$x^2 + y^2 = 1.$$  \hspace{1cm} (1)

**Solution:** The equation corresponds to the unit circle in the Cartesian plane. Since a vertical line may cut the circle at more than one point, this is not the graph of a function. However, it defines *two* differentiable functions $y = f(x)$ and $y = g(x)$, corresponding to the upper and lower branches of the graph, see Figure 1. In this case, it is possible solve for $y$ explicitly for the two branches:

$$f(x) = \sqrt{1 - x^2}, \quad g(x) = -\sqrt{1 - x^2}.$$  

When calculating $dy/dx$, one must specify which branch of the graph we are interested in. This is usually done by also specifying the $y$ value.

Figure 1: The two branches of the curve $x^2 + y^2 = 1$. 
Problem 1: Calculate $dy/dx$ at the point $(3/5, 4/5)$, where $y$ is defined implicitly by

$$x^2 + y^2 = 1.$$ 

Solution:

1. Check that the given point indeed lies on the curve (otherwise this does specify the branch of the function). We readily see that

$$\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = 1,$$

so the point indeed lies on the curve.

2. Since the $y$-value is positive, this corresponds to the upper branch, i.e., to

$$f(x) = \sqrt{1 - x^2}.$$ 

Instead of differentiating $f(x)$, we use Equation (1) to define the function

$$F(x) = x^2 + (y(x))^2 - 1.$$ 

Since $y$ is assumed to satisfy (1), the function $F(x)$ is identically zero, so its derivative is also zero everywhere. We now differentiate $F(x)$ using the chain rule:

$$\frac{dF}{dx} = 2x + 2y \cdot \frac{dy}{dx} = 0.$$ 

Isolating $dy/dx$ gives

$$\frac{dy}{dx} = -\frac{x}{y}.$$ (2)

Thus, the required derivative is calculated by substituting $x = 3/5$, $y = 4/5$ into the above expression:

$$\left.\frac{dy}{dx}\right|_{x=3/5} = -\frac{3}{4} = -\frac{3}{4}.$$ 

Remarks:

1. If we had differentiated $f(x)$ directly, we would have obtained

$$\left.\frac{df}{dx}\right|_{x=3/5} = -\frac{x}{\sqrt{1-x^2}}\bigg|_{x=3/5} = -\frac{3}{4},$$

which is the same as with implicit differentiation.

2. If we consider the point $(3/5, -4/5)$ instead, then we would be seeking the derivative of the lower branch of the graph. The result would become

$$\left.\frac{dy}{dx}\right|_{x=3/5} = -\frac{3}{4} = -\frac{3}{4}.$$ 

Thus, the dependence on $y$ in the expression (2) takes into account the fact that the equation defines more than one function.
**Problem 2:** Calculate \( \frac{dy}{dx} \) at the point \((0, 0)\), where \( y \) is defined implicitly by
\[
xy^2 = y - x. \tag{3}
\]

**Solution:** The graph of Equation (3) defines three functions, shown in red, blue and green in the figure on the right. The point \((0,0)\) falls on the blue branch of the graph. To calculate \( \frac{dy}{dx} \) at this point, we define
\[
F(x) = x(y(x))^2 - y(x) + x = 0
\]
and differentiate \( F(x) \) using the chain rule:
\[
F'(x) = y^2 + 2xy \frac{dy}{dx} - \frac{dy}{dx} + 1 = 0.
\]
Isolating \( \frac{dy}{dx} \) gives
\[
\frac{dy}{dx} = \frac{y^2 + 1}{1 - 2xy}.
\]
Thus, at \((0,0)\), the derivative is given by
\[
\left. \frac{dy}{dx} \right|_{(0,0)} = \frac{(0)^2 + 1}{1 - 2 \cdot 0 \cdot 0} = 1.
\]

**Problem 3:** How many functions defined on \((0, 1/2]\) are given implicitly by the equation \( xy^2 = y - x \)? For each such function, calculate its derivative at \( x = 1/3 \).

**Solution:** From the figure, we see that two functions (the green and blue branches) are defined implicitly by the equation on the interval \((0, 1/2]\). (The red one is defined over \([-1/2, 0)\).) Thus, there are two possible values of \( y \) at \( x = 1/3 \). We substitute \( x = 1/3 \) into the equation to get
\[
\frac{1}{3}y^2 = y - \frac{1}{3}
\]
\[
y^2 - 3y + 1 = 0
\]
\[
y = \frac{3 \pm \sqrt{5}}{2}.
\]
Thus, we need to calculate \( \frac{dy}{dx} \) at two points: \((1/3, 1/2(3 + \sqrt{5}))\) on the green curve, and \((1/3, 1/2(3 - \sqrt{5}))\) on the blue curve. At the first point, we have
\[
\left. \frac{dy}{dx} \right|_{(1/3, 1/2(3+\sqrt{5})} = \frac{(1/2(3 + \sqrt{5}))^2 + 1}{1 - 2(1/3)(1/2(3 + \sqrt{5}))} = -\frac{9(3 + \sqrt{5})}{2\sqrt{5}} \approx -10.537.
\]
At the second point, we have
\[
\left. \frac{dy}{dx} \right|_{(1/3, 1/2(3-\sqrt{5})} = \frac{(1/2(3 - \sqrt{5}))^2 + 1}{1 - 2(1/3)(1/2(3 - \sqrt{5}))} = \frac{9(3 - \sqrt{5})}{2\sqrt{5}} \approx 1.5374.
\]