More on the Generalized Fibonacci Numbers and Associated Bipartite Graphs

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Abstract
For a positive integer \( k \geq 2 \), the \( k \)-Fibonacci sequence \( \{ f_{n}^{(k)} \} \) is defined by
\[
f_{n}^{(k)} = f_{n-1}^{(k)} + f_{n-2}^{(k)} + \cdots + f_{n-k}^{(k)},
\]
for \( n \geq k \), with initial value \( f_{0}^{(k)} = f_{1}^{(k)} = \cdots = f_{k-2}^{(k)} = 0 \), \( f_{k-1}^{(k)} = 1 \). For a fixed \( \alpha = (a_{1}, a_{2}, \ldots, a_{m}) \), the \((k, \alpha)\)-sequence is defined by
\[
s_{(\alpha)}^{(k)}(n) = \sum_{i=1}^{m} a_{i} f_{n-k+i}^{(k)}
\]
for \( k \geq 2 \), \( m \geq 1 \) and \( n \geq 1 \). In this paper, we consider the relationship between \( s_{(\alpha)}^{(k)} \) and perfect matchings of a bipartite graph.

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1. Introduction

Let \( A = (a_{i,j}) \) be a square matrix of order \( n \) over a ring \( R \). The permanent of \( A \) is defined by
\[
\text{per}(A) = \sum_{\sigma \in S_{n}} \prod_{i=1}^{n} a_{i, \sigma(i)},
\]
where \( S_{n} \) denotes the symmetric group on \( n \) letters. It is easy to see that for any square matrix \( A \) and any permutation matrices \( P \) and \( Q \), \( \text{per}(A) = \text{per}(PAQ) \). Let
A_{i,j} \text{ be the matrix obtained from a square matrix } A = (a_{i,j}) \text{ by deleting the } i\text{-th row and the } j\text{-th column. Then it is also easy to see that } \text{per}(A) = \sum_{k=1}^{n} a_{i,k} \text{per}(A_{i,k}) = \sum_{k=1}^{n} a_{k,j} \text{per}(A_{k,j}) \text{ for any } i,j. \\

In this paper, all undefined terminologies and symbols of graph can be found in [1]. Let \( G \) be a bipartite graph with bipartition \((X,Y)\). If \( G \) contains a perfect matching, then \(|X| = |Y|\). Let \( A \) be an adjacency matrix of \( G \). It is known [7] that the number of perfect matchings (or 1-factor) of \( G \) is \( \sqrt{\text{per}(A)} \). Namely, if \(|X| = |Y| = n\) then \( A = \begin{pmatrix} O & B \\ B^T & O \end{pmatrix} \) for some square matrix \( B \) of order \( n \), where \( O \) is the zero matrix of order \( n \). Such matrix \( B \) is called a bipartite adjacency matrix. We shall denote the graph \( G \) as \( G(B) \). Note that the matrix \( B \) is not unique. The number of perfect matchings of \( G(B) \) is \( \text{per}(B) \), see [7].

Let \( \{F_n\} \) be the Fibonacci sequence, i.e., \( F_0 = 0, F_1 = 1 \) and \( F_n = F_{n-1} + F_{n-2} \) for \( n \geq 2 \).

The \textit{k-Fibonacci sequence} \( \{f_n^{(k)}\} \) for positive integer \( k \geq 2 \) is defined recursively by
\[
f_n^{(k)} = f_{n-1}^{(k)} + f_{n-2}^{(k)} + \cdots + f_{n-k}^{(k)}, \quad \text{for } n \geq k,
\]
with initial value \( f_0^{(k)} = f_1^{(k)} = \cdots = f_{k-2}^{(k)} = 0, f_{k-1}^{(k)} = 1 \). The number \( f_n^{(k)} \) is called the \textit{n-th k-Fibonacci number}. It is known that [6]
\[
f_j^{(k)} = 2^{j-k}, \quad \text{for } k \leq j \leq 2k-1.
\]

Note that \( \{f_n^{(2)}\} \) is the Fibonacci sequence.

The \textit{k-Lucas sequence} \( \{l_n^{(k)}\} \) is defined by \( l_{n}^{(k)} = f_{n-1}^{(k)} + f_{n-k}^{(k)} \) and \( l_n^{(k)} \) is called the \textit{n-th k-Lucas number}. It is known that \( l_j^{(k)} = 2^{j-1}, 1 \leq j \leq k-1 \), and \( l_{k}^{(k)} = 1 + 2^{k-1} \), see [6]. More about Lucas sequence can be found in [3]. Note that \( \{l_n^{(2)}\} \) is the Lucas sequence.

A matrix is said to be a \( (0,1)\)-matrix if each of its entries is either 0 or 1. Suppose \( n \) and \( k \) are positive integers. Let \( T_n = (t_{i,j}) \) be an \( n \times n \) tridiagonal \((0,1)\)-matrix, where \( t_{i,j} = 1 \) if and only if \( |j-i| \leq 1 \). Let \( U_n^{(k)} = (u_{i,j}) \) be an \( n \times n \) upper triangular \((0,1)\)-matrix, where \( u_{i,j} = 1 \) if and only if \( 2 \leq j-i \leq k-1 \) if \( k \leq n \) and \( u_{i,j} = U_{n}^{(n-k)} \), if \( k > n \). Let \( \mathcal{F}^{(n,k)} = T_n + U_n^{(k)} \) and let \( \mathcal{G}^{(n,k)} = \mathcal{F}^{(n,k)} + E_{1,k+1} - \sum_{j=2}^{k} E_{1,j} \) for \( n \geq 3 \), where \( E_{i,j} \) denotes the \( n \times n \) matrix with 1 at the \((i,j)\)-th entry and zeros elsewhere.

In [4, 5], Lee et al. found a class of bipartite graphs whose number of perfect matchings is \( f_n^{(k)} \) and prove the following result.

**Theorem 1.1:** For \( n \geq 2 \), the number of perfect matchings of \( G(\mathcal{F}^{(n,k)}) \) is \( f_{n-1+k}^{(k)} \).
In [2], Brualdi proved the following result:

**Theorem 1.2:** For \( n \geq 2 \), let \( A^{(n)} = I + U^{(n)} \). Then \( \text{per}(A^{(n)}) = 2^{n-1} \).

Making use of Theorem 1.2, Lee [6] proved the following result.

**Theorem 1.3:** For \( n \geq 3 \), the number of perfect matchings of \( G(\mathcal{G}^{(n,k)}) \) is \( l_{n-1}^{(k)} \).

In this paper, we shall show in Section 2 that the permanent of a special matrix is a linear combination of \( k \)-Fibonacci numbers. By making use of this result we obtain the number of perfect matchings of a larger class of bipartite graphs. Theorems 1.1 to 1.3 are special cases of this result. Moreover, in Section 3 we shall use this permanent to obtain the number of perfect matchings of certain bipartite graph which is not isomorphic to the graphs studied in [6].

2. Main results

For a fixed \( \alpha = (a_1, a_2, \ldots, a_m) \in R^m \), where \( R \) is a ring. We define the \((k, \alpha)\)-sequence by

\[
s(\alpha)_n^{(k)} = a_1 f_{n+k-2}^{(k)} + a_2 f_{n+k-3}^{(k)} + \cdots + a_m f_{n+k-m-1}^{(k)} = \sum_{i=1}^{m} a_i f_{n-1+k-i}^{(k)}, \quad k \geq 2, \ n \geq 1.
\]

The number \( s(\alpha)_n^{(k)} \) is called the \( n \)-th \((k, \alpha)\)-number. Note that, if \( \alpha = (1, \cdots, 1) \in \mathbb{Z}^k \), then \( s(\alpha)_n^{(k)} \) is the \((n-1+k)\)-th \( k \)-Fibonacci number \( f_{n-1+k}^{(k)} \); if \( \alpha = (1, 0, \cdots, 0, 1) \in \mathbb{Z}^{k+1} \), then \( s(\alpha)_n^{(k)} \) is the \((n-1)\)-st \( k \)-Lucas number \( l_{n-1}^{(k)} \).

**Theorem 2.1:** Suppose \( n, k \geq 2 \). Let

\[
B_n = \begin{pmatrix}
a_1 & a_2 & \cdots & a_n \\
1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0
\end{pmatrix},
\]

for some elements \( a_1, a_2, \ldots, a_n \) in a ring \( R \). Then \( \text{per}(B_n) = \sum_{i=1}^{n} a_i f_{n-1+k-i}^{(k)} \).

**Proof:** We shall prove the theorem by mathematical induction on \( n \). Since

\[
\text{per}(B_2) = a_1 + a_2 = a_1 f_k^{(k)} + a_2 f_{k-1}^{(k)},
\]

the theorem is true for \( n = 2 \).

Assume that the theorem is true for some \( n \geq 2 \). Expanding the permanent by the first column and by Theorem 1.1 and the induction assumption, we have
per(B_{n+1}) = a_1 \per(\mathcal{S}^{(n,k)}) + \per \left( \begin{array}{c|ccc} a_2 & a_3 & \cdots & a_{n+1} \\ \hline 1 & & & \\ 0 & & & \\ \vdots & & & \\ 0 & & & \mathcal{S}^{(n-1,k)} \end{array} \right)

= a_1 f_{n+k-1} + \sum_{i=1}^{n} a_i f_{n-1+k-i}

= a_1 f_{n+k-1} + \sum_{i=2}^{n+1} a_i f_{n+k-i} = \sum_{i=1}^{n+1} a_i f_{n+k-i}.

Thus, the theorem is true for each \( n \geq 2 \). \( \blacksquare \)

**Corollary 2.2:** For a fixed \( m \geq 1 \), suppose \( n, k \geq 2 \) and \( n \geq m \). Let

\[
\mathcal{S}^{(n,k)} = \begin{pmatrix} a_1 & a_2 & \cdots & a_m & 0 & \cdots & 0 \\ 0 & & & & \mathcal{S}^{(n-1,k)} & & \\ \vdots & & & & & & \\ 0 & & & \end{pmatrix}
\]

Then the number of perfect matching of \( G(\mathcal{S}^{(n,k)}_{\alpha}) \) is the \( n \)-th \((k, \alpha)\)-number with \( \alpha = (a_1, a_2, \ldots, a_m) \).

Applying Corollary 2.2 and Theorem 2.1 by choosing \( \alpha = (1, 1, \cdots, 1) \in \mathbb{Z}^k \) for \( n > k \) and by choosing \( a_i = 1 \) for all \( i = 1, 2, \ldots, n \) for \( n \leq k \) respectively, we get Theorem 1.1. Applying Corollary 2.2 and Theorem 2.1 by choosing \( \alpha = (1, 0, \cdots, 0, 1) \in \mathbb{Z}^{k+1} \) for \( n > k \) and by choosing \( a_1 = 1 \) and \( a_i = 0 \) for all \( i = 2, \ldots, n \) for \( n \leq k \) respectively, we get Theorem 1.3.

### 3. Other results

From Theorem 1.3, the number of perfect matchings of \( G(\mathcal{S}^{(n,2)}) \) is \( l_{n-1}^{(2)} \). In [6], there is a bipartite graph \( G \), which is not isomorphic to \( G(\mathcal{S}^{(n,2)}) \) and whose number of perfect matchings is also \( l_{n-1}^{(2)} \). Namely \( G = G(B^{(n)}) \), where \( B^{(n)} = T_n + E_{1,3} - E_{2,3} + E_{2,4} - E_{3,4} \) for \( n \geq 4 \). Now we shall show another bipartite graph whose number of perfect matchings is \( l_{n-1}^{(2)} \) too.

Let \( C^{(n)} = T_n - E_{2,3} + E_{1,5} \) for \( n \geq 5 \). It is easy to see that both \( G(C^{(n)}) \) and \( G(\mathcal{S}^{(n,2)}) \) contain exactly one vertex of degree 4 when \( n \geq 6 \). It is easy to show that \( G(C^{(6)}) \) is not isomorphic to \( G(C^{(6,2)}) \). Let \( a \) and \( b \) be the vertices of degree 4 in \( G(C^{(n)}) \) and \( G(\mathcal{S}^{(n,2)}) \) respectively. For \( n \geq 7 \), since \( b \) is adjacent to a vertex of degree 2 but \( a \) is not, \( G(C^{(n)}) \) is not isomorphic to \( G(\mathcal{S}^{(n,2)}) \). Since \( G(B^{(n)}) \) does not contain any vertex of degree 4 when \( n \geq 6 \), \( G(C^{(n)}) \) is not isomorphic to \( G(B^{(n)}) \). Note that \( G(C^{(5)}) \) is isomorphic to \( G(B^{(5)}) \).
Expanding the permanent by the first column and by Theorem 2.1 we have

\[
\text{per}(C^{(n)}) = \text{per}
\begin{pmatrix}
1 & 1 & 0 & 0 & 1 & 0 & 0 & \cdots \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix}
\]

\[
= \text{per}
\begin{pmatrix}
1 & 0 & 0 & \cdots \\
0 & \mathcal{F}(n-2,k) & \end{pmatrix}
+ \text{per}
\begin{pmatrix}
1 & 0 & 1 & 0 & \cdots \\
0 & \mathcal{F}(n-2,k) & \end{pmatrix}
\]

\[
= f_{n-1}^{(2)} + f_{n-1}^{(2)} + f_{n-4}^{(2)} = f_{n-1}^{(2)} + f_{n-2}^{(2)} + f_{n-3}^{(2)} + f_{n-4}^{(2)} = f_{n}^{(2)} + f_{n-2}^{(2)} = l_{n-1}^{(2)}
\]

Thus the number of perfect matchings of the bipartite graph \(G(C^{(n)})\) is \(l_{n-1}^{(2)}\).

**References**


