On Independent Domination Number of Regular Graphs\(^1\)

by

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Abstract

Let \( G \) be a simple graph. The independent domination number \( i(G) \) is the minimum cardinality among all maximal independent sets of \( G \). Haviland (1995) conjectured that any connected regular graph \( G \) of order \( n \) and degree \( \delta \leq n/2 \) satisfies \( i(G) \leq \lfloor 2n/3\delta \rfloor/2 \). In this paper, we will settle the conjecture of Haviland in the negative by constructing counterexamples. Therefore a larger upper bound is expected. We will also show that a connected cubic graph \( G \) of order \( n \geq 8 \) satisfies \( i(G) \leq 2n/5 \), providing a new upper bound for cubic graphs.

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1. Introduction

Let \( G = (V, E) \) be a simple graph of order \( n \) and minimum degree \( \delta \). For a nonempty set \( W \subset V \), its neighborhood \( N(W) \) denote the set of all elements of \( V \) adjacent with at least one element of \( W \). If \( W = \{v\} \), then \( N(W) \) is simply written as \( N(v) \). An independent set is a set of pairwise non-adjacent vertices of \( G \). A subset \( I \) of \( V \) is a dominating set if \( N(I) \cup I = V \). The independent domination number \( i(G) \) is the minimum cardinality among all independent dominating sets of \( G \). An independent set is dominating if and only if it is maximal, so \( i(G) \) is also the minimum cardinality of a maximal independent set in \( G \).

The parameter \( i(G) \) was introduced by Cockayne and Hedetniemi in [5] and some results on it can be found in [1-10]. Favaron [6] and Haviland [8] established upper bounds for \( i(G) \) in terms of \( n \) and \( \delta \). For regular graphs of degree different from zero, we can prove that \( i(G) \leq n/2 \).

However, for most values of \( \delta \) this is far from best possible. In [6] it was shown that for any graph with \( n/2 \leq \delta \leq n \), we have \( i(G) \leq n - \delta \), and this bound could be attained only by complete multipartite graphs with vertex classes all of the same order. By adapting arguments from [8], the following result can readily be proved (see [9]).

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Proposition 1.1. Let $G$ be a regular graph. If $n/4 \leq \delta \leq (3 - \sqrt{3})n/2$ then $i(G) \leq n - \sqrt{n\delta}$ and if $(3 - \sqrt{3})n/2 \leq \delta \leq n/2$ then $i(G) \leq \delta$.

If $n = 2m\delta$, then $i(mK_{\delta,\delta}) = n/2$ and $mK_{\delta,\delta}$ is disconnected for $m > 1$. Haviland [8] thought that if $G$ was connected then the upper bound for $i(G)$ could be a function of $n$ and $\delta$. She also stated the following Conjecture in [9].

Conjecture 1.2. If $G$ is a connected $r$-regular graph with $r = \delta \leq n/2$, then $i(G) \leq \lceil 2n/3\delta \rceil \delta/2$.

In section 2, we provide counterexamples to Conjecture 1.2. In section 3, we shall show that $i(G) \leq 2n/5$ for any connected cubic graphs, providing a new upper bound for $i(G)$ as a function of the number of vertices.

2. Counterexamples

Lemma 2.1 Given positive integers $r \geq 2$ and $s \geq 3$, let $G(r, s)$ be the family of graphs such that $V = \bigcup_{j=1}^{r} (X_j \cup Y_j \cup Z_j)$, and $E = (E_1 \cup E_2 \cup E_3 \cup E_4)$, where

1. $X_j = \{x_{j1}, x_{j2}, \cdots, x_{j(s-1)}\}$,
2. $Y_j = \{y_{j1}, y_{j2}, \cdots, y_{js}\}$,
3. $Z_j = \{z_{j1}, z_{j2}, \cdots, z_{js}\}$,
4. $E_1 = \bigcup_{j=1}^{r} \{x_{j1}y_{jl} \mid 1 \leq k \leq s - 1, 1 \leq l \leq s\}$,
5. $E_2 = \bigcup_{j=1}^{r} \{y_{jk}z_{jk} \mid 1 \leq k \leq s\}$,
6. $E_3 = \bigcup_{j=1}^{r} \{z_{jk}z_{jl} \mid 1 \leq k, l \leq s, k \neq l, \} \setminus \{z_{j1}z_{js}\}$, and
7. $E_4 = \{z_{js}z_{j+1}1 \mid 1 \leq j \leq r - 1\} \cup \{z_{rs}z_{11}\}$.

Then

1. $|V| = r(3s - 1)$

2. $G(r, s)$ is both connected and $s$-regular, and

3. $i(G(r, s)) = rs$.

( Note that $G(r, s)$ contains $r$ subgraphs, which we shall call blocks, isomorphic to each other. A typical block of $G(r, 5)$, consisting of 14 vertices, is shown in Fig. 2.1.)
Proof of Lemma 2.1: (1) and (2) are trivial. Because $\bigcup_{j=1}^{r} Y_j$ is an independent dominating set, $i(G(r,s)) \leq rs$. So (3) is also proved if we can show that $i(G(r,s)) \geq rs$.

We claim that for every $1 \leq j \leq r$, $|I \cap (X_j \cup Z_j)| \geq s$. Consider any such $j$ between 1 and $r$. If $X_j \cap I \neq \emptyset$, then $Y_j \cap I = \emptyset$. Since $I$ must dominate $X_j$, we have $X_j \subseteq I$. Now for any $1 < k < s$, $z_{j,k}$ is not adjacent to any vertex outside of $Y_j \cup Z_j$, and so in order for $I$ to dominate $z_{j,k}$, it must be that $Z_j \cap I \neq \emptyset$. Thus $|I \cap (X_j \cup Z_j)| \geq s$.

On the other hand, if $X_j \cap I = \emptyset$, then for each $1 \leq k \leq s$, exactly one of $y_{j,k}$ or $z_{j,k}$ is in $I$. And so in this case it also follows that $|I \cap (Y_j \cup Z_j)| \geq s$.

Thus, it follows that $i(G(r,s)) \geq rs$, and hence that $i(G(r,s)) = rs$.

Fig. 2.1

Theorem 2.2 If $r$ is sufficiently large and $s \geq 3$, then $G = G(r,s)$ is a connected $s$-regular graph with $i(G) \geq [2n/3s]s/2$, where $n$ is the order of $G$.

Proof: We have $|2r(3s - 1)/3s|s/2 \leq (2r - [2r/3s])s/2$

$= rs - ([2r/3s]s/2)$

$< i(G),$

provided $r$ is sufficiently large.

Theorem 2.2 settles Conjecture 1.2 in the negative for all $\delta \geq 3$. If $\delta = 3$, then by Theorem 2.2, the upper bound of $i(G)/n$ is at least $3/8$, as shown by the previous example with $s = 3$. However, if Conjecture 1.2 holds, then this upper bound would have been less than $(n + 4)/3n$, which is strictly less than $3/8$ if $n > 32$.

Note that in the above, $\delta$ is fixed and $n$ is large. In what follows, we shall construct connected
regular graphs $G$ with $\delta(G)$ small relative to $n$, but $i(G)/n$ is as close to $1/2$ as we wish.

**Lemma 2.3** Given positive integers $r \geq 1$ and $s \geq 2$, let $G^*(r,s)$ be the graph $(V,E)$ with $V = U \cup \bigcup_{j=1}^{r+1} (V_j \cup W_j)$, and $E = (E_1 \cup E_2 \cup E_3 \cup E_4)$, where

1. $U = \{u_1,u_2,\ldots,u_{2r+1}\}$,
2. $V_j = \{v_{j,1},v_{j,2},\ldots,v_{j,s+2r}\}$,
3. $W_j = \{w_{j,1},w_{j,2},\ldots,w_{j,s+2r+1}\}$,
4. $E_1 = \{u_ju_k| 1 \leq j < k \leq 2r+1\}$,
5. $E_2 = \bigcup_{j=1}^{r+1} \{u_jv_{j,k}| 1 \leq k \leq s\}$,
6. $E_3 = \bigcup_{j=1}^{r+1} \{v_{j,s+2k-1}v_{j,s+2k}| 1 \leq k \leq r\}$, and
7. $E_4 = \bigcup_{j=1}^{r+1} \{v_{j,k}w_{j,l}| 1 \leq k \leq s+2r, 1 \leq l \leq s+2r-1\}$.

Then

1. $|V| = 2(2r+1)(s+2r)$,
2. $G^*(r,s)$ is both connected and $(s+2r)$-regular, and
3. $i(G^*(r,s)) = 2r(s+r) + r + 1$.

**Proof:** (1) and (2) are trivial. Because $S = \left[\bigcup_{j=1}^{2r} (\{v_{j,k}| 1 \leq k \leq s\} \cup \{v_{j,s+2k}| 1 \leq k \leq r\})\right] \bigcup \{(u_{2r+1}) \cup \{v_{2r+1,s+2k}| 1 \leq k \leq r\}\}$ is a maximal independent set of $G^*(r,s)$, and $|S| = 2r(s+r) + r + 1$, we have $i(G^*(r,s)) \leq 2r(s+r) + r + 1$.

Suppose $I$ is a maximal independent set of order $i(G^*(r,s))$ and $I_j = I \cap \{V_j \cup W_j \cup \{u_j\}\}$ for $1 \leq j \leq 2r+1$. Clearly, $I = \bigcup_{j=1}^{2r+1} I_j$ and $|I| = \sum_{j=1}^{2r+1} |I_j|$. If $u_j \notin I$, then $|I \cap (V_j \cup W_j)| \geq s+r$, and if $u_j \in I$, then $|I \cap (V_j \cup W_j)| \geq r$. Because $I$ is independent and the induced subgraph on $U$ is complete, there is at most one $j$ with $u_j \in I$. It follows that $i(G^*(r,s)) \geq 2r(s+r) + r + 1$ and (3) follows.

**Theorem 2.4** Suppose $0 < \epsilon < 1$ and $N \geq 2$. Then there exists a connected $\delta$-regular graph of order $n$ with $\delta < n/N$ and $i(G) > n/(2+\epsilon)$.

**Proof:** Let $r_1$ be the smallest integer such that $2(2r_1+1) > N$. Because $\lim_{r \to \infty} \frac{r}{2r+1} = \frac{1}{2}$, we can find $r_2$ such that if $r \geq r_2$, then $\frac{r}{2r+1} > \frac{1}{2} - \frac{\epsilon}{12}$. Put $r = \max\{r_1, r_2\}$. Also, for fixed $r$, we
have \( \lim_{s \to \infty} \frac{2r(s + r) + r + 1}{2(2r + 1)(s + 2r)} = \frac{r}{2r + 1} \), so we can find \( s \) such that \( \frac{2r(s + r) + r + 1}{2(2r + 1)(s + 2r)} > \frac{r}{2r + 1} - \frac{\epsilon}{12} \).

Let \( G = G^*(r, s) \) and \( n = |G| \). Then \( G \) is a \( \delta \)-regular graph with \( \delta = s + 2r \).

By Lemma 2.3 and the definition of \( r \), \( \delta/n = 1/2(2r + 1) < 1/N \). Moreover, by Lemma 2.3 again and the definition of \( r \) and \( s \),

\[
\frac{i(G)}{n} = \frac{2r(s + r) + r + 1}{2(2r + 1)(s + 2r)} > \frac{r}{2r + 1} - \frac{\epsilon}{12} > \frac{1}{2} - \frac{\epsilon}{6} > \frac{1}{2 + \epsilon}
\]

provided \( 0 < \epsilon < 1 \). 

3. **Regular Cubic Graphs**

In this section, we obtain an upper bound for the independent domination number of a connected cubic graph.

**Theorem 3.1** If \( G \) is a connected cubic graph of order \( n \), where \( n \geq 8 \), then

\[
i(G) \leq \frac{2n}{5}.
\]

**Proof:** Let \( I \) be an independent dominating set (IDS) of cardinality \( i(G) \). Also let \( J = V \setminus I \) and \( B = (I, J) \) be the bipartite graph induced by edges of \( G \) joining a vertex in \( I \) to a vertex in \( J \). Among all such choices of \( I \), choose one so that \( B \) contains the smallest number of \( K_{2,3} \)'s. If \( v \in J \) is connected to \( u \in I \) by an edge in \( B \), we say that \( v \) is guarded by \( u \) and that \( u \) is a guardian of \( v \). For each \( t = 1, 2 \) and \( 3 \), let \( J_t = \{ v \in J : v \text{ has } t \text{ guardians} \} \). Since \( I \) is a dominating set, \( J \) is the disjoint union of \( J_1, J_2 \) and \( J_3 \). If \( |J_3| \leq |J_1| \), then

\[
3n = \sum_{v \in V} d_G(v) = 2 \sum_{v \in I} d_G(v) + 2|J_1| + |J_2| \\
\geq 6i(G) + |J_1| + |J_2| + |J_3| \\
= 6i(G) + (n - i(G)),
\]

and therefore \( i(G) \leq 2n/5 \). So the theorem is proved if we can construct an injective map \( f : J_3 \to J_1 \). A vertex \( v \in J \) is guarded by \( I' \subset I \) if it is guarded by at least one vertex \( u \in I' \). The set of guardians of a vertex \( v_0 \in J_3 \) shall be denoted by \( I_0 = N(v_0) = \{ u_1, u_2, u_3 \} \). A vertex \( v \) that is guarded only by vertices of \( I_0 \) is called exclusive (with respect to \( v_0 \)), otherwise not exclusive. \( V_{ex} \)
shall denote the set of exclusive vertices. Note that \(v_0\) is not adjacent to any vertex in \(V_ex \backslash \{v_0\}\).

If \(|Vex| \leq 2\), then \([I \cup Vex] \backslash \{I_0\}\) is a subminimal IDS. Henceforth, we suppose \(|Vex| \geq 3\). We have three possible cases.

**Case 1**: \(I_0\) does not guard any \(J_1\)-vertex.

In this case, \(Vex \subset J_2 \cup J_3\). If \(|Vex \cap J_3| = 3\), then \(n = 6\). So besides \(u_1\), there is at most one \(J_3\)-vertex in \(Vex\), and thus \(J_2 \cap Vex \neq \emptyset\). If \(|Vex \cap J_3| = 2\) and if \(w_1\) and \(w_2\) are two exclusive vertices in \(J_2\) and \(J_3 \backslash \{v_0\}\) respectively, then \([I \cup \{w_1\}] \backslash [N(w_1) \cap I]\) is a subminimal IDS. Hence \(Vex \cap J_3 = \{v_0\}\) and there are at least two \(J_2\)-vertices in \(Vex\), say \(v_1\) and \(v_2\). Suppose the guardian sets of \(v_1\) and \(v_2\) are not identical (see \(H_1\) of Fig. 3.1). The third vertex guarded by \(u_3\) must be guarded by a vertex in \(I \backslash \{u_2, u_3\}\) and therefore \([I \backslash \{u_2, u_3\}] \cup \{v_2\}\) is a subminimal IDS. So we assume that \(v_1\) and \(v_2\) have the same guardian set (see \(H_2\) of Fig. 3.1). Moreover, \(Vex = \{v_0, v_1, v_2\}\).

\[
\begin{array}{c}
\text{Fig. 3.1} \\
H_1 \quad H_2 \quad H_3'
\end{array}
\]

If \(z_1 = v_2\), then \((I \cup \{v_1\}) \backslash \{u_2, u_3\}\) is a subminimal IDS. Hence \(z_1 \neq v_2\). If \(z_1 \neq v_3\), then \(I' = I \cup \{v_1, v_2\} \backslash \{u_2, u_3\}\) is an IDS with \(|I| = |I'|\), but the bipartite graph \((I', V \backslash I')\) contains a smaller number of \(K_{2,3}\)s (compare \(H_2\) and \(H_3'\) in Fig. 3.1). Therefore \(z_1 = v_3 \in J_1\) and \(G\) contains the subgraph \(H_3\) in Fig. 3.2. We let \(f(v_0) = z_1\).

\[
\begin{array}{c}
\text{Fig. 3.2} \\
H_4 \quad H_5
\end{array}
\]

Suppose \(N^*(u_4)\) is the set of vertices which are guarded by \(u_4\) but not by any vertex in \(I \backslash [I_0 \cup \{u_4\}]\). If either \(v_4 \notin N^*(u_4)\) or \(v_5 \notin N^*(u_4)\), then \([I \cup \{v_0\} \cup N^*(u_4)] \backslash \{u_1, u_2, u_3, u_4\}\) is a subminimal IDS. Therefore \(|N^*(u_4)| = 3\). It follows that neither \(v_4\) nor \(v_5\) is in \(J_3\). Moreover, if
$v_4$ is in $J_2$, then it must be guarded by $u_1$. The same is true for $v_5$. If both $v_4$ and $v_5$ are in $J_2$ then $G$ contains the subgraph $H'_4$ in Fig. 3.2.

**Case 2:** $I_0$ guards exactly one $J_1$-vertex $v'$, which is guarded by $u_3 \in I_0$.

Besides $v_0$ and $v'$, there is an exclusive vertex in $J_2 \cup J_3$, because $|V_{ex}| \geq 3$. We have the following sub-cases.

**Sub-case 2.1:** $[V_{ex}\setminus\{v_0, v'\}] \subset J_2$ and there exists $v_2 \in V_{ex}\setminus\{v_0, v'\}$ guarded by $u_3$.

In this sub-case, $H_4$ appears (Fig. 3.3). Relabeling $v'$ as $z_2$, we let $f(v_0) = z_2$.

![Fig. 3.3](image)

**Sub-case 2.2:** $[V_{ex}\setminus\{v_0, v'\}] \subset J_2$ and no vertex in $V_{ex}\setminus\{v_0, v'\}$ is guarded by $u_3$.

Suppose $v_6 \in V_{ex}\setminus\{v_0, v'\}$ is guarded by $u_1$ and $u_2$. Because $G$ is cubic, $u_1$ guards a remaining vertex besides $v_0$ and $v_6$. The same is true for $u_2$. If these two remaining vertices $v_7$ and $v'_7$ are distinct, i.e. $G$ contains $H_5$ (Fig. 3.3), then since they are not in $J_1$, $I \cup \{v_6\}\setminus\{u_1, u_2\}$ is a subminimal IDS. Therefore both $u_1$ and $u_2$ guard the same remaining vertex, and $G$ contains $H'_5$ (Fig. 3.3). We have $v_6$ not adjacent to $v_7$, otherwise $I \cup \{v_6\}\setminus\{u_1, u_2\}$ is a subminimal IDS. We also have $z_3 = z'_3$, otherwise $I' = I \cup \{v_6, v_7\}\setminus\{u_1, u_2\}$, is an IDS with $|I| = |I'|$, but $(I', V\setminus I')$ contains less $K_{2,3}$'s than $B$ (compare $H'_5$ with $H''_5$ in Fig. 3.3). The vertex $z_3$ is different from $v'$.
for otherwise \((I \cup \{v_5, v_5\}) \setminus \{u_1, u_2, u_3\}\) is a subminimal IDS. Therefore the guardian of \(z_3\) is also different from the guardian of \(v'\), i.e. \(u_4 \neq u_3\), and \(G\) contains the subgraph \(H_6\).

If \(u_3\) also guards a \(J_3\)-vertex besides \(v_0\), then all vertices in \(N(u_4) \setminus J_1\) would be guarded by at least one vertex in \(I \setminus (I_0 \cup \{u_4\})\). Moreover \((I \cup \{v_0, v'\} \cup \{N(u_4) \cap J_1\}) \setminus \{u_1, u_2, u_3, u_4\}\) would be a subminimal IDS if \(|N(u_4) \cap J_1| = 1\). Therefore \(|N(u_4) \cap J_1| \geq 2\) and \(u_4\) guards another \(J_1\)-vertex besides \(z_3\). In this case, we let \(f(v_0) = z_3\). If \(u_3\) does not guard another \(J_3\)-vertex besides \(v_0\), i.e. the third vertex it guards is in \(J_2\), then we relabel \(v'\) as \(z_4\) and let \(f(v_0) = z_4\).

**Subcase 2.3** \([V_{ex} \setminus \{v_0, v'\}] \cap J_3 \neq \emptyset\).

Suppose \(v_2 \in V_{ex} \cap J_3\) and so \(G\) contains \(H_7\). The set \(I' = I \cup \{v'\}\setminus \{u_5\}\) is an IDS with \(|I'| = |I|\) but \((I', V \setminus I')\) contains a less \(K_{2,3}\)'s unless there are vertices \(u_4 \in I\) and \(u_5 \in I\), both of which guards \(v_3\) as well as \(v_4\) (compare \(H_7\) with \(H_7'\) in Fig. 3.4). Because \(G\) is cubic, \(u_4\), and similarly \(u_2\), cannot be \(u_4\) or \(u_5\). Therefore \(G\) must contain \(H_7''\) (Fig. 3.4).

Suppose \(v_5 = v_6 = w\). If \(N(w) \cap I \subseteq U = \{u_1, u_2, u_3, u_4, u_5\}\), then \(I \cup \{v', w\}\setminus \{N(w) \cup \{u_3\}\}\) is a subminimal IDS. If \(|N(w) \cap I| \setminus U \neq \emptyset\), then since \(u_1\) and \(u_2\) does not guard any \(J_1\)-vertex, \(I \cup \{v_0, v_2, v_3, v_4\}\setminus U\) is a subminimal IDS. Therefore \(v_5 \neq v_6\).

If both \(|N(v_5) \cap I| \setminus U\) and \(|N(v_6) \cap I| \setminus U\) are non-empty, then \(I \cup \{v_0, v_2, v_3, v_4\}\setminus U\) is a subminimal IDS. Therefore one of \(|N(v_5) \cap I| \setminus U\) and \(|N(v_6) \cap I| \setminus U\) must be empty. Without loss of generality, we may assume that \(|N(v_5) \cap I| \setminus U = \emptyset\). Then \(|I \cup \{v_5\}| \setminus \{N(v_5) \cap \{u_1, u_2, u_4\}\}\) is a subminimal IDS unless \(|N(v_5) \cap \{u_1, u_2, u_4\}| = 1\). So \(v_5\) is not guarded by \(u_1\) or by \(u_2\). Therefore \(v_5\) is a \(J_1\)-vertex. Relabeling \(v'\) and \(v_5\) as \(z_5\) and \(z_6\) respectively, we put \(f(v_0) = z_5\) and \(f(v_0) = z_6\).

![Fig. 3.4](image_url)

So far, the mapping is injective. Vertex \(z_1\) is guarded by a vertex which guards only \(J_1\)- or \(J_2\)-vertices. Vertex \(z_2\) is guarded by a vertex which also guards one \(J_2\)- and one \(J_3\)-vertex. The
latter is the pre-image of \( z_2 \). Similar argument can be applied to \( z_4 \). Vertex \( z_3 \) is guarded by a vertex which guards at least one other \( J_1 \)-vertex. Vertex \( z_5 \) is guarded by a vertex which guards two other \( J_3 \)-vertices. Vertex \( z_6 \) is guarded by a vertex which guards two other \( J_2 \)-vertices. If \( z_6 \) were the image of another \( J_3 \)-vertex as \( z_1 \), then \( G \) would contain \( H_6 \) in Fig. 3.5 (see also \( H'_3 \) of Fig. 3.2), and \( I \cup \{ z_5, z_6, v_6 \} \} \{ u_1 \mid 3 \leq i \leq 7 \} \) would be a subminimal IDS. Therefore \( z_6 \) will not be mapped as \( z_1 \).

![Diagram 3.5](image)

**Fig. 3.5**

**Case 3:** \( I_0 \) guards two or more \( J_1 \)-vertices.

Suppose \( u_1 \), one of the three guardians of \( v_0 \), guards two \( J_1 \)-vertices \( v_8 \) and \( v'_8 \). By examining the type of vertices guarded by \( u_1 \), we can conclude that neither \( v_8 \) nor \( v'_8 \) can possibly be mapped as \( z_i \) unless \( i = 3 \). If only one of \( v_8 \) and \( v'_8 \) (say \( v_8 \)) have been mapped as \( z_3 \), then we rename \( v'_8 \) as \( z_7 \) and define \( f(v_0) = z_7 \). If both \( v_8 \) and \( v'_8 \) have been mapped as \( z_3 \) according to Sub-case 2.2, then \( G \) contains the sub-graph in Fig 3.6. If \( v_0 \) is not guarded by both \( u_4 \) and \( u'_4 \), then \( I \cup \{ v_1, v_2, v_8, v'_1, v'_2, v'_8 \} \} \{ u_1, u_2, u_3, u_4, u_2', u_3', u_4' \} \) is a subminimal IDS. Therefore \( v_0 \) is guarded by both \( u_4 \) and \( u'_4 \) and we relabel \( v_2 \) as \( z_8 \) and let \( f(v_0) = z_8 \). We know that \( z_7 \) has not been mapped as \( z_3 \). Because \( z_7 \) is guarded by a vertex which guards two \( J_1 \)-vertices and one \( J_3 \)-vertex, its pre-image, it cannot be \( z_i \) for \( i = 1, \cdots, 6 \) and it cannot be the image of two distinct \( J_3 \)-vertices. The guardian of \( z_8 \) guards two \( J_3 \)-vertices and the \( J_1 \)-vertex \( z_8 \), but the guardian set of one of these two \( J_3 \)-vertices guards at least two \( J_1 \)-vertices. Among \( z_i, i = 1, \cdots, 7 \), only \( z_5 \) is guarded by a vertex which guards two \( J_3 \)-vertices, but both of these two \( J_3 \)-vertices has the same guardian set which guards exactly one \( J_1 \)-vertex.

![Diagram 3.6](image)

**Fig. 3.6**

9
Suppose \( v' \) is a \( J_3 \)-vertex whose guardian set guards two or more \( J_1 \)-vertices, but each guardian guards at most one \( J_1 \)-vertex. Let \( v \) be one of these \( J_1 \)-vertices. Because the guardian of \( v \) guards exactly one \( J_1 \)-vertex and at least one \( J_3 \)-vertex, \( v \) cannot have been mapped as \( z_1, z_3, z_6 \) and \( z_7 \). The guardian of \( z_2 \) guards exactly one \( J_3 \)-vertex whose guardian set guards exactly one \( J_1 \)-vertex. Because the guardian set of \( v \) guards one \( J_3 \)-vertex whose guardian set guards at least two \( J_1 \)-vertices, \( v \) cannot have been mapped as \( z_2 \). For the same reason, it cannot have been mapped as \( z_4 \) or \( z_5 \). The guardian of \( z_8 \) guards two \( J_3 \)-vertices \( v_0 \) and \( v_1 \), of Fig 3.6. The guardian set of \( v_1 \) guards exactly one \( J_1 \)-vertex, so \( v' \) cannot be \( v_1 \). The guardian of \( v_0 \) guards one \( J_3 \)-vertex and two \( J_1 \)-vertices, so \( v' \) cannot be \( v_0 \). Since the guardian of \( v \) guards the \( J_3 \)-vertex \( v' \), \( v \) cannot have been mapped as \( z_8 \).

Let \( W = \{w_1, w_2, \cdots, w_k\} \) be the set of \( J_3 \)-vertices whose guardian set guards two or more \( J_1 \)-vertices, but each guardian guards at most one \( J_1 \)-vertex; \( V_i \) be the set of \( J_1 \)-vertices guarded by the guardian set of \( w_i \), \( i = 1, \cdots, k \); and \( V^* = \bigcup_{i=1}^{k} V_i \). If the vertex \( v \) belongs to three distinct sets \( V_{i_1}, V_{i_2} \) and \( V_{i_3} \), then the guardian of \( v \) will guard \( w_{i_1}, w_{i_2} \) and \( w_{i_3} \). This is impossible because the graph is cubic. Therefore a vertex may belong to at most two distinct sets \( V_{i_1} \) and \( V_{i_2} \), and \( |V^*| \geq \frac{1}{2} \sum_{i=1}^{k} |V_i| \geq k \). We may now finish defining the injective map \( f \) from \( J_3 \) into \( J_1 \).

Note that the graph \( G' \) in Fig. 3.7 has 10 vertices and \( i(G') = 4 = 2n/5 \). For \( n \geq 12 \), we do not know if there exists a graph \( G'' \) such that \( i(G'') = 2n/5 \), but we suspect that such graph does not exist. Moreover, we do not know how close this upper bound is to being the best possible.

![Figure 3.7](image-url)

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References

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