On the $\ell$-distance face coloring of regular plane graphs

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Abstract

The $\ell$-distance face chromatic number of a connected plane graph is the minimum number of colors in a coloring of its faces so that whenever two different faces are at distance $\ell$ or less, they receive different colors. In this paper, we estimate the $\ell$-distance face chromatic numbers for connected 6-regular plane graphs. Also, we have a general result on $n$-regular plane graphs with $n \geq 6$.

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1 Introduction

In this paper, all graphs $G = (V, E, F)$ are connected plane graphs with at least two vertices, loops and multiple edges are allowed, where $V$, $E$ and $F$ are the sets of vertices, edges and faces of $G$ respectively. We denote the numbers of its vertices, edges and faces by $\nu$, $\varepsilon$ and $\phi$ respectively.

Let $G = (V, E, F)$. The degree of a face $f$ of $G$, denoted by $d_G(f)$ (or simply $d(f)$), is the number of edges incident with $f$ (edges incident with exactly one face are counted twice). Suppose $g_1, g_2 \in F$ and $u \in V$. The

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distance between $u$ and $g_2$, denoted by $d_G(u,g_2)$ (or simply $d(u,g_2)$), is the minimum distance $d_G(u,x_2)$ of all vertices $x_2$ incident with $g_2$. The distance between $g_1$ and $g_2$, denoted by $d_G(g_1,g_2)$ (or simply $d(g_1,g_2)$), is the minimum distance $d_G(x_1,g_2)$ of all vertices $x_1$ incident with $g_1$.

For $\ell \geq 0$, an $\ell$-distance face $k$-coloring of a graph $G = (V,E,F)$ is a mapping $\varphi : F \to \{1, 2, \ldots, k\}$ such that if $d(g_1,g_2) \leq \ell$ for $g_1 \neq g_2$, then $\varphi(g_1) \neq \varphi(g_2)$. The $\ell$-distance face chromatic number $\chi_{df}^\ell(G)$ is the minimum $k$ such that there is an $\ell$-distance face $k$-coloring of $G$.

The special face coloring was originally studied for cubic plane graphs by Bouchet et al. [3] and Bordin [2] as the Heawood face coloring.

An Heawood-coloring (or $h$-coloring for short) of $F$ is a mapping $h : F \to \{1, 2, \ldots, k\}$ such that for each edge $e$, the faces incident with the ends of $e$ have pairwise different colors, which in fact is $1$-distance face coloring. In [4], the $h$-coloring was generalized and studied by Hornak and Jendrol for 4-regular plane graphs and prove that $\chi_{df}^1(G) \leq 21$ for any 4-regular plane graph $G$. In this paper, we shall study the $\ell$-distance face coloring for connected $n$-regular plane graphs $G$ with $n \geq 6$ and give the upper bound and lower bound of $\chi_{df}^\ell(G)$. For simplicity proof, we first proof the result on 6-regular connected graphs.

2 Lemmas

We need several auxiliary lemmas for our main theorem.

**Lemma 1:** If $f$ is a face of a 6-regular graph $G$, then the number of faces of $G$ at distance at most $\ell$ from $f$ is at most $1 + 4d(f)5^\ell$.

**Proof:** Let $v_i(f)$ be the number of vertices and $\phi_i(f)$ the number of faces at distance $i$ from $f$. Any face of $G$ at distance $i$ from $f$ is incident with a vertex at distance $i$ from $f$. If $x$ and $y$ are vertices of $G$ at distance $i$ and $i-1$ from $f$, respectively, where $i \geq 1$, and if $xy$ is an edge of $G$, then faces of $G$ incident with $xy$ are at distance at most $i-1$ from $f$. Hence at most four among the faces incident with $x$ are at distance $i$ from $f$ and we have $\phi_i(f) \leq 4v_i(f)$ for every $i \geq 1$. Similarly, we have $v_i(f) \leq 5v_{i-1}(f)$ for $i \geq 2$ and $v_1(f) \leq 4v_0(f) = 4d(f)$. Since $\phi_0(f) \leq 1 + 4d(f)$ (note that $f$ is at distance 0 from itself), the number of faces at distance at most $\ell$
from $f$ is
\[
\sum_{i=0}^{\ell} \phi_i(f) \leq 1 + 4d(f) + \sum_{i=1}^{\ell} 4v_i(f) \\
= 1 + 4d(f) + 4 \sum_{i=1}^{\ell} v_i(f) \\
\leq 1 + 4d(f) + 4 \sum_{i=1}^{\ell} 4d(f) 5^{i-1} \\
= 1 + 4d(f) + 4d(f)(5^\ell - 1) \\
= 1 + 4d(f) 5^\ell.
\]

By a similar proof, we obtain the following general result.

**Lemma 2 [4]:** If $\varphi$ is a partial $\ell$-distance face $k$-coloring of a connected plane graph $G = (V, E, F)$ such that the set $H$ of the uncolored faces is $(\ell, k)$-colorable, then $\varphi$ can be extended to an $\ell$-distance face $k$-coloring of $G$.

**Lemma 3 [1]:** If $G$ is a simple connected plane graph, then $\varepsilon \leq 3\nu - 6$.

### 3 The main result

**Theorem 4:** Let $\ell \geq 1$. Suppose $G = (V, E, F)$ is a $6$-regular graph. Then
\[
6 \leq \chi_6^\ell(G) \leq \max\{1 + 8 \times 5^\ell, 2(\nu - 2)\}.
\]

**Proof:** Let $F_k(G)$ be the set of all faces of degree $k$ in $G$ and let $f_k(G)$ be its cardinality. Let $r = \max\{1 + 8 \times 5^\ell, \sum_{k=3}^{\infty} f_k(G)\}$. It is clear that there is a partial $\ell$-distance face $r$-coloring for faces of $G$ of degree at least 3 such that each color occurs at most once. Because $r \geq 1 + 8 \times 5^\ell \geq 1 + 4d(f) 5^\ell$
for any face \( f \in H \), and because of Lemma 1, the set \( H = \bigcup_{i=1}^{2} F_i(G) \) is \((\ell, r)\)-colorable. By Lemma 2, we have \( \chi_{df}(G) \leq r \).

By Lemma 3 and because \( \epsilon = 3\nu \), there exist at least 6 loops or multiple edges, so \(|H| \geq 6\). By Euler’s formula, \( \phi = \epsilon - \nu + 2 = 3\nu - \nu + 2 = 2\nu + 2 \), we have \( \sum_{i=3}^{\infty} f_k(G) \leq 2\nu + 2 - 6 = 2(\nu - 2) \), which implies that \( \chi_{df}(G) \leq \max\{1 + 8 \times 5^\ell, 2(\nu - 2)\} \).

Let \( xy \in E \). Let \( F' \) be the set of faces incident with either \( x \) or \( y \). Let \( K = (V', E', F') \) be the subgraph of \( G \) induced by \( F' \) (i.e., \( K \) is the subgraph induced by the edges incident with faces in \( F' \)). Since there are no vertices of degree 1, the degree of a face \( f \) is equal to the number of vertices occurred in its boundary. Since each vertex is incident with at least two faces, we have

\[
2|E'| = \sum_{f \in F'} d_K(f) \geq 2(|V'| - 2) + d_K(x) + d_K(y) = 2(|V'| - 2) + 12.
\]

This implies \(|E'| \geq |V'| + 4\). By Euler’s formula, \(|F'| = |E'| - |V'| + 2 \geq 6\). So \( \chi_{df}(G) \geq 6 \). This completes the proof of the theorem.

**Remark:** We conjecture that \( \chi_{df}(G) \leq 41 \) for any 6-regular graph \( G \), but the method of transforming graphs in [4] does not work.

By a proof similar to that of Theorem 4, we have the following theorem.

**Theorem 5:** Let \( \ell \geq 1 \). Suppose \( G = (V, E, F) \) is an \( n \)-regular graph with \( n \geq 6 \). Then \( n \leq \chi_{df}(G) \leq \max\{1 + 2(n - 2)(n - 1)^\ell, 2(\nu - 2)\} \).

**References**


