Lecture 4:  
Collocation: VIEs with smooth solutions

Collocation solution $\mathbf{u}_{h} \in \mathcal{S}_{m-1}^{(-1)}(I_{h})$ for

$$u(t) = g(t) + \int_{0}^{t} K(t, s) u(s) \, ds, \quad t \in [0, T]:$$

$\rightarrow$ **Collocation equation:** find $\mathbf{u}_{h} \in \mathcal{S}_{m-1}^{(-1)}(I_{h})$ so that

$$\mathbf{u}_{h}(t) = g(t) + \int_{0}^{t} K(t, s) \mathbf{u}_{h}(s) \, ds, \quad t \in X_{h}. \quad (1)$$

$\leftarrow$ **Collocation points:** (given $0 < c_{1} < \cdots < c_{m} \leq 1$)

$$X_{h} := \{t_{n} + c_{i}h_{n} : i = 1, \ldots, m \ (0 \leq n \leq N - 1)\}.$$  
For $t = t_{n} + vh_{n} \in e_{n} \ (v \in (0, 1])$ we have

$$\mathbf{u}_{h}(t) = \sum_{j=1}^{m} L_{j}(v) \mathcal{U}_{n,j}, \quad v \in (0, 1], \quad (2)$$

where the $\mathcal{U}_{n,j} := \mathbf{u}_{h}(t_{n} + c_{j}h_{n})$ are unknown.

$\leftarrow$ **Computation** of collocation solution $\mathbf{u}_{h}(t)$?

Insert (2) into collocation equation (1) with $t = t_{n} + c_{i}h_{n}$:

$\Rightarrow$ System of linear algebraic equations for

$$\mathcal{U}_{n} := (\mathcal{U}_{n,1}, \ldots, \mathcal{U}_{n,m})^{T}:$$

$$[\mathbb{I}_{m} - h_{n}A_{n}] \mathcal{U}_{n} = g_{n} + \sum_{\ell=0}^{n-1} h_{\ell}A_{n,\ell} \mathcal{U}_{\ell}$$

where $A_{n} \in \mathbb{R}^{m \times m}$; $\mathbb{I}_{m}$ is the identity matrix in $\mathbb{R}^{m \times m}$.  

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Some details:
If \( t = t_n + c_i h_n \) (\( i = 1, \ldots, m \)) in the collocation equation
\[
    u_h(t) = g(t) + \int_t^t K(t, s)u_h(s) \, ds,
\]
then
\[
    u_h(t) = g(t) + \int_0^{t_n} K(t, s)u_h(s) \, ds + F_{n,i},
\]
where
\[
    \int_0^{t_n} K(t, s)u_h(s) \, ds = \sum_{\ell=0}^{n-1} h_\ell \int_0^1 K(t, t_\ell + sh_\ell)u_h(t_\ell + sh_\ell) \, ds
\]
and
\[
    F_{n,i} := h_n \int_0^{c_i} K(t, t_n + sh_n)u_h(t_n + sh_n) \, ds.
\]
Since the local representation of \( u_h \) on \( [t_n, t_{n+1}] \) is
\[
    u_h(t_n + vh_n) = \sum_{j=1}^m L_j(v)U_{n,j}, \quad v \in (0, 1],
\]
we obtain
\[
    \int_0^1 K(t, t_\ell + sh_\ell)u_h(t_\ell + sh_\ell) \, ds
\]
\[
    = \sum_{j=1}^m \int_0^1 K(t, t_\ell + sh_\ell)L_j(s) \, ds \cdot U_{\ell,j}
\]
and
\[
    F_{n,i} = h_n \sum_{j=1}^m \int_0^{c_i} K(t, t_n + sh_n)L_j(s) \, ds \cdot U_{n,j}.
\]
For each \( n = 0, \ldots, N - 1 \) solve the linear system

\[
[I_m - h_n A_n] U_n = g_n + \sum_{\ell=0}^{n-1} h_\ell A_{n,\ell} U_\ell,
\]

for \( U_n := (U_{n,1}, \ldots, U_{n,m})^T \in \mathbb{R}^m \), where
\( U_{n,j} := u_h(t_n + c_j h_n) \) and
\( g_n := (g(t_n + c_i h_n) \ (i = 1, \ldots, m))^T \in \mathbb{R}^m \).

The matrices \( A_n \) and \( A_{n,\ell} \) have the forms

\[
A_n := \left[ \int_0^{c_i} K(t_n + c_i h_n, t_n + s h_n) L_j(s) \, ds \right]_{(i, j = 1, \ldots, m)},
\]

and, for \( 0 \leq \ell < n \),

\[
A_{n,\ell} := \left[ \int_0^1 K(t_n + c_i h_n, t_\ell + s h_\ell) L_j(s) \, ds \right]_{(i, j = 1, \ldots, m)}.
\]

When we know the solution \( U_n \) of (3), then the collocation solution on \( (t_n, t_{n+1}] \) is given by

\[
u_h(t_n + v h_n) = \sum_{j=1}^m L_j(v) U_{n,j}, \quad v \in (0, 1]. \]

**Question:**
Numerical) evaluation of integrals in the matrices \( A_n \) and \( A_{n,\ell} \) \( (\ell < n) \)?
The fully discretized collocation method

In general, the integrals in the collocation equation for \( u_h \in S_{m-1}^{(-1)}(I_h) \),

\[
    u_h(t) = g(t) + \int_0^t K(t, s) u_h(s) \, ds,
\]

where \( t = t_n + c_i h_n \) \((i = 1, \ldots, m; \ 0 \leq n \leq N - 1)\) cannot be evaluated exactly and thus have to be approximated by suitable numerical quadrature.

**Assume:** Quadrature formulas are \( m \)-point interpolatory quadrature rules with *abscissas* given by the collocation points:

\[
    \int_0^1 K(t, t_\ell + s h_\ell) u_h(t_\ell + s h_\ell) \, ds = \sum_{\nu=1}^{m} w_\nu K(t, t_\ell + c_\nu h_\ell) u_h(t_\ell + c_\nu h_\ell) + E_{n,\ell}^{(i)} = U_{n,\ell}
\]

and (note that \( K(t, s) \) is defined only for \( s \leq t \))

\[
    \int_0^{c_i} K(t, t_n + s h_n) u_h(t_n + s h_n) \, ds = \sum_{\nu=1}^{m} w_{i,\nu} K(t, t_n + c_i c_\nu h_n) u_h(t_n + c_i c_\nu h_n) + E_n^{(i)}.
\]

The fully discretized collocation equation is obtained by replacing the integrals by the above quadrature approximations (*without* the quadrature errors \( E_{n,\ell}^{(i)} \) and \( E_n^{(i)} \)). ⇒ The resulting discrete collocation solution \( \hat{u}_h \in S_{m-1}^{(-1)}(I_h) \) possesses the same superconvergence properties as the exact collocation solution \( u_h \).

(Brunner (2004))
**VIEs:**

**Optimal order of convergence of** $u_h$ **and** $u_{h}^{it}$

**Recall:** Collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$:

$$u_h(t) = g(t) + \int_0^t K(t,s)u_h(s)\,ds, \quad t \in X_h,$$

with

$$X_h := \{t_n + c_i h_n : i = 1, \ldots m \ (0 \leq n \leq N - 1)\}.$$  

$\hookrightarrow$ Iterated collocation solution $u_{h}^{it} \in C(I)$:

$$u_{h}^{it}(t) := g(t) + \int_0^t K(t,s)u_h(s)\,ds, \quad t \in [0,T].$$

$\hookrightarrow$ **Note:** $u_{h}^{it}(t) = u_h(t)$ for all $t \in X_h$.

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**Theorem 4.1:**

(a) If $u \in C^d(I)$ $(d \geq m)$, then for general $\{c_i\}$,

$$\|u - u_h\|_\infty \leq Ch^m.$$  

(b) If $u \in C^d(I)$ $(d \geq m + 1)$, and if the $\{c_i\}$ are so that

$$J_0 := \int_0^1 \prod_{i=1}^m (s - c_i)\,ds = 0,$$

then

$$\|u - u_{h}^{it}\|_\infty \leq Ch^{m+1}$$

*(global superconvergence on I: Brunner & Yan (1996)).*
Remark:
We have $J_0 = \int_0^1 \prod (s - c_i) \, ds = 0$ when the $\{c_i\}$ are the Gauss points, the Radau I points, or the Radau II points. For these collocation parameters the iterated collocation solution $u_{ih}^{it}$ is globally superconvergent:

**Corollary 4.2:** If $u_h \in S_{m-1}^{(-1)}(I_h)$ is the collocation solution corresponding to the Gauss, Radau I or Radau II points, then the iterated collocation solution satisfies

$$\|u - u_{ih}^{it}\|_\infty \leq C h^{m+1} \quad (m \geq 2).$$

**Exercise 4.1:**
Does the statement of Corollary 4.2 remain true for $m = 1$?

The key elements in establishing global (on $I$) and local (on $I_h$) superconvergence results are

- the resolvent representation of the collocation error, and
- the theory of interpolatory quadrature formulas of optimal degree of precision.
Local superconvergence on $I_h$

Recall: $u_h^{it}(t) = u_h(t)$ for all $t \in X_h$!

$\Rightarrow u_h^{it}(t_n) = u_h(t_n)$ for all $X_h$ with $c_m = 1$.

**Theorem 4.3:** Let $u_h \in S_m^{(-1)}(I_h)$ be the collocation solution of VIE with respect to the collocation parameters $\{c_i\}$. Assume that the exact solution of the VIE satisfies $u \in C^d(I)$.

- If the $\{c_i\}$ are the *Radau I* points, and if $d \geq 2m - 1$, then
  \[
  \max_{1 \leq n \leq N} |u(t_n) - u_h^{it}(t_n)| \leq C h^{2m-1},
  \]
  but:
  \[
  \max_{1 \leq n \leq N} |u(t_n) - u_h(t_n)| \leq C h^m.
  \]

- If the $\{c_i\}$ are the *Radau II points*, and if $d \geq 2m - 1$, then
  \[
  \max_{1 \leq n \leq N} |u(t_n) - u_h(t_n)| \leq C h^{2m-1}.
  \]

- If the $\{c_i\}$ are the *Gauss points*, and if $d \geq 2m$, then
  \[
  \max_{1 \leq n \leq N} |u(t_n) - u_h^{it}(t_n)| \leq C h^{2m},
  \]
  but:
  \[
  \max_{1 \leq n \leq N} |u(t_n) - u_h(t_n)| \leq C h^m.
  \]
Proof: Setting $\mathcal{V}u(t) := \int_0^t K(t, s)u(s) \, ds$, the collocation equation for

$$u(t) = g(t) + (\mathcal{V}u)(t), \quad t \in I,$$

may be written as

$$u_h(t) = g(t) - \delta_h(t) + (\mathcal{V}u_h)(t), \quad t \in I,$$

where $\delta_h(t) = 0$ for all $t \in X_h$.

Thus, $e_h := u - u_h$ and $e_h^{it} := u - u_h^{it}$ satisfy

$$e_h(t) = \delta_h(t) + (\mathcal{V}e_h)(t), \quad t \in I, \quad (4)$$

and

$$e_h^{it}(t) = e_h(t) - \delta_h(t), \quad t \in I. \quad (5)$$

The solution of the VIE (4) is given by

$$e_h(t) = \delta_h(t) + \int_0^t R(t, s)\delta_h(s) \, ds, \quad t \in I,$$

where $R(t, s)$ is the resolvent kernel of the kernel $K(t, s)$ of $\mathcal{V}$. It follows from (5) that

$$e_h^{it}(t) = \int_0^t R(t, s)\delta_h(s) \, ds, \quad t \in I.$$
Recall:

\[ e_h(t) = \delta_h(t) + \int_0^t (R(t, s)\delta_h(s)) \, ds, \quad t \in I, \]

and

\[ e^{it}_h(t) = \int_0^t R(t, s)\delta_h(s) \, ds, \quad t \in I, \]

where \( \delta_h(t) = 0 \) for all \( t \in X_h \).

(a) \( t = t_n + vh_n \) (\( v \in (0, 1] \), \( 0 \leq n \leq N - 1 \)):

\[ e_h(t) = \delta_h(t) + \sum_{\ell=0}^{n-1} h_\ell \int_0^1 R(t, t_\ell + sh_\ell)\delta_h(t_\ell + sh_\ell) \, ds \]

\[ + h_n \int_0^v R(t, t_n + sh_n)\delta_h(t_n + sh_n) \, ds. \]

(b) \( t = t_{n+1} \) (\( 0 \leq n \leq N - 1 \)):

\[ e_h(t_{n+1}) = \delta_h(t_{n+1}) \]

\[ + \sum_{\ell=0}^{n-1} h_\ell \int_0^1 R(t_{n+1}, t_\ell + sh_\ell)\delta_h(t_\ell + sh_\ell) \, ds \]

\[ + h_n \int_0^1 R(t_{n+1}, t_n + sh_n)\delta_h(t_n + sh_n) \, ds. \]

Connection with optimal m-point interpolatory quadrature (with quadrature abscissas chosen to be the collocation points \( \{t_\ell + c_i h_\ell\} \)).
Recall $m$-point interpolatory quadrature formula with abscissas $\{d_j\}$ with $0 \leq d_1 < \cdots < d_m \leq 1$:

$$
\int_0^1 f(t_n + s h_n) \, ds = \sum_{j=1}^{m} w_j f(t_n + d_j h_n) + E_n(f),
$$

with quadrature weights

$$
w_j := \int_0^1 L_j(s) \, ds \quad (j = 1, \ldots, m)
$$

Quadrature error $E_n(f)$:

- For arbitrary abscissas $\{d_j\}$ (and $f \in C^m[0, 1]$):

$$
|E_n(f)| \leq Q h_n^m.
$$

- If the $\{d_j\}$ satisfy

$$
J_\nu := \int_0^1 s^\nu \prod_{i=1}^{m} (s - d_j) \, ds = 0 \quad (\nu = 0, \ldots, \kappa - 1)
$$

and $J_\kappa \neq 0$ ($1 \leq \kappa \leq m$), then

$$
|E_n(f)| \leq Q h_n^{m+\kappa},
$$

provided that $f \in C^{m+\kappa}[0, 1]$.

- $\kappa = m$: The $\{d_j\}$ are the Gauss (-Legendre) points (zeros of $P_m(2s - 1)$).
- $\kappa = m - 1$ and $d_m = 1$:

The $\{d_j\}$ are the Radau II points.

- $\kappa = m - 2$ and $d_1 = 0$, $d_m = 1$ ($m \geq 2$):

The $\{d_j\}$ are the Lobatto points.
(Proof of Theorem 4.3: Completion)

I. **Optimal order** of $e_h(t)$ for $t \in I_h$:

(Recall that $\delta_h(t) = 0$ for all $t \in X_h$.)

At $t = t_{n+1}$ ($0 \leq n \leq N - 1$):

$$e_h(t_{n+1}) = \delta_h(t_{n+1})$$

$$+ \sum_{\ell=0}^{n-1} h_{\ell} \int_0^1 R(t_{n+1}, t_{\ell} + s h_{\ell}) \delta_h(t_{\ell} + s h_{\ell}) \, ds$$

$$= E_{n,\ell}$$

$$+ h_n \int_0^1 R(t_{n+1}, t_n + s h_n) \delta_h(t_n + s h_n) \, ds.$$  

$$= E_n$$

$$\Rightarrow e_h(t_{n+1}) = \delta_h(t_{n+1}) + \sum_{\ell=0}^{n-1} h_{\ell} E_{n,\ell} + h_n E_n$$

(where $|E_{n,\ell}| \leq Q h^{m+\kappa}$, $|E_n| \leq Q h^{m+\kappa}$).

- If $c_m = 1$ : $\delta_h(t_{n+1}) = 0$ ($n = 0, \ldots, N - 1$).
- If $c_m < 1$ : $|\delta_h(t_{n+1})| = O(h^m)$!
II. **Optimal order** of $e^{it}(t)$ for $t \in I_h$:
(Recall: that $e^{it}_h = e^h - \delta_h$ and
\[
e^{it}_h(t_{n+1}) = \sum_{\ell=0}^{n-1} h_\ell \int_0^1 R(t_{n+1}, t_\ell + s h_\ell) \delta_h(t_\ell + s h_\ell) \, ds
+ h_n \int_0^1 R(t_{n+1}, t_n + s h_n) \delta_h(t_n + s h_n) \, ds.
\]
⇒ (using the same quadrature approximations as in the expression for $e^{it}_h(t_{n+1})$)
\[
e^{it}_h(t_{n+1}) = \sum_{\ell=0}^{n-1} h_\ell E_{n,\ell} + h_n E_n.
\]

**Note:** Since $n \leq N - 1$, we have $\sum_{\ell=0}^{n-1} h_\ell \leq T$.

**Remark:** The representation of the **iterated** collocation error $e^{it}_h(t)$ for VIEs is identical with the representation of the **collocation error** $e^{it}_h(t)$ for ODEs, except that $r(t,s)$ has now been replaced by the **resolvent kernel** $R(t,s)$ of the VIE kernel $K(t,s)$.
Remarks:

1. If the collocation parameters \( \{c_i\} \) are such that
\[
0 = c_1 < \cdots < c_m = 1 \quad (m \geq 2),
\]
then \( u_h \in C(I) : \)
\[
u_h \in S_{m-1}^{-1}(I_h) \cap C(I) =: S_{m-1}^{(0)}(I_h).
\]
\implies \text{Optimal order on } I_h:\n\[
\max_{(n)} |u(t_n) - u_h(t_n)| \leq Ch^{2m-2}.
\]

2. **Smoother** piecewise polynomial spaces:
\( u_h \in S_{m}^{(d)}(I_h) \) with \( d \geq 1 \) (\( d < m \)) ?
- \( m = 3, \ d = 2 : \) \( u_h \) is divergent (as \( h \to 0 \)).
  (Hung (1970))
- \( m = 3, \ d = 1 : \) If \( 0 < c_1 < c_2 = 1 \), then \( u_h \) is divergent if \( (1 - c_1)/c_1 > 1 \).
  (Oja (2001))
- \( m \geq 3, \ d = 1 : \) If the \( \{c_i\} \) are the Radau II points, then \( u_h \) is divergent! (Brunner (2004))

(\( \hookrightarrow \) General divergence theory : Brunner (BIT, 2004)).
Collocation for **Fredholm integral equations**

Let the linear *Fredholm integral operator* $F : C(I) \to C(I)$ be given by

$$(F(u))(t) := \int_0^T K(t,s)u(s) \, ds, \quad t \in I := [0,T].$$

The *collocation solution* $u_h \in S_{m-1}^{(-1)}(I_h)$ to the FIE

$$u(t) = g(t) + \lambda (Fu)(t), \quad t \in I \ (\lambda \neq 0)$$

is determined by the *collocation equation*

$$u_h(t) = g(t) + \lambda (Fu)(t), \quad t \in X_h,$$

and the corresponding *iterated* collocation solution $u_h^{it}$ is defined by

$$u_h^{it}(t) := g(t) + \lambda (Fu_h)(t), \quad t \in I.$$

→ **Assume:** $\lambda^{-1} \notin \sigma(F)$ (i.e. $\lambda^{-1}$ is not an eigenvalue of $F$).

**Theorem 4.4:**

If the $\{c_i\}$ are the *Gauss* points, then

$$\max\{|u(t) - u_h(t)| : t \in X_h\} \leq Ch^{2m}.$$

Moreover, the *iterated* collocation solution is **globally superconvergent** on $I$:

$$\|u - u_h^{it}\|_\infty \leq Ch^{2m}.$$

(Chandler (1979), Chatelin & Lebbar (1981); see also Brunner (1984), Atkinson (1997))
Remark:
Theorem 4.4 shows that for Fredholm integral equations, superconvergence of $u^i_h$ is possible at the collocation points $X_h$:

If the $\{c_i\}$ are the Gauss points, then

$$\max\{|u(t) - u_h(t)| : t \in X_h\} \leq Ch^{2m}.$$  

Question:
Is superconvergence on $X_h$ possible for Volterra integral equations; that is, what is the optimal value of $p^*$ in

$$\max\{|u(t) - u_h(t)| : t \in X_h\} \leq Ch^{p^*} ?$$

(Exercise 4.3)
Superconvergence results for non-standard VIEs:

\[ u(t) = g(t) + \int_0^t K(t, s)G(u(t), u(s)) \, ds, \quad t \in [0, T]. \]

(\text{\(\text{\textendash}\text{\textendash}\) Zhang Ran et al. (2010)})

**Auto-convolution VIEs**

Nonlinear VIEs of the form

\[ u(t) = g(t) + \int_0^t K(t, s)u(t - s)u(s) \, ds, \quad t \in [0, T], \]

are called (quadratic) VIEs of auto-convolution type (von Wolfersdorf & Janno (1995), ...).

**Generalization:**

\[ u(t) = g(t) + \int_0^t K(t, s)G(u(t - s), u(s)) \, ds. \]

\(\text{\(\text{\textendash}\text{\textendash}\)} Superconvergence analysis of collocation solutions in \(S_{m-1}^{(-1)}(I_h)\) and corresponding \textit{iterated} collocation solution: current work by Zhang Ran and Brunner.
Volterra integral equations of the **first kind**

\[(Va u)(t) := \int_0^t (t - s)^{-\alpha} K(t, s) u(s) \, ds = g(t), \]

\(t \in I := [0, T],\ 0 \leq \alpha < 1,\) with kernel \(K \in C(D)\)

\((D := \{(t, s) : 0 \leq s \leq t \leq T\}),\) and \(g \in C(I).\)

**Theorem 4.5:** (Volterra (1896))

Assume that

(i) \(K \in C^1(D),\ K(t, t) \neq 0\) on \(I,\) and

(ii) \(g \in C^1(I),\ g(0) = 0.\)

Then there exists, for \(any\ \alpha \in [0, 1),\) a unique solution \(u \in C(I).\)

**Regularity** of solutions on \(I:\)

- \(\alpha = 0:\) If (i), (ii), and

  (iii) \(K \in C^{d+1}(D),\)

  (iv) \(g \in C^{d+1}(I)\)

hold, then \(u \in C^d(I).\)

- \(0 < \alpha < 1:\) If (i)-(iv) hold, then

  \(u \in C(I) \cap C^d((0, T])\) for \(any\ \ d \geq 1.\)

At \( t = 0^+: \ u'(t) \sim Ct^{\alpha-1}.\)
Numerical methods for first-kind Volterra integral equations \((\alpha = 0)\)

- **Direct quadrature (DQ) methods:**

\[
\int_0^t K(t,s)u(s) \, ds = g(t), \quad t \in I := [0,T] : 
\]

Assume that \(K(t,t) \neq 0\) for all \(t \in I\). Let \(I_h := \{t_n := nh : n = 0,1,\ldots,N \ (t_N = T)\}\) (uniform mesh on I). For \(t = t_n \ (1 \leq n \leq N)\):

\[
h \sum_{j=0}^{n} w_{n,j} K(t_n,t_j) U_j = g(t_n),
\]

where \(U_j\) is an approximation for \(u(t_j)\).

**Theorem 4.6:** (Gladwin & Jeltsch (1974), Gladwin (1979,1982); Holyhead *et al.* (1975); Wolkenfelt (1981); Dixon *et al.* (1986); Lubich (1987): \(0 < \alpha < 1\))

(a) DQ methods based on (composite) interpolatory quadrature formulas of order \(> 2\) are unstable (divergent), as \(h \to 0\).

(b) **Stable** (convergent) DQ methods of higher order exist.

*In particular:* \((\rho, \sigma)\)-reducible DQ methods (based on linear multistep methods for ODEs) (Wolkenfelt (1981)).

**Problem:** Construction of higher-order (stable) DQ methods on non-uniform meshes?
Collocation methods:
Let \( I_h := \{ t_n : 0 = t_0 < t_1 < \cdots < t_N = T \} \), with
\( e_n := (t_n, t_{n+1}] \), \( h_n := t_{n+1} - t_n \), \( h := \max\{h_n\} \).

Collocation spaces: For given \( m \geq 1 \), use either
\[
S_m^{(0)}(I_h) := \{ v \in C(I) : v|_{e_n} \in \pi_m \ (0 \leq n \leq N-1) \}
\]
(globally continuous piecewise polynomials of degree \( m \geq 1 \) \( \Rightarrow \) \( \dim S_m^{(0)}(I_h) = Nm + 1 \)), or
\[
S_m^{(-1)}(I_h) := \{ v : v|_{e_n} \in \pi_{m-1} \ (0 \leq n \leq N-1) \}
\]
(discontinuous piecewise polynomials of degree \( m - 1 \geq 0 \) \( \Rightarrow \) \( \dim S_m^{(-1)}(I_h) = Nm \)).

Collocation points:
\[
X_h := \{ t_n + c_i h_n : 0 \leq n \leq N - 1 \},
\]
corresponding to prescribed collocation parameters \( \{c_i\} \) with \( 0 < c_1 < \cdots < c_m \leq 1 \)
(\( \Rightarrow |X_h| = Nm \)).

Collocation equations:
Find either: \( u_h \in S_m^{(0)}(I_h) \) so that
\[
(Vu_h)(t) = g(t) \quad \text{for all} \quad t \in X_h,
\]
using the initial condition \( u_h(0) = u(0) = \frac{g'(0)}{K(0,0)} \),
or: Find \( u_h \in S_m^{(-1)}(I_h) \) so that
\[
(Vu_h)(t) = g(t), \quad t \in X_h
\]
(here, no initial condition is needed).
Collocation in $S_{m-1}^{(-1)}(I_h)$ for V1s

$(\forall u)(t) := \int_0^t K(t,s)u(s) \, ds = g(t), \ t \in I := [0, T]$

(with $K \in C^{d+1}(D), \ K(t,t) \neq 0; \ g \in C^{d+1}(I), \ g(0) = 0$):

The collocation solution $u_h$ corresponds to collocation points $\{t_n + c_i h_n\}$ with given collocation parameters $\{c_i\}$ satisfying $0 < c_1 < \cdots < c_m \leq 1$.

**Theorem 4.7:** (Brunner (1978, 2004))

Let $u_h \in S_{m-1}^{(-1)}(I_h)$ be the collocation solution for V1, and assume that $I_h$ is a uniform mesh on $I$. Define

$$\rho_m := (-1)^m \prod_{i=1}^m \frac{1 - c_i}{c_i}.$$

Then: $\lim_{h \to 0} \|u - u_h\|_\infty = 0 \iff \rho_m \in [-1, 1]$.

If $d \geq m + 1$ the attainable (global) order of convergence of $u_h$ is given by

$$\|u - u_h\|_\infty \leq C \begin{cases} h^m & \text{if } -1 \leq \rho_m < 1, \\ h^{m-1} & \text{if } \rho_m = 1. \end{cases}$$

**Note:** If the $\{c_i\}$ are the Gauss points, then $\rho_m = (-1)^m \Rightarrow$ Order reduction for even $m$!
Collocation in $S^{(0)}_m(I_h)$ for V1s, with collocation parameters $\{c_i\}$ satisfying $0 < c_1 < \cdots < c_m \leq 1$, and using the (exact) initial condition $u_h(0) = u(0) = \frac{g'(0)}{K(0,0)}$.

**Theorem 4.8:** *(Kauthen & B. 1997)*
Let $u_h \in S^{(0)}_m(I_h)$ be the collocation solution for V1, and assume that $I_h$ is a uniform mesh on $I$. Define

$$r_m := (-1)^m \prod_{i=1}^{m-1} \frac{1 - c_i}{c_i} \leq 1.$$ 

- **Case I:** $c_m = 1$
If $d \geq m + 2$ the attainable (global) order of convergence of $u_h$ is given by

$$\|u - u_h\|_\infty \leq C \begin{cases} h^{m+1} & \text{if } r_m \in [-1, 1), \\ h^m & \text{if } r_m = 1. \end{cases}$$

- **Case II:** $c_m < 1$
If the $\{c_i\}$ are symmetric ($c_i = 1 - c_{m+1-i}$), then the collocation solution $u_h$ is unstable (divergent) as $h \to 0$. 
Corollary 4.9:

• $u_h \in S_m^{(0)}(I_h)$ with $\{c_i\}$ given by the Radau II points (zeros of $(P_m - P_{m-1})(2s - 1)$ ($m \geq 2$), with $c_m = 1$):
  $\Rightarrow u_h$ is divergent.

• $u_h \in S_m^{(0)}(I_h)$ with $\{c_i\}$ given by the Gauss (-Legendre) points (zeros of $P_m(2s - 1)$ ($m \geq 1$), with $c_m < 1$):
  $\Rightarrow u_h$ is divergent.

Remarks:

• There exist collocation parameters $\{c_i\}$ with $c_m < 1$ for which the collocation solution $u_h$ is stable (convergent). (Kauthen & Brunner (1997)).

• Collocation in smoother piecewise polynomial spaces?
  (i) $u_h \in S_2^{(1)}(I_h)$ is divergent for any $c_1 \in (0, 1]$ (Hung (1970)).
• Discontinuous Galerkin methods

\[(\mathcal{V} u)(t) := \int_0^t K(t, s)u(s) \, ds, \quad t \in I := [0, T] :\]

For given mesh \( I_h \) (with \( h_n := t_{n+1} - t_n \), \( e_n := (t_n, t_{n+1}] \) and local basis functions \( \{\phi_{n,i}\}_{i=0}^m \) on \( e_n \), the discontinuous Galerkin (DG) solution \( u_h \in S_m^{(-1)}(I_h) \) is defined by

\[\langle \mathcal{V} u_h, \phi_{n,i} \rangle = \langle g, \phi_{n,i} \rangle \quad i = 0, 1, \ldots, m \quad (0 \leq n \leq N-1).\]

The time-stepping form of the (exact) DG method is given by the local DG equations

\[
\int_0^1 \left( \int_0^v K(t, t_n + sh_n)u_h(t_n + sh_n) \, ds \right) \phi_{n,i}(v) \, dv
\]

\[= h_n^{-1} \left( \int_0^1 g(t_n + vh_n)\phi_{n,i}(v) \, dv - H_{n,i} \right)\]

\((i = 0, 1, \ldots, m; \ t = t_n + vh_n \ (v \in (0, 1]). \quad \text{The } H_{n,i} \text{ denote the (exact) lag terms:}

\[H_{n,i} := \int_0^1 \left( \int_0^{t_n} K(t_n + vh_n, s)u_h(s) \, ds \right) \phi_{n,i}(v) \, dv.\]

The DG equations define the DG solution recursively on \( e_0, e_1, \ldots, e_{N-1}. \) On \( e_n \) the local representation of \( u_h \) is

\[u_h(t_n + vh_n) = \sum_{j=0}^m \alpha_{n,j}\phi_{n,j}(v), \quad v \in (0, 1],\]

with the \( \{\alpha_{n,j}\} \) determined by the solution of a linear algebraic system in \( \mathbb{R}^{m+1} \).
Recall: $u_h$ is called the exact DG solution to the solution of

$$(\forall u)(t) := \int_0^t K(t, s)u(s)\, ds = g(t)$$

if the inner products in the DG equations are evaluated exactly.

**Theorem 4.10:** (B., Davies & Duncan (2009))

Let $u_h \in S^{(-1)}_m(I_h)$ be the exact DG solution for

$$\int_0^t K(t, s)u(s)\, ds = g(t), \quad t \in I := [0, T],$$

and assume that $K \in C^{d+1}(D)$, $g \in C^{d+1}(I)$, with $K(t, t) \neq 0$ ($t \in I$) and $g(0) = 0$.

Then:

(i) $u_h$ converges unconditionally to $u$:

$$\lim_{h \to 0} \|u - u_h\|_\infty = 0.$$

(ii) For $d \geq m + 1$,

$$\|u - u_h\|_\infty \leq C \begin{cases} 
  h^{m+1} & \text{if } m \text{ is even,} \\
  h^m & \text{if } m \text{ is odd.}
\end{cases}$$
• Semi-discretised DG methods

The time-stepping form of the exact DG method is given by

\[
\int_0^1 \left( \int_0^v K(t, t_n + sh_n)u_h(t_n + sh_n) \, ds \right) \phi_{n,i}(v) \, dv
\]

\[
= h_n^{-1} \left( \int_0^1 g(t_n + vh_n)\phi_{n,i}(v) \, dv - H_{n,i} \right)
\]

\[(i = 0, 1, \ldots, m; \ t = t_n + vh_n (v \in (0, 1])), \text{ with the}
\]

\[H_{n,i} := \int_0^1 \left( \int_0^{t_n} K(t_n + vh_n, s)u_h(s) \, ds \right) \phi_{n,i}(v) \, dv.
\]

\(\hookrightarrow\) Approximation of the inner products in the Galerkin equation: use interpolatory \(q + 1\)-point quadrature formulas,

\[
\int_0^1 f(s) \, ds = \sum_{\nu=0}^{q} w_{q,\nu}f(d_\nu) + E_q[f],
\]

with given quadrature points \(0 \leq d_0 < \cdots < d_q \leq 1\). Denote the resulting semi-discrete DG solution by \(\hat{u}_h\).

**Theorem 4.11:** (Brunner, Davies & Duncan (2010))

Let \(\hat{u}_h \in S_m^{(-1)}(I_h)\) be the semi-discrete DG solution for V1, and assume that \(q = m\). Then \(\hat{u}_h\) coincides with the (exact) collocation solution in \(S_m^{(-1)}(I_h)\) corresponding to the collocation points \(\{d_i\}\), provided that \(d_0 > 0\).
Remark:

- Discrete DG methods with **more dense** quadrature approximations:
  Suppose that \( 0 \leq d_0 < d_1 < \cdots < d_q \leq 1 \) with \( q > m \), and interpolatory \((q + 1)\)-point quadrature rules of the form

\[
\int_0^1 f(s) \, ds \approx \sum_{\nu=0}^{q} w_{q,\nu} f(d_{\nu}),
\]

with **degree of precision** \( \geq q \), are used to approximate the inner products in the Galerkin equations for \( V_1 \).

**Question:**
Do there exist **unconditionally stable** (convergent) semi-discrete DG methods based on such more dense quadrature approximations?
(Current work with **Davies** and **Duncan**)
Applications related to collocation for V1s

- **Integral-algebraic equations** (IAEs / IDAEs): (see Lecture 7)
  (i) System of Volterra integral equations including *second-kind* and *first-kind* integral equations
  (ii) System consisting of ODEs (or Volterra *integro-differential* equations) and *first-kind* Volterra integral equations
- **Boundary integral equations** (single-layer potential): ⇐ *Boundary integral method* for homogeneous diffusion equation (on bounded $\Omega \subset \mathbb{R}^2$ with smooth boundary $\Gamma$), Dirichlet boundary data $g$ and vanishing initial data (Hamina & Saranen (1994)):

  $$ \int_0^t \int_0^1 E(x(\theta) - x(\varphi), t - \tau)u(\varphi, \tau) \, d\varphi \, d\tau = f(\theta, t) $$

  on $\mathbb{R} \times [0, T]$. Here, $x(\theta)$ is a smooth 1-periodic representation of $\Gamma$, $f(\theta, t) := g_\Gamma(x(\theta), t)$ and

  $$ E(x, t) := \begin{cases} 
  (4\pi t)^{-1}\exp(-|x|^2/(4t)), & t > 0 \\
  0, & t \leq 0.
  \end{cases} $$

  ⇐ First-kind *Volterra-Fredholm* integral equation!
Concluding remarks:

• *Iterative correction* of low-order collocation solutions for second-kind VIEs: Brunner, Lin & Yan (1996).


• *Sequential regularization methods* (based on collocation) for first-kind VIEs with noisy data: Lamm (2000), Lamm & Dai (2005).
Basic references:


(↩ See also the handout "References: Lecture IV" for additional papers and books on collocation and related methods for Volterra integral equations.)
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Lecture 5 of

Theory and numerical solution of Volterra functional integral equations

Hermann Brunner

Department of Mathematics and Statistics
Memorial University of Newfoundland
St. John’s, NL
Canada

Department of Mathematics
Hong Kong Baptist University
Hong Kong SAR
P.R. China
Lecture 5:
**Collocation: VIEs with non-smooth solutions**

Volterra integral operators with weakly singular kernels, \( \mathcal{V}_\alpha : C(I) \to C(I) \) \((0 < \alpha \leq 1)\):

\[
(\mathcal{V}_\alpha u)(t) := \int_0^t (t - s)^{-\alpha} K(t, s) u(s) \, ds \quad \text{if } 0 < \alpha < 1,
\]
and

\[
(\mathcal{V}_1 u)(t) := \int_0^t \log(t - s) K(t, s) u(s) \, ds \quad \text{if } \alpha = 1.
\]

**Assume:** \( K \in C(D), \ K(t, t) \neq 0 \) \((t \in I := [0, T])\).

**Remark:** These singularities \((t - s)^{-\alpha}\) and \(\log(t - s)\) are called weak singularities (or: integrable singularities) because their integrals over \([0, T]\) are finite.

We study the *collocation solution* \( u_h \in S_{m-1}(-1)(I_h) \) and the corresponding *iterated* collocation solution for the VIE

\[
u(t) = g(t) + (\mathcal{V}_\alpha u)(t), \quad t \in I,
\]
for \( \alpha \in (0, 1) \) (algebraic singularity) and \( \alpha = 1 \) (logarithmic singularity).
Regularity of solutions of VIEs with weakly singular kernels

\[ K_\alpha(t, s) := (t - s)^{-\alpha}K(t, s) \quad (0 < \alpha \leq 1). \]

**Theorem 5.1:** (Theorem 1.9 / Lecture 1)
Let \(0 < \alpha < 1\). Assume that \(g \in C^d(I), \ K \in C^d(D)\) for some \(d \geq 0\).

(a) If \(d = 0\) the VIE

\[ u(t) = g(t) + \int_0^t K_\alpha(t, s)u(s) \, ds, \quad t \in I, \]
possesses a unique solution \(u \in C(I)\). This solution has the representation

\[ u(t) = g(t) + \int_0^t R_\alpha(t, s)g(s) \, ds, \quad t \in I, \]
where the resolvent kernel \(R_\alpha(t, s)\) of the kernel \(K_\alpha(t, s)\) has the form

\[ R_\alpha(t, s) = (t - s)^{-\alpha}Q_\alpha(t, s). \]

Here, \(Q_\alpha(t, s)\) is **continuous** on \(D\).

(b) For \(d \geq 1\) every nontrivial solution has the property that \(u \in C(I) \cap C^d(0, T)\): as \(t \to 0^+\) the solution behaves like

\[ u'(t) \sim Ct^{-\alpha}. \]
Logarithmic kernel singularity: $\alpha = 1$

$$u(t) = g(t) + \int_0^t \log(t - s)K(t, s)u(s)\, ds, \quad t \in I,$$

with $g \in C^d(I)$, $K \in C^d(D)$, $K(t, t) \neq 0 \ (t \in I)$.  

**Theorem 5.2:**
If $d = 0$ the VIE (6) possesses a unique solution $u \in C(I)$.
For $d \geq 1$ the solution of (6) has the property that

$$u \in C(I) \cap C^d(0, T],$$

with

$$u'(t) \sim C \log(t) \quad \text{as} \quad t \to 0^+.$$

**Exercise 5.1:**
Does the solution of the VIE (6) possess a resolvent representation similar to the one in Theorem 5.1; that is, does the resolvent kernel $R_1(t, s)$ of the kernel $\log(t - s)K(t, s)$ have the form

$$R_1(t, s) = \log(t - s)Q_1(t, s), \quad \text{with} \quad Q_1 \in C(D) ?$$
**Collocation and iterated collocation**

Collocation solution \( u_h \in S^{(-1)}_{m-1}(I_h) \) for the VIE

\[
u(t) = g(t) + \int_0^t K_\alpha(t,s)u(s) \, ds, \quad t \in [0,T],
\]

with weakly singular kernel

\[
K_\alpha(t,s) := \begin{cases}
(t-s)^{-\alpha}K(t,s) & \text{if } 0 < \alpha < 1 \\
\log(t-s)K(t,s) & \text{if } \alpha = 1.
\end{cases}
\]

\( \hookrightarrow \) **Collocation equation**: find \( u_h \in S^{(-1)}_{m-1}(I_h) \) so that

\[
u_h(t) = g(t) + \int_0^t K_\alpha(t,s)u_h(s) \, ds, \quad t \in X_h.
\]

\( \hookrightarrow \) **Collocation points**: (given \( 0 < c_1 < \cdots < c_m \leq 1 \))

\[
X_h := \{ t_n + c_i h_n : i = 1, \ldots, m \ (0 \leq n \leq N-1) \}.
\]

For \( t = t_n + v h_n \ (v \in (0,1]) \) we have

\[
u_h(t) = \sum_{j=1}^{m} L_j(v) U_{n,j}, \quad v \in (0,1],
\]

where the values \( U_{n,j} := u_h(t_n + c_j h_n) \) are unknown.

\( \hookrightarrow \) **Iterated collocation solution**:

\[
u_{h}^{it}(t) := g(t) + \int_0^t K_\alpha(t,s)u_h(s) \, ds, \quad t \in I.
\]
For \( n = 0, \ldots, N - 1 \) solve \((0 < \alpha < 1)\)

\[
[I_m - h_n^{1-\alpha} A_n(\alpha)] U_n = g_n + \sum_{\ell=0}^{n-1} h_\ell^{1-\alpha} A_n,\ell(\alpha) U_\ell
\]

(7)

(where \( 0 < \alpha < 1 \)) for \( U_n := (U_{n,1}, \ldots, U_{n,m})^T \in \mathbb{R}^m \).

Here \( I_m \) is the identity matrix in \( \mathbb{R}^{m \times m} \); \( A_n(\alpha), A_n,\ell(\alpha) \) are matrices in \( \mathbb{R}^{m \times m} \); \( U_{n,j} := u_h(t_n + c_j h_n) \), and \( g_n := (g(t_n + c_i h_n) (i = 1, \ldots, m))^T \) is in \( \mathbb{R}^m \).

The matrices \( A_n(\alpha) \) and \( A_n,\ell(\alpha) \) have the forms

\[
A_n(\alpha) := \left[ \int_0^{c_i} K_\alpha(t_{n} + c_i h_n, t_{n} + s h_n) L_j(s) \, ds \right]_{(i, j = 1, \ldots, m)}
\]

and, for \( 0 \leq \ell < n \),

\[
A_{n,\ell} := \left[ \int_0^1 K_\alpha(t_{n} + c_i h_n, t_\ell + s h_\ell) L_j(s) \, ds \right]_{(i, j = 1, \ldots, m)}
\]

where \( K_\alpha(t, s) := (t - s)^{-\alpha} K(t, s) \) \((0 < \alpha \leq 1)\).

(Exercise: Write down (7) for the case \( \alpha = 1 \) \((\logarithmic kernel singularity)\)!)}

When we know the solution \( U_n \) of (7), then the collocation solution on \( (t_n, t_{n+1}] \) is given by

\[
u_h(t_n + v h_n) = \sum_{j=1}^m L_j(v) U_{n,j}, \quad v \in (0, 1].\]
Recall (Lecture 4): For VIEs with $\alpha = 0$, the optimal orders of convergence of $u_h \in S_{m-1}^{(-1)}(I_h)$ and $u_{ht}^i$ on uniform meshes $I_h$ are given by

**Theorem 4.1:** (Global superconvergence on $I$)
(a) If $u \in C^d(I) \ (d \geq m)$, then
$$\|u - u_h\|_{\infty} \leq C h^m.$$  
(b) If $u \in C^d(I) \ (d \geq m + 1)$ and
$$J_0 := \int_0^1 \prod_{i=1}^m (s - c_i) \, ds = 0,$$
then
$$\|u - u_{ht}^i\|_{\infty} \leq C h^{m+1}.$$  

**Theorem 4.3:** (Local superconvergence on $I_h$)
Assume that $u \in C^d(I)$.
(a) If the $\{c_i\}$ are the Gauss points, and if $d \geq 2m$, then
$$\max_{1 \leq n \leq N} |u(t_n) - u_{ht}^i(t_n)| \leq C h^{2m}.$$  
(b) If the $\{c_i\}$ are the Radau II points, and if $d \geq 2m - 1$, then
$$\max_{1 \leq n \leq N} |u(t_n) - u_h(t_n)| \leq C h^{2m-1}.$$  

→ **Note:** The local superconvergence order is (much) higher than the global superconvergence order.
Let the mesh $I_h$ be uniform.

If $u_h \in S^{(-1)}_{m-1}(I_h)$ is the collocation solution for

$$u(t) = g(t) + \int_0^t K_\alpha(t,s)u(s)\,ds, \quad 0 < \alpha \leq 1,$$

and $u^{it}_h$ the \textit{iterated} collocation solution, what are the \underline{optimal} orders $p$ and $p^*$ in

$$\|u - u^{it}_h\|_\infty \leq Ch^p$$

and

$$\max_{1 \leq n \leq N} |U(t_n) - u^{it}_h(t_n)| \leq Ch^{p^*} \, ?$$

(\textit{Recall:} If $g$ and $K$ are smooth, the solution of the VIE has an \underline{unbounded derivative} at $t = 0^+$:

$$u'(t) \sim C \begin{cases} 
    t^{-\alpha} & \text{if } 0 < \alpha < 1, \\
    \log(t) & \text{if } \alpha = 1. 
\end{cases}$$

Since the optimal orders in \textbf{Theorem 4.1} and \textbf{Theorem 4.2} are only true if $u \in C^d$ on the \underline{closed} interval $[0, T]$, we cannot expect the same high orders when the mesh $I_h$ is \underline{uniform}.)

For $0 < \alpha < 1$ this is related to a result from \textbf{classical approximation theory} on the approximation of a function like $f(t) = t^\beta$ ($0 < \beta < 1$) on $[0, 1]$. 

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Uniform approximation of non-smooth functions:

**Theorem 5.3:**
Let \( f(t) = t^{\beta} \) \((0 < \beta < 1)\) and \( t \in [0, 1] \). Let \( f_h \in S_1^{(0)}(I_h) \) be the unique interpolant of \( f \) with respect to the points \( \{t_n\} \) of the mesh \( I_h := \{t_n : 0 = t_0 < t_1 < \cdots < t_N = 1\} \).

(a) If \( I_h \) is a **uniform** mesh \(( h = 1/N \)), then

\[
\|f - f_h\|_{\infty} \leq CN^{-\beta}
\]

(b) If \( I_h \) is a **graded** mesh with mesh points given by

\[
t_n = \left(\frac{n}{N}\right)^r, \ \text{with} \ r = r(\beta) = 2/\beta,
\]

then

\[
\|f - f_h\|_{\infty} \leq C(r)N^{-2}.
\]

(References: See for example


Graded meshes on \( I = [0, T] \):

**Definition:**
A mesh \( I_h = \{ t_n : 0 = t_0 < \cdots < t_N = T \} \) is called a **graded mesh** if the mesh points are given by

\[
t_n = \left( \frac{n}{N} \right)^r T \quad (n = 0, 1, \ldots, N),
\]

with **grading exponent** \( r > 1 \).

**Theorem 5.4:**
Assume that \( g \in C^m(I) \), \( K \in C^m(D) \) in

\[
u(t) = g(t) + \int_0^t (t - s)^{-\alpha} K(t, s) u(s) \, ds, \quad t \in I,
\]

and \( 0 < \alpha < 1 \). Let the solution \( u \) be approximated by collocation in \( S^{(-1)}_{m-1}(I_h) \).

(a) If \( I_h \) is a **uniform** mesh (\( r = 1 \)), then

\[
\| u - u_h \|_\infty \leq Ch^{1-\alpha}.
\]

(b) If \( I_h \) is a **graded** mesh with \( r \geq m/(1 - \alpha) \), then

\[
\| u - u_h \|_\infty \leq CN^{-m}.
\]

(\textit{Brunner (1985, 2004); B., Pedas \\& Vainikko (1999))}
Convergence for logarithmic kernel singularity 

(Recall: Graded meshes on \( I = [0, T] \) : 

\[ \rightarrow \ \text{A mesh } I_h = \{ t_n : 0 = t_0 < \cdots < t_N = T \} \text{ is called a } \textbf{graded mesh} \text{ if the mesh points are given by} \]

\[ t_n = \left( \frac{n}{N} \right)^r T \quad (n = 0, 1, \ldots, N), \]

with \textbf{grading exponent } \( r > 1 \).

\[
\textbf{Theorem 5.5:} \\
\text{Assume that } g \in C^m(I), \ K \in C^m(D) \text{ in } \\
u(t) = g(t) + \int_0^t \log(t - s)K(t, s)u(s) \, ds. \\
\text{Let the solution } u \text{ be approximated by collocation in } S^{(-1)}_{m-1}(I_h). \\
(a) \text{If } I_h \text{ is a } \textbf{uniform} \text{ mesh, then} \\
\|u - u_h\|_{\infty} \leq Ch(1 + |\log(h)|). \\
(b) \text{If } I_h \text{ is a } \textbf{graded} \text{ mesh with} \\
r \begin{cases} > 1 & \text{if } m = 1 \\ \geq m & \text{if } m \geq 2, \end{cases}
\text{then} \\
\|u - u_h\|_{\infty} \leq CN^{-m}. \\
\text{The global convergence orders in Theorems 5.4 and 5.5 hold for any choice of the collocation parameters } \{c_i\}. \\
(\text{Brunner, Pedas & Vainikko (1999)})
Is superconvergence on \( I \) or \( I_h \) possible?

\[ \leftrightarrow \text{Yes, but for } 0 < \alpha < 1 \text{ the optimal order is only } p^* = m + 1 - \alpha ! \]

**Theorem 5.6:** *(Superconvergence on \( I \) and \( I_h \))

Assume:

(a) \( g \in C^d(I), \ K \in C(D) \ (d \geq m+1), \ 0 < \alpha < 1. \)

(b) \( J_0 := \int_0^1 \prod_{i=1}^{m} (s - c_i) \ ds = 0 . \)

(c) \( u_h \in S_m^{-1}(I_h) , \) with corresponding *iterated* collocation solution \( u_h^{it} . \)

- If \( I_h \) is a uniform mesh \( (r = 1) \), then
  \[ \|u - u_h^{it}\|_{\infty} \leq CN^{-2(1-\alpha)}. \]

- If \( I_h \) is a graded mesh with \( r \geq m/(1 - \alpha) \), then
  \[ \|u - u_h^{it}\|_{\infty} \leq CN^{-(m+1-\alpha)}. \]

- At the mesh points \( t_n \ (1 \leq n \leq N) \) of the graded mesh \( I_h \) the optimal order of local superconvergence for \( u_h^{it} \) on \( I_h \) is the same as its global order of superconvergence on \( I \).

*(Brunner, Pedas & Vainikko (1999)).*
Implicitly linear collocation
(for nonlinear VIE of Hammerstein type (VHIE)):

\[ u(t) = g(t) + \int_0^t K_\alpha(t,s)G(s,u(s)) \, ds, \quad t \in I, \]

with

\[ K_\alpha(t,s) := \begin{cases} 
(t-s)^{-\alpha}K(t,s) & \text{if } 0 \leq \alpha < 1, \\
\log(t-s)K(t,s) & \text{if } \alpha = 1,
\end{cases} \]

and smooth \( G(s,u) \).

If we set \( z(t) := G(t,u(t)) \) then this VHIE may be written as the pair of equations (8),(9):

\[ z(t) = G \left( t, g(t) + \int_0^t K_\alpha(t,s)z(s) \, ds \right), \quad t \geq 0, \quad (8) \]

and

\[ u(t) = g(t) + \int_0^t K_\alpha(t,s)z(s) \, ds, \quad t \geq 0. \quad (9) \]

(I) Collocation solution \( z_h \in S_{m-1}^{-1}(I_h) \) for solution \( z \) of (8). Then:

(II) Approximation \( w_h \) to \( u \) in (9):

\[ w_h(t) := g(t) + \int_0^t K_\alpha(t,s)z_h(s) \, ds. \]

Question: Direct collocation (and iterated collocation) vs. implicitly linear collocation: \( \leftrightarrow \) Numerical comparison? (See also: Brunner (1991).)
Open problem:
Convergence analysis of collocation solution $u_h \in S_{m-1}^{-1}(I_h)$ and corresponding iterated collocation solution $u_{h}^{\text{it}}$ for the weakly singular non-standard VHIEs

$$u(t) = g(t) + \int_{0}^{t} K_{\alpha}(t,s)G(u(t),u(s))\,ds$$
and

$$u(t) = g(t) + \int_{0}^{t} K_{\alpha}(t,s)G(u(t-s),u(s))\,ds$$
(with $0 < \alpha \leq 1$)?
Concluding remarks
The following papers describe some alternative approaches for solving VIEs with weakly singular kernels:

- **Non-polynomial** spline collocation:

- **Hybrid** collocation schemes:
Cao, Herdman & Xu (2003), Huang & Xu (2006)

- **Smoothing transformations**
Basic references:


(↩ See also the handout "References: Lecture VI" for additional papers and books on collocation and related methods for Volterra integral equations with weakly singular kernels.)
Lecture 6 of

Theory and numerical solution of Volterra functional integral equations

Hermann Brunner
Department of Mathematics and Statistics
Memorial University of Newfoundland
St. John’s, NL
Canada

Department of Mathematics
Hong Kong Baptist University
Hong Kong SAR
P.R. China
Volterra functional integral equations
(VFIEs: VIEs with delays)
Volterra integral operators with delay functions:

- \((\mathcal{V}_\theta u)(t) := \int_0^{\theta(t)} K_1(t,s)u(s) \, ds, \quad t \in I\)
- \((\mathcal{W}_\theta u)(t) := \int_{\theta(t)}^t K(t,s)u(s) \, ds, \quad t \in I\)

(where \(I := [0,T]\).) The delay function (or: lag function) \(\theta\) has the form

\[\theta(t) := t - \tau(t), \quad \tau(t) \geq 0.\]

We shall refer to \(\tau = \tau(t)\) as the delay.

- **Non-vanishing delay:** \(\tau(t) \geq \tau_0 > 0\) for \(t \in I\).
- **Vanishing delay:** \(\tau(0) = 0, \quad \tau(t) > 0\) for \(t > 0\).
- **State-dependent delay:** \(\tau = \tau(t, u(t))\)
Recall:

$$(\mathcal{V}u)(t) := \int_0^t K(t, s)u(s) \, ds,$$

$$(\mathcal{V}_\theta u)(t) := \int_0^{\theta(t)} K(t, s)u(s) \, ds,$$

$$(\mathcal{W}_\theta u)(t) := \int_{\theta(t)}^t K(t, s)u(s) \, ds.$$

- **Second-kind Volterra functional integral equations (VFIEs):**

  $$u(t) = g(t) + (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t).$$

  $\hookrightarrow$ **Special cases:**

  $$u(t) = g(t) + (\mathcal{V}_\theta u)(t),$$

  $$u(t) = g(t) + (\mathcal{W}_\theta u)(t).$$

  Also:

  $$u(t) = b(t)u(\theta(t)) + g(t) + (\mathcal{W}_\theta u)(t).$$

  (Denisov & Lorenzi (1997) $\hookrightarrow$ Ill-posed problem!)

- **First-kind VFIEs:**

  $$(\mathcal{W}_\theta u)(t) = g(t), \quad t \in I.$$

  $\hookrightarrow$ **Special case:** (Volterra (1897))

  $$\theta(t) = qt \quad \text{with} \quad 0 < q < 1$$

  (proportional delay).
I. VFIEs with non-vanishing delays

Assume: The delay function \( \theta(t) = t - \tau(t) \)
is such that:
(D1) \( \tau(t) \geq \tau_0 > 0 \) for \( t \in I := [0, T] \);
(D2) \( \theta \) is strictly increasing on \( I \);
(D3) \( \tau \in C^d(I) \) for some \( d \geq 0 \).

Definition:
The points \( \{\xi_{\mu}\} \) generated by

\[ \theta(\xi_{\mu}) = \xi_{\mu - 1}, \quad \mu \geq 1 \quad (\xi_0 := 0), \]

are called the **primary discontinuity points**
(or: **breaking points**) induced by the delay function \( \theta \).

\( \hookrightarrow \) By (D1):
\[ \xi_{\mu} - \xi_{\mu - 1} \geq \tau_0 \quad (\mu \geq 1). \]

(Assume without loss of generality:
\( T \) is such that \( T = \xi_{M+1} \) for some \( M \geq 1 \).)

**Example 1:** Constant delay \( \tau > 0 \):
\[ \Rightarrow \xi_{\mu} = \mu \cdot \tau \quad (\mu = 0, \ldots, M + 1). \]

**Example 2:** Non-vanishing proportional delay,
for \( t \in [t_0, T] \) with \( t_0 > 0 \):

\[ \theta(t) = qt = t - (1 - q)t \quad \Rightarrow \quad \tau(t) \geq (1 - q)t_0 > 0. \]

\[ \Rightarrow \quad \xi_{\mu} = q^{-\mu}t_0 \quad (\mu = 0, \ldots, M + 1). \]
Remarks:

● Computation of breaking points $\xi_\mu$:
In general, the equation

$$\theta(\xi_\mu) = \xi_{\mu-1} \quad (\mu = 0, 1, \ldots; \xi_0 = t_0)$$
cannot be solved exactly $\leftrightarrow$ Numerical computation?
(Guglielmi & Hairer (2009); also: Bellen & Zennaro (2003))

● Variable delays $\tau(t)$ for which $\tau(t) \geq \tau_0 > 0$
is replaced by $\tau(t) > 0$? (clustering of breaking points \{\xi_\mu\} ?)

● Piecewise continuous delay functions:
Cooke & Wiener (1991): piecewise constant $\theta(t)$;
Liang & Brunner (2010): piecewise linear $\theta(t)$.
Recursive solution: ‘Method of steps’
(Solve VFIE on \( \mathbf{I}^{(\mu)} := [\xi_\mu, \xi_{\mu+1}], \; \mu = 0, \ldots, M \).)

**Illustration:** Consider the VFIE

\[
\begin{align*}
    u(t) &= g(t) + \int_{\theta(t)}^{t} K(t, s) u(s) \, ds, \quad t \in [0, T] \\
\end{align*}
\]

with initial condition \( u(t) = \phi(t) \) for \( t \leq 0 \).

- Solve a sequence of VIEs:
  - For \( t \in \mathbf{I}^{(0)} := [0, \xi_1] \) (\( \Rightarrow \theta(t) \in [\theta(0), 0] \)):
    \[
    u(t) = \Phi_0(t) + g(t) + \int_{0}^{t} K(t, s) u(s) \, ds,
    \]
    where
    \[
    \Phi_0(t) := \int_{\theta(t)}^{0} K(t, s) \phi(s) \, ds
    \]
    is known.
  - For \( t \in \mathbf{I}^{(\mu)} := [\xi_\mu, \xi_{\mu+1}] \) \( (\mu = 1, \ldots, M) \):
    \[
    u(t) = \Phi_\mu(t) + g(t) + \int_{\xi_{\mu}}^{t} K(t, s) u(s) \, ds,
    \]
    with known
    \[
    \Phi_\mu(t) := \int_{\theta(t)}^{\xi_{\mu}} K(t, s) u(s) \, ds.
    \]

**Question:**
**Regularity** of solution \( u(t) \) at \( t = \xi_\mu \) ?
Regularity: VFIEs with non-vanishing delays

- Assume that $\theta(t) = t - \tau(t)$ satisfies:
  
  (D1) $\tau(t) \geq \tau_0 > 0$ for $t \in I := [0, T]$;
  
  (D2) $\theta$ is strictly increasing on $I$;
  
  (D3) $\tau \in C^d(I)$, with arbitrarily large $d$.

**Theorem 6.1:** (Smoothing of solutions)

Assume that the given functions $g, K, \theta, \phi$ in the VFIE

$$u(t) = g(t) + \int_0^{\theta(t)} K(t, s)u(s) \, ds, \quad t \in I,$$

$$u(t) = \phi(t) \text{ if } t \leq 0,$$

are arbitrarily smooth. Then:

(a) The VFIE has a unique solution $u \in C([0, T])$.

In general, $u$ has a (finite) discontinuity at $t = \xi_0 = 0$.

(b) On each (left-open) subinterval $(\xi_\mu, \xi_{\mu+1}]$ the solution is arbitrarily smooth.

(c) At the breaking points $t = \xi_\mu$ with $\mu \geq 1$ the regularity of $u$ is described by

$$u \in C^{\mu-1} \quad \text{but} \quad u \not\in C^\mu.$$

(Brunner (2004))
**Regularity:** VFIEs with vanishing delays

- Assume that on $I = [0, T]$, $\theta(t)$ satisfies:
  (D1) $\theta(0) = 0$, $\theta(t) \leq \bar{q}t$ for some $\bar{q} \in (0, 1)$;
  (D2) $\theta$ is **strictly increasing** on $I$;
  (D3) $\theta \in C^d(I)$ for arbitrarily large $d \geq 0$.

**Theorem 6.2:** ($\hookrightarrow$ **Globally smooth solutions**)

Assume that the given functions $g$, $K$, $\theta$ in the VFIE

$$u(t) = g(t) + \int_0^{\theta(t)} K(t, s)u(s) \, ds, \quad t \in I := [0, T],$$

are in $C^d$ for some $d \geq 0$, and $\theta(t)$ satisfies (D1) and (D2). Then:

(a) The VFIE has a unique solution $u \in C^d(I)$.

The solution is **globally smooth**: there are no **breaking points**!

(b) This solution is given by

$$u(t) = g(t) + \sum_{j=1}^{\infty} \int_0^{\theta_j(t)} K_j(t, s)g(s) \, ds, \quad t \in I$$

(where $\theta_j := \theta \circ \cdots \circ \theta$. The **iterated kernels** are defined by $K_1(t, s) := K(t, s)$ and

$$K_{j+1}(t, s) := \int_{\theta-j(s)}^{\theta(t)} K(t, v)K_j(v, s) \, dv \ (j \geq 1).$$
Solution representation for VFIE with vanishing $\theta(t)$:

$$u(t) = g(t) + \sum_{j=1}^{\infty} \int_{0}^{\theta^j(t)} K_j(t,s)g(s) \, ds, \quad t \in I$$

(where $\theta^j := \theta \circ \cdots \circ \theta$).

- The solution representation given in Theorem 6.2 is due to Andreoli (1914), Chambers (1990) (for $\theta(t) = qt$) and Brunner & Hu (2005) (for general nonlinear $\theta(t)$). It can be proved by using Picard iteration, as for VIEs.

- Special case: $\theta(t) = qt$ ($0 < q < 1$)

Here, $\theta^j(t) = q^j t$ and $\theta^{-j}(s) = q^{-j} s$.

- If $\theta(t) = t$ ($q = 1$):

$$\Rightarrow u(t) = g(t) + \sum_{j=1}^{\infty} \int_{0}^{t} K_j(t,s)g(s) \, ds$$

$$= \int_{0}^{t} \left( \sum_{j=1}^{\infty} K_j(t,s) \right) g(s) \, ds,$$

with resolvent kernel given by the Neumann series (Lecture 1),

$$R(t,s) := \sum_{j=1}^{\infty} K_j(t,s).$$
Remarks:

• Theorem 6.2(a) (*global smoothness of solution*) remains true for the VFIE

\[ u(t) = g(t) + \int_{\theta(t)}^{t} K(t,s)u(s) \, ds. \]

• **Open problem:** If the nonlinear vanishing delay function \( \theta(t) \) satisfies only assumptions (D1) (\( \theta(0) = 0, \theta(t) \leq \bar{q}t \) for some \( \bar{q} \in (0,1) \)) and (D3) (\( \theta(t) \) is smooth), but \( \theta(t) \) is not strictly increasing):

\[ \leftrightarrow \quad \text{Representation of solution and regularity analysis?} \]
Collocation solutions for VFIEs

(I) **Non-vanishing** delays (↔ *Illustration*):

\[ u(t) = g(t) + \int_{\theta(t)}^{t} K(t, s)u(s) \, ds, \quad t \in [0, T], \]

with \( u(t) = \phi(t) \ (t \leq 0), \ \theta(t) = t - \tau(t) \ (\tau(t) \geq \tau_0 > 0). \)

**Primary discontinuity points:** \( \{\xi_{\mu} : \mu \geq 0\} \)

↔ *Assume:* \( T = \xi_{M+1} \) for some \( M \geq 1 \).

Let \( I^{(\mu)} := [\xi_{\mu}, \xi_{\mu+1}] \ (0 \leq \mu \leq M). \)

**Definition:** A mesh \( I_{h} \) on \( I = [0, T] \) is called **a constrained mesh** if \( I_{h} := \bigcup_{\mu=0}^{M} I_{h}^{(\mu)}, \) with the **local meshes** \( I_{h}^{(\mu)} \) (on \( I^{(\mu)} = [\xi_{\mu}, \xi_{\mu+1}] \)) given by

\[ I_{h}^{(\mu)} := \{t_{n}^{(\mu)} : \xi_{\mu} = t_{0}^{(\mu)} < t_{1}^{(\mu)} < \cdots < t_{N}^{(\mu)} = \xi_{\mu+1}\}. \]

Let \( h_{n}^{(\mu)} := t_{n+1}^{(\mu)} - t_{n}^{(\mu)}, \ h^{(\mu)} := \max_{(n)}\{h_{n}^{(\mu)}\}. \)

↔ The **collocation points** \( X_{h} := \bigcup_{\mu=0}^{M} X_{h}^{(\mu)} \)

are defined by the **local** sets

\[ X_{h}^{(\mu)} := \{t_{n}^{(\mu)} + c_{i}h_{n}^{(\mu)} : i = 1, \ldots, m; \ 0 \leq n \leq N-1\}, \]

corresponding to given \( 0 < c_{1} < \cdots < c_{m} \leq 1 \).
Local representation of collocation solution \( u_h \in S_{m-1}^{(-1)}(I_h) \) on \((t_n^{(\mu)}, t_{n+1}^{(\mu)})\):

\[
u_h(t_n^{(\mu)} + v h_n^{(\mu)}) = \sum_{j=1}^{m} L_j(v) U_{n,j}^{(\mu)}, \ v \in (0, 1),\]

where \( U_{n,j}^{(\mu)} := u_h(t_n^{(\mu)} + c_j h_n^{(\mu)}) \).

For \(\mu = 0, \ldots, M\): generate \( u_h \in S_{m-1}^{(-1)}(I_h) \) recursively on \( I(\mu) \):

\[
u_h(t) = g(t) + \int_{\theta(t)}^{t} K(t, s) u_h(s)) \) ds, \ t \in X_h^{(\mu)}.
\]

If \(\mu = 0\) then

\[
u(\theta(t_n^{(0)} + c_i h_n^{(0)})) = \phi(\theta(t_n^{(0)} + c_i h_n^{(0)})).
\]

**Definition:** A constrained mesh \( I_h \) is called a \(\theta\)-invariant mesh if the local meshes \( I_h^{(\mu)} \) are such that

\[
\theta(I_h^{(\mu)}) = I_h^{(\mu-1)} \text{ for } \mu = 1, \ldots, M.
\]

If \(\theta\) is linear:

\[
\theta(t_n^{(\mu)} + c_i h_n^{(\mu)}) = t_n^{(\mu-1)} + c_i h_n^{(\mu-1)} \quad (\mu \geq 1).
\]

If \(\theta\) is nonlinear:

\[
\theta(t_n^{(\mu)} + c_i h_n^{(\mu)}) = t_n^{(\mu-1)} + \tilde{c}_i h_n^{(\mu-1)}
\]

for some \(0 < \tilde{c}_1 < \cdots < \tilde{c}_m \leq 1\).
\( \textbf{Collocation solution} \ u_h \in S_{m-1}^{(-1)}(I_h) : \)

Local representation of \( u_h \) on \((t_n^{(\mu)}, t_{n+1}^{(\mu)})\) is given by

\[
u_h(t_n^{(\mu)} + vh_n^{(\mu)}) = \sum_{j=1}^{m} L_j(v)U_{n,j}^{(\mu)}, \quad v \in (0, 1].
\]  \( (10) \)

Set \( t_{n,i}^{(\mu)} := t_n^{(\mu)} + c_i h_n^{(\mu)} \); substitute (10) into the collocation equation \( u_h(t) = g(t) + \int_{\theta(t)}^t K(t,s)u_h(s) \, ds \)

\( \Rightarrow \) System of linear algebraic equations for \( U_n^{(\mu)} := (U_{n,1}^{(\mu)}, \ldots, U_{n,m}^{(\mu)})^T : \)

\[
U_{n,i}^{(\mu)} = g(t_{n,i}^{(\mu)}) + \Psi_{n,i}^{(\mu)}
\]

\[+ h_n^{(\mu)} \int_{0}^{c_i} K(t_{n,i}^{(\mu)}, t_n^{(\mu)} + sh_n^{(\mu)})u_h(t_n^{(\mu)} + sh_n^{(\mu)}) \, ds \]

\((i = 1, \ldots, m; \quad n = 0, \ldots, N - 1), \) with

\[
\Psi_{n,i}^{(\mu)} := \int_{\theta(t_{n,i}^{(\mu)})}^{t_{n,i}^{(\mu)}} K(t_{n,i}^{(\mu)}, s)u_h(s) \, ds.
\]

If the mesh \( I_h \) is \( \theta \)-invariant and \( \theta \) is linear:

\[
u_h(\theta(t_{n,i}^{(\mu)})) = u_h(t_{n,i}^{(\mu)-1}).
\]

\( \leftarrow \textbf{Iterated} \) collocation solution:

\[
u_{h}^{\text{it}}(t) := g(t) + \int_{\theta(t)}^t K(t,s)u_h(s) \, ds, \quad t \in I.
\]
Some details:
For $t = t_n^{(\mu)} + \nu h_n^{(\mu)} \ (\nu \in (0,1) \Rightarrow \theta(t) \in (t_n^{(\mu-1)}, t_n^{(\mu-1)})]$ the VFIE

$$u(t) = g(t) + \int_{\theta(t)}^{t} K(t,s) u(s) \, ds$$

may be written as

$$u(t) = g(t) + \Phi_{\mu}(t) + \int_{\xi_{\mu}}^{t} K(t,s) u(s) \, ds, \quad t \in I^{(\mu)},$$

(11)

where

$$\Phi_{\mu}(t) := \int_{\theta(t)}^{\xi_{\mu}} K(t,s) u(s) \, ds.$$

$\hookrightarrow$ The collocation equation at $t_n^{(\mu)} + c_i h_n^{(\mu)}$ ($i = 1, \ldots, m; \ 0 \leq n \leq N - 1$) corresponds to the collocation equation for the VIE (11) on the interval $I^{(\mu)} = [\xi_{\mu}, \xi_{\mu} + 1]$!

$\Rightarrow$ Find the collocation solution for the VFIE (11) on each $I_{h}^{(\mu)} \ (\mu = 0, 1, \ldots, M)$.

$\hookrightarrow$ Collocation equations and (super-) convergence analysis as in Lecture 4!
**Optimal global convergence estimates:**

**Assume:** The delay function \( \theta(t) = t - \tau(t) \) satisfies:

1. \( \tau(t) \geq \tau_0 > 0, \ t \in I := [0, T]; \)
2. \( \theta \) is strictly increasing on \( I; \)
3. \( \tau \in C^d(I) \) for some \( d \geq 0. \)

**Theorem 6.3:** ((Brunner (1994, 2004)))

Suppose that the mesh \( I_h \) is \( \theta \)-invariant, and let \( u_h \in S_{m-1}(-1)(I_h) \) be the collocation solution for the VFIE.

(a) For arbitrary collocation parameters \( \{c_i\} \):

\[
\|u - u_h\|_\infty \leq Ch^m \quad (h := \max_{(\mu)}\{h(\mu)\}).
\]

(b) If the collocation parameters \( \{c_i\} \) satisfy

\[
J_0 := \int_0^1 \prod_{i=1}^m (s - c_i) \, ds = 0,
\]

then

\[
\|u - u_h^{it}\|_\infty \leq Ch^{m+1}.
\]

Here, the iterated collocation solution is defined by

\[
u_h^{it}(t) := g(t) + (Vu_h)(t) + (V\theta u_h)(t), \ t \in I.
\]
Summary: Constrained and \( \theta \)-invariant meshes

- **Primary discontinuity points** (or: breaking points) \( \{\xi_\mu\} \) induced by the delay function \( \theta \):
  \[
  \theta(\xi_\mu) = \xi_{\mu-1} \quad (\mu \geq 1, \; \xi_0 := 0),
  \]
  with \( \xi_\mu - \xi_{\mu-1} \geq \tau_0 > 0 \) for all \( \mu \geq 1 \).

  - Assume: \( T = \xi_{M+1} \) for some \( M \geq 1 \).

- **Definition:**
  A mesh \( I_h \) on \( I := [0, T] \) is called a constrained mesh if it contains the primary discontinuity points \( \{\xi_\mu\} \) induced by \( \theta \); that is,
  \[
  I_h := \bigcup_{\mu=0}^{M} I_h^{(\mu)}
  \]
  is defined by the local meshes
  \[
  I_h^{(\mu)} := \{t_n^{(\mu)} : \xi_\mu = t_0^{(\mu)} < t_1^{(\mu)} < \cdots < t_N^{(\mu)} = \xi_{\mu+1}\}.
  \]

- **Definition:**
  A constrained mesh \( I_h \) is said to be \( \theta \)-invariant if the local meshes \( I_h^{(\mu)} \) satisfy
  \[
  \theta : \; I_h^{(\mu)} \rightarrow I_h^{(\mu-1)} \quad \text{for} \quad \mu = 1, \ldots, M;
  \]
  that is, if
  \[
  \theta(t_n^{(\mu)}) = t_n^{(\mu-1)} \quad (n = 0, 1, \ldots, N)
  \]
  for \( \mu = 1, \ldots, M \).
VFIEs with **non-vanishing** delays:

→ **Local superconvergence results**

**Theorem 6.4:** (Brunner (2004))

If the delay function $\theta(t) = t - \tau(t)$ satisfies

(D1) $\tau(t) \geq \tau_0 > 0$, $t \in I := [0, T]$;
(D2) $\theta$ is **strictly increasing** on $I$;
(D3) $\tau \in C^d(I)$ for some $d \geq 0$;

and if the given functions (including the initial function $\phi$) are sufficiently smooth, then the collocation solution $u_h \in S^{(-1)}_{m-1}(I_h)$ and the corresponding **iterated** collocation solution $u_{h}^{it}$ for the VFIE possess the **same optimal order** of **local superconvergence** on $I_h$ as the ones for the **classical** (non-delay) VIE if, and only if, the underlying mesh $I_h$ is $\theta$-**invariant**.

**Example:** If the $\{c_i\}$ are the **Gauss** (-Legendre) points:

$$\max_{1 \leq n \leq N} |u(t_n^{(\mu)}) - u_{h}^{it}(t_n^{(\mu)})| \leq Ch^{2m}.$$ 

For the **Radau II** points:

$$\max_{1 \leq n \leq N} |u(t_n^{(\mu)}) - u_{h}(t_n^{(\mu)})| \leq Ch^{2m-1}.$$ 

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Collocation solutions for VFIEs

(II) VFIEs with vanishing delay functions:

\[ u(t) = g(t) + (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t), \]
\[ u(t) = g(t) + (\mathcal{W}_\theta u)(t), \]

where \( t \in I := [0, T] \).

Volterra integral operators \((C(I) \to C(I))\):

\[ (\mathcal{V}u)(t) := \int_0^t K_0(t, s)u(s)\,ds, \]
\[ (\mathcal{V}_\theta u)(t) := \int_0^{\theta(t)} K_1(t, s)u(s)\,ds, \]
\[ (\mathcal{W}_\theta u)(t) := \int_{\theta(t)}^t K(t, s)u(s)\,ds. \]

Assume that the delay function \( \theta \) satisfies:

(D1) \( \theta(0) = 0, \ \theta(t) \leq \bar{q}t \) for some \( \bar{q} \in (0, 1) \);
(D2) \( \theta \) is strictly increasing on \( I \);
(D3) \( \theta \in C^d(I) \) for some \( d \geq 0 \).

Also:

\[ u(t) = b(t)u(\theta(t)) + g(t) + (\mathcal{W}_\theta u)(t). \]

(Current work with Xie Hehu and Zhang Ran)
**Illustration:** VFIE with $\theta(t) = qt$ ($0 < q < 1$)

$$u(t) = g(t) + \int_0^{qt} K(t, s) u(s) \, ds, \quad t \in [0, T].$$

Since the solution is *smooth* on $I$ (there are no breaking points $\xi_\mu$!), we choose a uniform mesh $I_h$ for the collocation solution $u_h \in S_{m-1}(I_h)$:

$$u_h(t) = g(t) + (\mathcal{V}_\theta u)(t), \quad t \in X_h.$$ 

Location of points $\theta(t_n + c_i h)$?

Define (for $\theta(t) = qt$, $0 < q < 1$):

$$q^I := \left\lceil \frac{q}{1-q} c_1 \right\rceil, \quad q^{II} := \left\lceil \frac{q}{1-q} c_m \right\rceil.$$

For the collocation points $t = t_n + c_i h \in e_n$, the images $q(t_n + c_i h)$ satisfy:

- **Phase I:** $0 \leq n < q^I$

  $$q(t_n + c_i h) \in (t_n, t_{n+1}) \quad \text{for all} \quad i = 1, \ldots, m.$$

- **Phase II:** $q^I \leq n < q^{II}$

  $$q(t_n + c_i h) \leq t_n \quad (i = 1, \ldots, \nu_n)$$

  for some $\nu_n \in \{1, \ldots, m-1\}$.

- **Phase III:** $q^{II} \leq n \leq N - 1$

  $$q(t_n + c_i h) \leq t_n \quad \text{for all} \quad i = 1, \ldots, m.$$
Collocation solutions for VFIEs

\[ u(t) = g(t) + (V u)(t) + (V_\theta u)(t), \quad t \in I, \quad (12) \]

with delay function \( \theta(t) = qt \) (0 < q < 1).

Collocation equation \( \Rightarrow \) Systems of linear algebraic equations for \( U_n := (U_{n,1}, \ldots, U_{n,m})^T \)
in the local representation of \( u_h \) on \((t_n, t_{n+1}]\),

\[ u_h(t) = \sum_{j=1}^{m} L_j(v) U_{n,j}, \quad v \in (0, 1], \]
of the collocation solution \( u_h \in S^{(-1)}_{m-1}(I_h) \) (with uniform mesh \( I_h \)) for the VFIE (12).

- **Phase I:** \( 0 \leq n < q^I := \left\lceil \frac{q}{1-q}c_1 \right\rceil \)
  \[ [I_m - h(\mathcal{A}_n + B_n^I(q))] U_n = H_n. \]
- **Phase II:** \( q^I \leq n < q^{II} := \left\lceil \frac{q}{1-q}c_m \right\rceil \)
  \[ [I_m - h(\mathcal{A}_n + B_n^{II}(q))] U_n = H_n + h\tilde{\beta}^{II}_n(q) U_{n-1}. \]
- **Phase III:** \( q^{II} \leq n \leq N - 1 \)
  \[ [I_m - h\mathcal{A}_n] U_n = H_n + h\tilde{\beta}^{III}_n(q) U_{\tilde{n}}, \]

for some \( \tilde{n} = \tilde{n}(q) < n \). (\( H_n : \) History of \( \mathcal{V} \) for \( t \in [0, t_n] \).)
Recall: Linear algebraic systems for $U_n = (U_{n,1}, \ldots, U_{n,m}^T)$:

**Phase I:**

$$[\mathcal{I}_m - h(\mathcal{A}_n + B^I_n(q))]U_n = H_n.$$  

**Phase II:**

$$[\mathcal{I}_m - h(\mathcal{A}_n + B^{II}_n(q))]U_n = H_n + h\tilde{B}^{II}_n(q)U_{n-1}.$$  

**Phase III:**

$$[\mathcal{I}_m - h\mathcal{A}_n]U_n = H_n + h\tilde{B}^{III}_n(q)U_{\tilde{n}},$$  

for some $\tilde{n} = \tilde{n}(q) \leq n - 1.$

When $q(t_n + c_i h) > t_n$ the matrices $B^I_n(q)$, $B^{II}_n(q)$ represent the contributions of the delay integral $(V_\theta u_h)(t)$; for $q(t_n + c_i h) \leq t_n$ the contributions are described by the matrices $\tilde{B}^{II}_n(q)$, $\tilde{B}^{III}_n(q).$ For example,

$$B^I_n(q) := \left[ \int_0^{\gamma_{n,i}} K_1(t_{n,i}, t_n + s h)L_j(s) \, ds \right]_{(i,j = 1, \ldots, m)}$$

(where $q(t_n + c_i h) = t_{q,n,i} + \gamma_{n,i} h$).

Remark:

(Comparison with classical VIEs (Lecture 4))

In **Phase III**, the matrix $\mathcal{I}_m - h\mathcal{A}_n$ on the left-hand side of the linear system for $U_n$ is the same as for VIEs without delay.
Comparison: **Classical VIE** (no delay):

→ For each \( n = 0, \ldots, N - 1 \), solve the linear system

\[
[I_m - h_n A_n] U_n = g_n + \sum_{\ell=0}^{n-1} h_\ell A_{n,\ell} U_\ell =: H_n,
\]

for \( U_n := (U_{n,1}, \ldots, U_{n,m})^T \in \mathbb{R}^m \), where

\( U_{n,j} := u_h(t_n + c_j h_n) \) and

\( g_n := (g(t_n + c_i h_n) \ (i = 1, \ldots, m))^T \in \mathbb{R}^m \).

The matrices \( A_n \) and \( A_{n,\ell} \) have the forms

\[
A_n := \left[ \int_0^{c_i} K(t_n + c_i h_n, t_n + sh_n) L_j(s) \, ds \right],
\]

\( (i, j = 1, \ldots, m) \)

and, for \( 0 \leq \ell < n \),

\[
A_{n,\ell} := \left[ \int_0^1 K(t_n + c_i h_n, t_\ell + s h_\ell) L_j(s) \, ds \right].
\]

(13)

\( (i, j = 1, \ldots, m) \)

When we know the solution \( U_n \) of (13), then the collocation solution on \( [t_n, t_{n+1}] \) is given by

\[
u_h(t_n + v h_n) = \sum_{j=1}^m L_j(v) U_{n,j}, \quad v \in (0, 1).
\]
VFIEs with vanishing delays: Local superconvergence on uniform $I_h$

Collocation solution $u_h \in S_{m-1}^{-1}(I_h)$ (with uniform mesh $I_h$ on $I = [0, T]$) for

$$u(t) = g(t) + (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t).$$

Delay function: $\theta(t) = qt = t - (1-q)t \ (0 < q < 1)$.

$\Rightarrow$ Special case: $K_1 = -K_0 =: -K :$

$$u(t) = g(t) + (\mathcal{W}_\theta u)(t),$$

where

$$(\mathcal{W}_\theta u)(t) := \int_{\theta(t)}^{t} K(t,s)u(s) \, ds.$$

**Theorem 6.5:** (Brunner & Hu (2005))

If the $\{c_j\}$ satisfy $\int_0^1 \prod_{i=1}^{m} (s - c_i) \, ds = 0$ (e.g.: Gauss or Radau II points), then

$$\|u - u_{hit}\|_\infty \leq Ch^{m+1} \ (m \geq 2),$$

and

$$\max_{1 \leq n \leq N} |u(t_n) - u_{hit}(t_n)| \leq C \begin{cases} h^{2m} & \text{if } m = 1, 2 \\ h^{m+2} & \text{if } m > 2 \text{ and } q = 1/2 \\ h^{m+1} & \text{otherwise.} \end{cases}$$

$\Rightarrow$ Comparison: For $q = 1$ (classical VIEs),

$$\max_{1 \leq n \leq N} |u(t_n) - u_{hit}(t_n)| \leq Ch^{2m}$$

for all $m \geq 1$, if the $\{c_i\}$ are the Gauss points.
Proof of Theorem 6.5
(based on representation of collocation error):
The collocation error \( e_h := u - u_h \) for
\[
    u(t) = g(t) + (\mathcal{V}u)(t) + (\mathcal{V}_\theta u)(t), \ t \in I,
\]
with vanishing delay function \( \theta(t) \) (e.g. \( \theta(t) = qt \)) satisfies the VFIE
\[
e_h(t) = \delta_h(t) + (\mathcal{V}e_h)(t) + (\mathcal{V}_\theta e_h)(t), \ t \in [0, T].
\]
The defect function \( \delta_h(t) \) is piecewise smooth, with \( \delta_h(t) = 0 \) for all \( t \in X_h \).
For \( \mathcal{V} \equiv 0 \) the solution of the error equation is given by (Brunner & Hu (2005))
\[
e_h(t) = \delta_h(t) + \sum_{j=1}^{\infty} \int_0^{\theta_j(t)} K_j(t, s) \delta_h(s) \, ds, \ t \in [0, T],
\]
where the iterated kernels \( K_j(t, s) \) of \( K_1(t, s) \) in \( \mathcal{V}_\theta \) are smooth. For \( t = t_n \) (uniform mesh \( I_h \)), \( \theta_j(t_n) = t_{q_n,j} + \gamma_{n,j}h \), where
\[
    q_{n,j} := \lfloor \theta_j(t_n)/h \rfloor \in \mathbb{N},
\]
and
\[
    \gamma_{n,j} := \theta_j(t_n)/h - q_{n,j} \in [0, 1).
\]
For $\theta(t) = qt$, $t = t_n = nh$ ($1 \leq n \leq N$):

$$e_h(t_n) = \delta_h(t_n) + \sum_{j=1}^{\infty} \int_0^{q^j t_n} K_j(t_n, s) \delta_h(s) \, ds.$$ 

Also:

$$e^{i t h}(t_n) = \sum_{j=1}^{\infty} \int_0^{q^j t_n} K_j(t_n, s) \delta_h(s) \, ds.$$ 

(Recall that $\delta_h(t) = 0$ for $t \in X_h$.)

Since $\theta_j(t_n) = t_{q_{n,j}} + \gamma_{n,j} h$, we have

$$\int_0^{q^j t_n} K_j(t_n, s) \delta_h(s) \, ds = \int_0^{t_{q_{n,j}}} K_j(t_n, s) d_h(s) \, ds$$

$$+ h \int_0^{\gamma_{n,j}} K_j(t_n, t_{q_{n,j}} + sh) \delta_h(t_{q_{n,j}} + sh) \, ds.$$ 

(etc.)
Collocation on (quasi-) geometric meshes

• Non-vanishing delay techniques:
  On $[0, t_0]$ (with suitable small $t_0 = t_0(q; N) > 0$), assume given initial approximation to $u(t)$.
  \[ 
  \begin{align*}
  \text{Choose geometric macro-mesh given by} \\
  \{\xi_\mu := q^{\kappa-\mu}T : 0 \leq \mu \leq \kappa\}, \quad \kappa = \kappa(q; N),
  \end{align*}
  \]
with appropriate $\kappa$ such that $\xi_0 := t_0 \to 0$ as $N \to \infty$. \[ \text{Local (uniform) meshes:} \]
  \[ 
  I_h^{(\mu)} := \{t_n^{(\mu)} : \xi_\mu = t_0^{(\mu)} < t_1^{(\mu)} < \cdots < t_N^{(\mu)} = \xi_{\mu+1}\}. 
  \]
  \[ \Rightarrow \text{Collocation solution } u_h \in S_m(I_h^{(\kappa-1)}) \text{ (for the Gauss points, with global } \theta \text{-invariant mesh } I_h := \bigcup_{\mu=0}^{\kappa-1} I_h^{(\mu)} \text{) for the VFIDE} \]
  \[ u'(t) = a(t)u(t) + b(t)u(\theta(t)) + (\nabla u)(t) + (\nabla \theta u)(t) : \]
  \[ \Rightarrow \max_{t \in I_h} |u(t) - u_h(t)| \leq C^*_m(q)N^{-2m}. \]
  \[ \text{(Bellen, Brunner, Maset & Torelli (2006))} \]

\[ \text{For VFIEs: convergence analysis on quasi-geometric meshes remains open.} \]
Vanishing delay techniques: (Brunner, Hu & Lin (2001), B. & Hu (2007))

Global mesh:

\[ I_h := \{ t_n = t_n^{(N)} := d^{n}T : 0 \leq n \leq N \}, \]

with suitably chosen \( d = d(q; m, N) \in (0, 1) \).

Collocation in \( S_m^{(0)}(I_h) \) for VFIDEs,

\[ u'(t) = a(t)u(t) + b(t)u(\theta(t)) + (Vu)(t) + (V_\theta u)(t), \]

using the Gauss points, yields

\[ \max_{t \in I_h} |u(t) - u_h(t)| \leq C^*_m(q)N^{-(2m-\epsilon_N)}, \]

where \( \epsilon_N \to 0 \), as \( N \to \infty \).
Questions:
• Numerical comparison of collocation solutions on quasi-geometric meshes (approach of Bellen et al.) and on geometric meshes (approach of Brunner & Hu)?
• Analysis of collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ and corresponding iterated collocation solution $u_{h}^{it}$ based on geometric meshes for VFIEs with vanishing delay functions?

Remark:
The variable stepsize code **RADAR5** (Guglielmi & Hairer (2001); see www.unige.ch/~hairer), when applied to delay differential equations with delay function $\theta(t) = qt$ ($0 < q < 1$), generates meshes $I_h$ with stepsize sequences \{$h_n$\} that appear to show exponential-like growth (Guglielmi (2006)).
Remarks:

- Superconvergence analysis for nonlinear delay functions that are not strictly increasing: 

- VFIEs with *multiple delays*:
  \[ \theta_j(t) = q_j t \quad (0 < q_1 < \cdots < q_r < 1) : \]

  \[ u(t) = g(t) + \sum_{j=1}^{r} \int_{0}^{\theta_j(t)} K_j(t, s) u(s) \, ds, \quad t \in [0, T]. \]

  (Brunner (2009))

- More general VIEs with vanishing delays:

  \[ u(t) = b(t) u(\theta(t)) + g(t) + (\mathcal{W}_\theta u)(t) \]

  (current work with Xie Hehu and Zhang Ran).

- VFIEs with *weakly singular kernels*:

  \[ (\mathcal{W}_{\theta, \alpha} u)(t) := \int_{\theta(t)}^{t} (t - s)^{-\alpha} K(t, s) u(s) \, ds \]

  with \( 0 < \alpha < 1 \):

  (i) *Non-vanishing delays*: Brunner (2004: Section 6.5

  (ii) *Vanishing delays*: \( \leftrightarrow \) Ongoing work: Brunner & Bai.
Basic references:


(↩ See also the handout “References: Lecture VI” for additional papers and books on collocation and related methods for DDEs and Volterra functional integral equations.)
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Lecture 7 of

Theory and numerical solution of Volterra functional integral equations

Hermann Brunner

Department of Mathematics and Statistics
Memorial University of Newfoundland
St. John’s, NL
Canada

Department of Mathematics
Hong Kong Baptist University
Hong Kong SAR
P.R. China
Lecture 7:
Additional topics and future research

- VIEs with power-law nonlinearities
- Spectral collocation methods for VIEs
- Systems of integral-algebraic VIEs
- Singularity perturbed VIEs
- VIEs of the third kind
VIEs with **power-law nonlinearity:**

The existence and uniqueness of *non-trivial* solutions of the nonlinear VIE

\[ u^\beta(t) = \int_0^t (t - s)^{-\alpha} K(t, s) u(s) \, ds, \quad t \in [0, 1], \tag{14} \]

with \( \beta > 1, \ 0 \leq \alpha < 1, \ K(t, s) \geq 0, \) was studied by Buckwar (1997, 2005): in addition to the solution \( u(t) \equiv 0 \) the VIE may possess a positive solution.

**Open problems:**

(i) Assume that (1) has a unique *positive* solution \( u(t) \). Can this solution be found by collocation in \( S_{m-1}^{(-1)}(I_h) \) or in \( S_m^{(0)}(I_h) \)? Does the collocation equation

\[ u_h^\beta(t) = \int_0^t (t - s)^{\alpha} K(t, s) u_h(s) \, ds, \quad t \in X_h, \]

always (i.e. for every mesh with mesh diameter \( h > 0 \)) have a positive solution \( u_h(t) \)?

(ii) What happens if in (1), \( \beta \in (0, 1), \) with \( \beta = 1/q \) (\( q \in \mathbb{N} \))?
Spectral collocation methods for VIEs
(Tang, Xu & Chen (2008), Chen & Tang (2010))

In a spectral collocation method the exact solution $u(t)$ of the VIE

$$u(t) = g(t) + \int_{-1}^{t} K(t, s) u(s) \, ds, \quad t \in [-1, 1]$$

(15)

(the standard interval for orthogonal polynomials like Legendre or Jacobi polynomials) is approximated by a *global* polynomial $U^N(t)$ of degree $N$ (with suitably large $N$) that is forced to satisfy (2) at some distinct points $\{t_i\}_{i=0}^{N}$ in $[-1, 1]$. These points are usually chosen to be the Gauss(-Legendre) points, the Radau points, or the Lobatto points) in $[-1, 1]$.

If the solution of (2) is in $C^d[-1, 1]$) for some (large) $d$, then one typically obtains

$$\|u - U^N\|_\infty \leq CN^{\mu-d} \quad \text{(with } \mu \in \{1/2, 3/4\})$$

(Tang, Xu & Cheng (2008)).

If the kernel contains the *weakly singular* factor $(t - s)^{-\alpha}$ $(0 < \alpha < 1)$, the $\{t_i\}_{i=0}^{N}$ are chosen as the zeros of some Jacobi polynomial (Chen & Tang (2010)).
Remark:
In the practical application of the spectral collocation method, polynomials $U^N(t)$ of degree up to about $N = 40$ are used. The *numerical comparison* (computational efficiency) of this method with *collocation* in $S_{m-1}^{(-1)}(I_h)$ (and corresponding *iterated collocation*) remains to be carried out.
Integral-algebraic VIEs and VIDEs
(ODEs, VIEs and VIDEs with non-local constraints)

Illustrations:

• ODE with non-local (integral) constraint:

\[ u'(t) = F(t, u(t), w(t)), \quad t \in I := [0, T], \]

\[ 0 = g(t) + \int_0^t k(t - s)G(s, u(s), w(s)) \, ds, \quad t \in I, \]

with \( g(0) = 0 \).

• Second-kind VIE with non-local constraint (I):

\[ u(t) = f(t) + \int_0^t [K_{11}(t, s)u(s) + K_{12}(t, s)w(s)] \, ds, \]

\[ 0 = g(t) + \int_0^t K_{22}(t, s)w(s) \, ds, \quad t \in I, \]

where \( g(0) = 0, \ g \in C^1(I); \ |K_{22}(t, t)| \geq \kappa_0 > 0 \) and \( K_{22} \in C^1(D) \).

• Second-kind VIE with non-local constraint (II):

\[ u(t) = f(t) + \int_0^t [K_{11}(t, s)u(s) + K_{12}(t, s)w(s)] \, ds, \]

\[ 0 = g(t) + \int_0^t K_{21}(t, s)u(s) \, ds, \quad t \in I, \]

where \( g(0) = 0, \ g \in C^1(I); \ |K_{21}(t, t)| \geq \kappa_0 > 0 \) and \( K_{21} \in C^1(D) \).
Collocation for IAEs and IDAEs

Let

\((V_{k,\ell}w)(t) := \int_0^t K_{k,\ell}(t,s)w(s)\,ds \quad (k, \ell = 1, 2)\).

(I) System of integral-algebraic Volterra equations (IAEs):

\[ u(t) = f(t) + (V_{1,1}u)(t) + (V_{1,2}z)(t) \]

\[ 0 = g(t) + (V_{2,1}u)(t) + (V_{2,2}z)(t), \]

where \(u(\cdot) \in \mathbb{R}^{d_1}, z(\cdot) \in \mathbb{R}^{d_2} \quad (d_1, d_2 \geq 1).\)

(II) System of integro-differential-algebraic Volterra equations (IDEAs):

\[ u'(t) + B_{1,1}(t)u(t) + B_{1,2}(t)z(t) \]

\[ = f(t) + (V_{1,1}u)(t) + (V_{1,2}z)(t) \]

\[ 0 = g(t) + (V_{2,1}u)(t) + (V_{2,2}z)(t), \]

where \(u(\cdot) \in \mathbb{R}^{d_1}, z(\cdot) \in \mathbb{R}^{d_2}.\)

Collocation approximations for the solutions \((u(t), z(t))\) of the above systems of Volterra IAEs and IDAEs?
**IAEs:** Convergence of collocation solutions for

\[ u(t) = A_1(t) + (\mathcal{V}_{1,1} u)(t) + (\mathcal{V}_{1,2} z)(t) \]  
\[ 0 = A_2(t) + (\mathcal{V}_{2,1} u)(t) + (\mathcal{V}_{2,2} z)(t). \]

**Assume:** The system of IAEs has **index 1** (i.e. the first-kind integral equation (4) is **uniquely** solvable for \( z(t) \) on \( I = [0, T] \)).

**Theorem 1:** (Kauthen (2001))

Let the solution \( u, z \) of (3),(4) be approximated by the collocation solutions \( u_h, z_h \) in \( S_{m-1}(-1)(I_h) \), with collocation points \( \{c_i\} \) given by the **Radau II** points. Then:

\[ \max_{1 \leq n \leq N} \{|u(t_n) - u_h(t_n)|\} \leq C_1 h^{2m-1}, \]
\[ \max_{1 \leq n \leq N} \{|z(t_n) - z_h(t_n)|\} \leq C_2 h^m. \]

**Remark:** Example of IAE system with **index 2:**

\( \mathcal{V}_{2,2} = 0 \) and (4) is uniquely solvable for \( u(t) \).

\( \rightarrow \) Liang & Brunner (2010)
**IDAEs:** Convergence of collocation solutions for

\[ u'(t) + B_{1,1}(t)u(t) + B_{1,2}(t)z(t) = f(t) + (\mathcal{V}_{1,1}u)(t) + (\mathcal{V}_{1,2}z)(t) \]

\[ 0 = g(t) + (\mathcal{V}_{2,1}u)(t) + (\mathcal{V}_{2,2}z)(t). \]

**Assume:** The system of IDAEs has index 1.

**Theorem 2:** (B. (2004))

(a) Let \( u \) and \( z \) be approximated by the collocation solutions \( u_h \in S_m^0(I_h) \) and \( z_h \in S_{m-1}^{(-1)}(I_h) \), respectively. If the \( \{c_i\} \) are the Radau II points, then

\[
\max_{1 \leq n \leq N} \{|u(t_n) - u_h(t_n)|\} \leq C_1 h^{2m-1}
\]

\[
\max_{1 \leq n \leq N} \{|z(t) - z_h(t)|\} \leq C_2 h^m.
\]

(b) Let the solution \( u, z \) be approximated by the collocation solutions \( u_h, z_h \) in \( S_m^0(I_h) \). If the \( \{c_i\} \) are the Gauss points, then \( u_h, z_h \) are **divergent** as \( h \to 0 \).
**Singularly perturbed Volterra equations**

\[ \varepsilon u'(t) = g(t) + \int_0^t K(t,s)G(s,u(s)) \, ds, \quad t \in I, \]

and

\[ \varepsilon u(t) = g(t) + \int_0^t K(t,s)G(s,u(s)) \, ds, \quad t \in I, \]

with \( 0 < \varepsilon \ll 1 \) ⇔ \( \varepsilon = 0 \): Reduced equation:

\[ 0 = g(t) + \int_0^t K(t,s)G(s,u(s)) \, ds, \quad t \in I. \]

→ Volterra integral equation of the first kind.

**Illustration:** (Kauthen (1997))

Consider the singularly perturbed VIE

\[ \varepsilon u(t) = \sin(t) - \int_0^t u(s) \, ds, \quad t \in [0, \pi/2]. \quad (18) \]

The reduced equation \((\varepsilon = 0)\) is the first-kind VIE

\[ 0 = \sin(t) - \int_0^t u_0(s) \, ds. \]

Its solution is \( u_0(t) = \cos(t), \) with \( u_0(0) = 1. \)

If \( 0 < \varepsilon \ll 1 \) the solution of (5) satisfies \( u(0) = 0 \)

and has the form

\[ u(t) = \frac{1}{1 + \varepsilon^2} [\cos(t) + \varepsilon \sin(t) - e^{-t/\varepsilon}]. \]

→ **Observation:** \( u_0(t) \) does not uniformly approximate \( u(t) \) near \( t = 0^+ \), since \( u(t) \) contains a boundary layer term, \( e^{-t/\varepsilon}. \)
Collocation solutions for singularly perturbed VIEs:

Choice of *collocation points* \( \{ c_i \} \) and *meshes* \( I_h \)? (Note that as \( \varepsilon \to 0^+ \), the singularly perturbed VIE or VIDE assumes the character of a *first-kind* VIE!)

*Recall* (Lecture 4):

**Theorem:** (Kauthen & B. (1997))

Let \( u_h \in S_m^{(0)}(I_h) \) be the collocation solution to

\[
\int_0^t K(t,s)u(s)\,ds = g(t), \quad t \in I,
\]

with \( 0 < c_1 < \cdots < c_m \leq 1 \) and uniform mesh \( I_h \). If \( c_m = 1 \), then

\[
\lim_{h \to 0} \| u - u_h \|_\infty = 0 \quad \text{iff} \quad \prod_{i=1}^{m-1} \frac{1 - c_i}{c_i} \leq 1.
\]

\( \Rightarrow \) Collocation solution \( u_h \) corresponding to the *Radau II* points is *divergent* (as \( h \to 0 \))!

\( c_m < 1 \): If the \( \{ c_i \} \) are the *Gauss* points, then \( u_h \in S_m^{(0)}(I_h) \) is *divergent*, too.

\( \leftarrow \) **Open problem**: Analysis (including the selection of the mesh \( I_h \)) of the convergence properties of collocation solutions in \( S_{m-1}^{(-1)}(I_h) \) for *singularly perturbed* VIEs (and VIDEs) with kernels \( K_\alpha = (t-s)^{-\alpha}K(t,s) \) \( (0 \leq \alpha < 1) \)?

(Ongoing work (DG method): **Tao Xia** (Hunan Normal University))
VIEs of the **third kind**
*(Pereverzev & Prössdorf (1997), Berg & von Wolfersdorff (2005))*

Example:

\[
a(t)u(t) = g(t) + \int_0^t (t - s)^{-\alpha}K(t, s)u(s)\,ds, \quad t \in I,
\]

with \(0 \leq \alpha < 1\), \(a, g \in C(I)\), \(K \in C(D)\). The function \(a(t)\) vanishes either at one point in \(I\) (e.g. at \(t = 0\)) or on a subset of some ‘small’ subinterval \(I_0 := [\tau_0, \tau_1] \setminus \{0, T\}\).

This problem is **ill-posed**, and thus some **regularization technique** has to be employed for computing numerical approximations to the exact solution \(u(t)\).

**Open problem:** Construction of high-order (regularized) methods (e.g.: collocation or discontinuous Galerkin methods?) for (6)?
(If (6) is a system of VIEs, with **singular matrix** \(a(t)\): connection with system of **IAEs**?)
Basic references:


(↩ See also the handout ”References: Lecture VII” for additional papers on topics described in Lecture 7 and on related open problems.)