Exercises

LECTURE 1:

Exercise 1.1:
(a) Show that \( z(t) \) is a solution of the VIE
\[
    u(t) = g(t) + \int_0^t K(t,s)u(s) \, ds, \quad t \in [0,T].
\]
(b) Show that this is the only solution: \( z(t) = u(t) \).

Exercise 1.2: Consider the Fredholm integral equation
\[
    u(t) = g(t) + \lambda \int_0^T K(t,s)u(s) \, ds, \quad t \in [0,T],
\]
with \( K(t,s) = A(t)B(s) \), where \( A, B \in C(I) \).
Does this equation have a (unique) solution \( u \in C(I) \) for any given function \( g \in C(I) \) and and (real or complex) \( \lambda \)?
Generalize your result to equations with kernel
\[
    K(t,s) = \sum_{j=1}^r A_j(t)B_j(s), \quad A_j, B_j \in C(I).
\]
(Hint: Remember the basic result from Linear Algebra about the (non-)existence and uniqueness of solutions for the linear system \( Cx = b \) where \( C \in \mathbb{R}^{n \times n} \) and \( b \in \mathbb{R}^n \).)

Exercise 1.3:
(a) Is the condition \( H(t,t) \neq 0 \) for all \( t \in [0,T] \) necessary for the existence of a unique solution of
\[
    \int_0^t H(t,s)u(s) \, ds = f(t) ?
\]
b) Consider the first-kind Volterra integral equation
\[
    \int_0^t \frac{(t-s)^k-1}{(k-1)!} u(s) \, ds = f(t), \quad t \in [0,T],
\]
where \( k \) is an integer with \( k \geq 1 \), and \( f(t) \) has continuous derivatives of at least order \( k \). Does this VIE possess a unique (continuous) solution?
(c) Does the VIE
\[
    \int_0^t (2t-3s)u(s) \, ds = t^2, \quad t \in [0,T],
\]
possess a unique (continuous) solution on \([0,T]\)?

Exercise 1.4:
(a) Prove Theorem 1.8 by using Picard iteration. Show that for any \( \alpha \in (0,1) \) we have \( u \not\in C^1(I) \).
(b) Find the resolvent kernel corresponding to the kernel \( K_\alpha(t,s) := \lambda(t-s)^{-\alpha} \) (\( 0 < \alpha < 1 \), \( \lambda = \text{const} \)) of the VIE
\[
    u(t) = g(t) + \lambda \int_0^t K_\alpha(t,s)u(s) \, ds.
\]
Exercise 1.5:
(a) Prove the generalized Gronwall lemma (Lemma 1.10) for
\[ z(t) \leq g(t) + \lambda \int_0^t (t-s)^{-\alpha} z(s) \, ds, \quad t \in I, \]
with \( \lambda > 0 \) and \( 0 < \alpha < 1 \).
(b) State and prove the analogue of Lemma 1.10 for the integral inequality
\[ z(t) \leq g(t) + \lambda \int_0^t \log(t-s) z(s) \, ds, \quad t \in I. \]

Exercise 1.6: Complete the proof Theorem 1.9.

Exercise 1.7: Analyze the regularity of the solution of the VIE
\[ u(t) = t^\beta + \lambda \int_0^t (t-s)^{-\alpha} u(s) \, ds, \quad t \in I, \]
when \( \beta > 0, \beta \notin \mathbb{N} \) and \( 0 < \alpha < 1 \).

Exercise 1.8: Prove Theorem 1.11.

Exercise 1.9: Prove Theorem 1.12.

Exercise 1.10: Determine the solution of the VIDE
\[ u'(t) = g(t) + \lambda \int_0^t (t-s)^{-\alpha} u(s) \, ds, \quad 0 < \alpha < 1, \]
satisfying \( u(0) = u_0 \).

LECTURE 2:

Exercise 2.1:
Let \( A_j = A_j(t) \) and \( b_j = b_j(s, u) \) \( (j = 1, \ldots, r) \) be given continuous functions. Show that the nonlinear (Volterra-Hammerstein) integral equation
\[ u(t) = g(t) + \int_0^t \sum_{j=1}^r A_j(t)b_j(s, u(s)) \, ds, \quad t \in I, \]
can be reduced to an initial-value problem for a system of \( r \) nonlinear ordinary differential equations.

Exercise 2.2:
(a) Let \( p > 1 \) and \( u_0 > 0 \). Does the solution of the nonlinear VIE
\[ u(t) = u_0 + \int_0^t (t-s)^p u(s) \, ds, \quad t \geq 0, \]
blow up in finite time \( T_b > 0 \)?
(b) (hard!) Solve (a) for the nonlinear VIE with weakly singular kernel,
\[ u(t) = u_0 + \int_0^t (t-s)^{-\alpha} u^p(s) \, ds, \]
where \( 0 < \alpha < 1 \).
Exercise 2.3:
Extend the (local) existence/uniqueness result of Theorem 2.2 to VIEs with weakly singular kernels:
\[ u(t) = g(t) + \int_0^t (t-s)^{-\alpha} k(t,s,u(s)) \, ds, \quad 0 < \alpha < 1. \]

LECTURE 3:
Exercise 3.1:
Prove that quadrature formulas
\[ Q_m(f) := \sum_{j=1}^{m} w_{m,j} f(c_j) \quad (0 \leq c_1 < \cdots < c_m \leq 1) \]
with maximum degree of precision are interpolatory.

Exercise 3.2:
Suppose that the initial-value problem
\[ u'(t) = a(t)u(t) + b(t), \quad t \in I := [0,T], \quad u(0) = u_0, \]
is solved by collocation in \( S_m^{(0)}(I_h) \), with collocation parameters \( \{ c_i \} \) satisfying
\[ 0 = c_1 < \cdots < c_m = 1 \quad (m \geq 2). \]
Prove that the resulting collocation solution \( u_h \) is in \( C^1(I) \):
\[ u_h \in S_m^{(0)}(I_h) \cap C^1(I) =: S_m^{(1)}(I_h). \]

Exercise 3.3:
Derive the collocation method in \( S_m^{(0)}(I_h) \) for the initial-value problem in Exercise 3.2. Use \( m = 2 \), and (i) \( c_1 = 1/3, \ c_2 = 1 \), (ii) \( c = 0, \ c_2 = 1 \).

LECTURE 4:
Exercise 4.1:
Let \( m \geq 2 \) and suppose that \( u_h \in S_{m-1}^{(-1)}(I_h) \) is the collocation solution for the VIE
\[ u(t) = g(t) + \int_0^t K(t,s)u(s) \, ds, \quad t \in I := [0,T]. \]
Show that if \( g \in C(I) \), \( K \in C(D) \) and the collocation points
\[ X_h := \{ t_n + c_i h_n : 0 \leq c_1 < \cdots < c_m \leq 1 \} \]
are such that \( c_1 = 0 \) and \( c_m = 1 \), then \( u_h \in C(I) \):
\[ u_h \in S_{m-1}^{(-1)}(I_h) \cap C(I) =: S_{m-1}^{(0)}(I_h). \]

Exercise 4.2:
Is the global superconvergence result of Corollary 4.2 true when \( m = 1 \)?
Exercise 4.3:
Suppose that $u_h \in S_m^{(-1)}(I_h)$ is the collocation solution to a linear VIE with smooth solution, and $u_h^{it}$ is the corresponding iterated collocation solution. If the collocation parameters $\{c_i\}$ are the Gauss points, is local superconvergence of $u_h$ and/or $u_h^{it}$ at the collocation points $X_h$ possible?

Exercise 4.4:
Consider the VIE
\[
  u(t) = g(t) + \int_t^T K(t,s)u(s) \, ds, \quad t \in I,
\]
with $g \in C(I)$, $K \in C(\bar{D})$ ($\bar{D} := \{(t,s) : 0 \leq t \leq s \leq T\}$).
(a) Show that this VIE has a unique solution $u \in C(I)$, and find its representation in terms of the resolvent kernel of $K$.
(b) Derive the results on the optimal superconvergence of the iterated collocation solution $u_h^{it}$ (corresponding to the collocation solution $u_h \in S_m^{(-1)}(I_h)$) on $I$ and on $I_h$.

Exercise 4.5:
Let $u_h \in S_m^{(-1)}(I_h)$, with uniform mesh $I_h$, be the collocation solution for the first-kind VIE
\[
  \int_0^t K(t,s)u(s) \, ds = g(t), \quad t \in I = [0,T].
\]
If $m = 1$, prove that $\|u - u_h\|_\infty \to 0$, as $h \to 0$, if and only if the collocation parameters $c_1$ satisfies $1/2 \leq c_1 \leq 1$.

LECTURE 5:

Exercise 5.1:
Consider the VIE with kernel $K_1(t,s) := \log(t-s)K(t,s)$ (see Theorem 5.2),
\[
  u(t) = g(t) + \int_0^t K_1(t,s)u(s) \, ds, \quad t \in I.
\]
Does the solution possess the representation
\[
  u(t) = g(t) + \int_0^t R_1(t,s)g(s) \, ds, \quad t \in I,
\]
and does the resolvent kernel of $K_1(t,s)$ have the form
\[
  R_1(t,s) = \log(t-s)Q_1(t,s), \quad \text{with} \quad Q_1 \in C(D)?
\]

Exercise 5.2:
Consider the weakly singular Volterra-Hammerstein IE
\[
  u(t) = g(t) + \int_0^t K_\alpha(t,s)G(s,u(s)) \, ds, \quad t \in I, \tag{1}
\]
with
\[
  K_\alpha(t,s) := \begin{cases} (t-s)^{-\alpha}K(t,s) & \text{if } 0 < \alpha < 1, \\ \log(t-s)K(t,s) & \text{if } \alpha = 1. \end{cases}
\]
We have seen (Lecture 2, Lecture 5) that setting $z(t) := G(t,u(t))$ this VHIE may be written as the pair of equations
\[
  z(t) = G \left( t, g(t) + \int_0^t K_\alpha(t,s)z(s) \, ds \right), \quad t \geq 0, \tag{2}
\]
and
\[ u(t) = g(t) + \int_0^t K_\alpha(t, s)z(s)\,ds, \quad t \geq 0. \] (3)

Assume:
(a) \( u^I_h \) is the iterated collocation solution corresponding to the collocation solution \( u_h \in S_{m-1}(I_h) \) of the VHIE (1).
(b) Collocation solution \( z_h \in S_{m-1}(I_h) \) for solution \( z \) of (2), followed by approximation \( w_h \) to \( u \) in (3) is given by
\[ w_h(t) := g(t) + \int_0^t K_\alpha(t, s)z_h(s)\,ds. \]

How are \( u^I_h \) and \( w_h \) related?
(Hint: Consider first the case where \( \alpha = 0 \) and \( m = 1 \).)

LECTURE 6:

Exercise 6.1:
Consider the VFIEs
\[ u(t) = g(t) + \int_0^{\theta(t)} K(t, s)u(s)\,ds \] (4)
and
\[ y(t) = f(t) + \int_0^t H(t, s)y(\theta(s))\,ds \] (5)
on \( I = [0, T] \), and with \( \theta(t) = t - \tau (\tau > 0) \) or \( \theta(t) = qt (0 < q < 1) \).
(i) How are the VFIEs (4) and (5) related?
(ii) Write down the collocation equations for \( y_h \in S_{m-1}(I_h) \) for the VFIE (5).

Exercise 6.2:
Suppose that a VFIE with non-vanishing delay function \( \theta(t) = t - \tau(t) \) is to be solved numerically on an interval \( I = [0, t] \) with large \( T \), and assume that \( \theta(t) \) is defined for all \( t > 0 \) and such that
\[ \lim_{t \to \infty} =: \bar{\theta} < \infty, \]
(i) Give an example of such a \( \theta(t) \) (for which conditions (D1)-(D3) hold).
(ii) What can be said about the breaking points \( \{\xi_\mu\} \)?

Exercise 6.3:
Assume that the delay function \( \theta(t) = t - \tau(t) \) does not satisfy condition (D1) \( (\tau(t) \geq \tau_0 > 0, \ t \in I) \) but only the weaker condition \( \tau(t) > 0, \ t \in I \). Is it possible that the resulting breaking points \( \{\xi_\mu\} \) can cluster (that is, can there exist subintervals of \( I = [0, T] \) containing a large number of such points \( \xi_\mu \) that are arbitrarily close to each other)?