STAT 3820 Homework 3

3.1 Assume that a decision maker’s current wealth is 10,000. Assign $u(0) = -1$ and $u(10,000) = 0$.

a. When facing a loss of $X$ with probability 0.5 and remaining at current wealth with probability 0.5, the decision maker would be willing to pay up to $G$ for complete insurance. The values of $X$ and $G$ in three situations are given below:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10,000</td>
<td>6000</td>
</tr>
<tr>
<td>6000</td>
<td>3300</td>
</tr>
<tr>
<td>3300</td>
<td>1700</td>
</tr>
</tbody>
</table>

Determine three values on the decision maker’s utility of wealth function $u$.

b. Calculate the slopes of the four line segments joining the five points determined on the graph $u(w)$. Determine the rates of change of the slopes from segment to segment.

c. Put yourself in the role of a decision maker with wealth 10,000. In addition to the given values of $u(0)$ and $u(10,000)$, elicit three additional values on your utility of wealth function $u$.

d. On the basis of the five values of your utility function, calculate the slopes and the rates of change of the slopes as done in part (b).

3.2 A utility function is given by

$$u(w) = \begin{cases} 
  e^{-(w-100)^2/200}, & w < 100 \\
  2 - e^{-(w-100)^2/200}, & w \geq 100 
\end{cases}$$

a) Is $u'(w) \geq 0$?

b) For what range of $w$ is $u''(w) < 0$?

3.3 A decision maker has utility function $u(w) = k \log w$. The decision maker has wealth $w, w > 1$, and face a random loss $X$, which has a uniform distribution on the interval $(0,1)$. Use formula $u(w - G) = E[u(w - X)]$ to show that the maximum insurance premium that the decision maker will pay for complete insurance is

$$G = w - \frac{w^w}{e(w-1)^{w-1}}.$$ 

3.4 The decision maker has a utility function $u(w) = -e^{-\alpha w}$ and faced with a random loss that has a chi-square distribution with $n$ degrees of freedom. If $0 < \alpha < 1/2$, use formula $u(w - G) = E[u(w - X)]$ to obtain an expression for $G$, the maximum insurance premium the decision maker will pay, and prove $G > n = \mu$. 

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3.5 a. An insurer with net worth 100 has accepted (and collected the premium for) a risk \( X \) with the following probability distribution:

\[
\Pr(X = 0) = \Pr(X = 51) = \frac{1}{2}.
\]

What is the maximum amount \( G \) it should pay another insurer to accept 100% of this loss? Assume the first insurer’s utility function of wealth is \( u(w) = \log w \).

b. An insurer, with wealth 650 and the same utility function \( u(w) = \log w \), is considering accepting the above risk. What is the minimum amount \( H \) this insurer would accept as premium to cover 100% of the loss.

3.6 Obtain the mean and variance of the claim random variable \( X \) where \( q = 0.05 \) and the claim amount random variable \( B \) is uniformly distribution between 0 and 20.

3.7 Let \( X \) be the number of heads observed in five tosses of a true coin. Then, \( X \) true dice are thrown. Let \( Y \) be the sum of the numbers showing on the dice. Determine the mean and variance of \( Y \).

3.8 The probability of a fire in a certain structure in a given time period is 0.02. If a fire occurs, the damage to the structure is uniformly distributed over the interval \((0, a)\) where \( a \) is its total value. Calculate the mean and variance of fire damage to the structure within the time period.

3.9 A fire insurance company covers 160 structures against fire damage up to an amount stated in the contract. The number of contracts at the different contract amounts are give below.

<table>
<thead>
<tr>
<th>Contract Amount</th>
<th>Number of Contracts</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 000</td>
<td>80</td>
</tr>
<tr>
<td>20 000</td>
<td>35</td>
</tr>
<tr>
<td>30 000</td>
<td>25</td>
</tr>
<tr>
<td>50 000</td>
<td>15</td>
</tr>
<tr>
<td>100 000</td>
<td>5</td>
</tr>
</tbody>
</table>

Assume that for each of the structures, the probability of one claim within a year is 0.04, and the probability of more than one claim is 0. Assume that fire in the structures are mutually independent events. Furthermore, assume that the conditional distribution of the claim size, given that a claim has occurred, is uniformly distributed over the interval from 0 to the contract amount. Let \( N \) be the number of claims and let \( S \) be the amount of Claims in a 1-year period.

a. Calculate the mean and variance of \( N \).

b. Calculate the mean and variance of \( S \).

c. What relative security loading, \( \theta \), should be used to so the company can collect an amount equal to the 99th percentile of the distribution of total claim? (Use a normal approximation.)
Consider a portfolio of 32 policies. For each policy, the probability \( q \) of a claim is \( 1/6 \) and \( B \), the benefit amount given that there is a claim, has p.d.f.

\[
f(y) = \begin{cases} 
2(1 - y), & 0 < y < 1 \\
0, & \text{elsewhere.}
\end{cases}
\]

Let \( S \) be the total claims for portfolio. Using a normal approximation, estimate \( \Pr(S > 4) \).