A Spectral Domain Decomposition Approach for Steady Navier-Stokes Problems in Circular Geometries

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Abstract

In this study, a spectral collocation domain decomposition method is developed for the numerical solution of second and fourth order problems in circular domains. The method is applied to the Navier-Stokes equations and its performance is investigated in the cases of the stream function and the stream function-vorticity formulations.

1 Introduction

The numerical solution of the steady Navier-Stokes (NS) equations with circular geometry has been the subject of several approaches. These include an iterative-integral equation technique using the biharmonic Green's function [11], an integro-differential method based on the Biot-Savart law [15], and a series expansion method based on the use of truncated Fourier series [5]. In the numerical treatment of the inflow-outflow problem with iterative methods, difficulties arise in determining boundary values of the vorticity. The problem has been discussed in detail by several authors. There has been interest in the numerical use of integral representations for the vorticity and velocity. The integral representations come from the classical theorems of viscous fluid motion expressed in suitable forms. Mills [11] used integral formulations for the velocity components and the vorticity based on an application of a Green's function method. Wang and Wu [15] used Fourier-series expansion and integro-differential methods to determine the velocity and vorticity components. Dennis et al. [5] presented a method which lies in between the purely finite-difference method, with calculation of boundary vorticity locally, and the integral method of Wang and Wu. In these studies the boundary vorticity was determined by applying a constraint condition derived from the integral representation.

A polar coordinate system on the unit circle can be obtained through the change of variable $x = r\cos\theta, y = r\sin\theta, r \in [0,1], \theta \in [0, 2\pi)$. A spectral approach may use Fourier expansions in the θ- and Chebyshev expansions in the r-direction. By using a Fourier-series expansion method (see e.g., [5, 15]), it is possible to solve the NS equations with a circular geometry. Since most practical problems have boundary singularities at $r = 1$, the Fourier spectral method requires a large number of collocation points (or trigonometric functions), especially for high Reynolds numbers. In a recent study, Bernardi and Karageorghis [2] apply a Legendre Galerkin spectral method to second and fourth order problems in a portion of a disk.

The object of the present study is to investigate the applicability of a spectral domain decomposition technique based on a Chebyshev collocation method for the solution of second and fourth order problems in circular domains. In Section 2 we illustrate the basic idea of our spectral domain decomposition approach by considering a second order problem. Fourth order problems and the Navier-Stokes equations will be discussed in Section 3. The final section presents some numerical examples which illustrate the convergence behavior of the spectral domain decomposition method.

2 Second Order Problems

2.1 The general problem

We consider the Dirichlet problem

$$\nabla^2 \phi(x, y) = F(x, y) \qquad \text{in} \quad \tilde{\Omega}, \tag{1}$$

where $\tilde{\Omega} = \{(x,y); x^2+y^2<1\}$ given ϕ on the boundary $\partial\tilde{\Omega}$. In plane polar coordinator (r,θ), the problem is

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = F(r,\theta) \quad \text{in} \quad \Omega, \tag{2}$$

where $\Omega = \{(r,\theta); 0<r<1, 0\le\theta<2\pi\}$ given ϕ on the boundary $\partial\Omega$.

Clearly, Eq. (2) is singular at the origin and we therefore need to impose numerical boundary conditions there [4, 6, 7]. From Huang and Sloan [7] we require the Laplacian of ϕ to be C^2 at the origin. This yields the pole conditions

$$\frac{\partial^2\phi}{\partial\theta^2}(0,\theta) = 0, \tag{3}$$

$$\frac{\partial\phi}{\partial r}(0,\theta) + \frac{\partial^3\phi}{\partial r\partial\theta^2}(0,\theta) = 0, \tag{4}$$

for $0\le\theta<2\pi$.

2.2 Numerical method

The circle Ω is decomposed into two subregions Ω_I and Ω_{II} where $\Omega_I = \{(r,\theta); 0<r<1, 0\le\theta<\beta\}$ and $\Omega_{II} = \{(r,\theta); 0<r<1, \beta\le\theta<2\pi\}$, and $\beta\in(0,2\pi)$. In $\Omega_s, s=I$ or II, the solution is approximated by

$$\phi^s(r,\theta) = \sum_{m=0}^{M_s}\sum_{n=0}^{N_s} a^s_{mn}\tilde{T}^s_m(r)\hat{T}^s_n(\theta) \tag{5}$$

where $T^s_m(r)$ and $\hat{T}^s_n(\theta)$ are appropriately shifted Chebyshev polynomials. As the two subregions have common interfaces we choose $M_I = M_{II}$.

In order to determine the $(M_I+1)(N_I+1)+(M_{II}+1)(N_{II}+1)$ unknown coefficients we follow a collocation strategy similar to the one described in Karageorghis [9]. We define the sets of points $\{r^s_i\}_{i=0}^{M_s}, \{\theta^s_j\}_{j=0}^{N_s}, s=I$ or II to be the Gauss-Lobatto points on the intervals $[0,1],[0,1],[0,\beta]$ and $[\beta,2\pi]$, respectively. In region I, say, we follow the strategy:

- Eq. (2) is collocated at $(r^I_i,\theta^I_j), 2\le i\le M_I-1, 1\le j\le N_I-1$. The boundary condition is collocated at $(1,\theta^I_j), 0\le j\le N_I$ (which means that if it is a polynomial in θ on that part of the boundary, it is satisfied identically).
- On the interface $\theta=\beta$, we impose $\phi^I(r^I_i,\beta) = \phi^{II}(r^I_i,\beta)$ for $1\le i\le M_I-1$ and $\frac{\partial\phi^I}{\partial\theta}(r^I_i,\beta) = \frac{\partial\phi^{II}}{\partial\theta}(r^I_i,\beta)$ for $1\le i\le M_I-1$.
- At the origin we impose the pole conditions

  $$\frac{\partial^2\phi^I}{\partial\theta^2}(0,\theta_j) = 0, \qquad \frac{\partial\phi^I}{\partial r}(0,\theta_j) + \frac{\partial^3\phi^I}{\partial r\partial\theta^2}(0,\theta_j) = 0$$

  for $1\le j\le N_I-1$.

- In order to get $(M_I+1)(N_I+1)$ equations we need two more equations which we obtain by imposing the satisfaction of the governing equation (2) at the points $(r_1, \theta_{[N_I/2]})$ and $(r_1, \theta_{[N_I/2]+1})$.

An identical strategy gives $(M_{II}+1)(N_{II}+1)$ equations in Ω_{II}.

3 Fourth Order Problems

3.1 Stream function formulation

We consider the Dirichlet problem

$$\bigtriangledown^4\, \psi(x,y) = F(x,y) \qquad \text{in} \quad \tilde{\Omega}, \tag{6}$$

where $\tilde{\Omega}$ is the circle defined in Section 2.1, given ψ and $\frac{\partial\psi}{\partial n}$ on the boundary $\partial\tilde{\Omega}$. In plane polar coordinates, equation (6) becomes

$$\begin{aligned}\frac{\partial^4\psi}{\partial r^4} + \frac{2}{r}\frac{\partial^3\psi}{\partial r^3} + \frac{1}{r^2}\left(2\frac{\partial^4\psi}{\partial r^2\partial\theta^2} - \frac{\partial^2\psi}{\partial r^2}\right) + \frac{1}{r^3}\left(\frac{\partial\psi}{\partial r} - 2\frac{\partial^3\psi}{\partial r\partial\theta^2}\right)\\ +\frac{1}{r^4}\left(4\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^4\psi}{\partial\theta^4}\right) = f(r,\theta).\end{aligned} \tag{7}$$

Eq. (7) is singular at the origin and we therefore need to impose pole conditions. By requiring $\bigtriangledown^4\psi$ to be C^2 at the origin, we obtain

$$4\frac{\partial^2\psi}{\partial\theta^2}(0,\theta) + \frac{\partial^4\psi}{\partial\theta^4}(0,\theta) = 0, \tag{8}$$

$$\frac{\partial\psi}{\partial r}(0,\theta) + 2\frac{\partial^3\psi}{\partial r\partial\theta^2}(0,\theta) + \frac{\partial^5\psi}{\partial r\partial\theta^4}(0,\theta) = 0, \tag{9}$$

$$2\frac{\partial^4\psi}{\partial r^2\partial\theta^2}(0,\theta) + \frac{1}{2}\frac{\partial^6\psi}{\partial r^2\partial\theta^4}(0,\theta) = 0, \tag{10}$$

$$\frac{3}{2}\frac{\partial^3\psi}{\partial r^3}(0,\theta) + \frac{5}{3}\frac{\partial^5\psi}{\partial r^3\partial\theta^2}(0,\theta) + \frac{1}{6}\frac{\partial^7\psi}{\partial r^3\partial\theta^4}(0,\theta) = 0, \tag{11}$$

for $0 \le \theta < 2\pi$.

In order to obtain numerical solutions for the fourth-order problem, the circular region Ω is decomposed into the subdomains Ω_I and Ω_{II}, as in the second order case. With the approximations (5) and by choosing the collocation points in each region as in Sect. 2.2 we adopt a collocation strategy similar to the one described in Karageorghis [9]. In region I, the governing equation is collocated at the points $(r_i, \theta_j), 3 \le i \le M_I - 3; 2 \le j \le N_I - 2$. The boundary conditions are both collocated at the points $(1, \theta_j^I), 0 \le j \le N_I$ (which means that if both are polynomials on the boundary segment $\partial\Omega_I$, they are satisfied identically). On the interface $\theta = \beta$, we impose the conditions

$$\psi^I(r_i^I,\beta)=\psi^{II}(r_i^I,\beta), \qquad \frac{\partial\psi^I}{\partial\theta}(r_i^I,\beta)=\frac{\partial\psi^{II}}{\partial\theta}(r_i^I,\beta), \tag{12}$$

$$\frac{\partial^2\psi^I}{\partial\theta^2}(r_i^I,\beta)=\frac{\partial^2\psi^{II}}{\partial\theta^2}(r_i^I,\beta), \qquad \frac{\partial^3\psi^I}{\partial\theta^3}(r_i^I,\beta)=\frac{\partial^3\psi^{II}}{\partial\theta^3}(r_i^I,\beta), \tag{13}$$

for $2\le i\le M_I-2$. Each of the pole conditions (8)-(11) is satisfied at the points $(0,\theta_j^I), 2\le j\le N_I-2$. Finally, we collocate the governing equation at the extra points $(r_i,\theta_j), i=M_I-1, M_I-2$ and $j=[N_I/2]-1, [N_I/2], [N_I/2]+1, [N_I/2]+2$ to get a total of $(M_I+1)(N_I+1)$ equations. An identical strategy yields $(M_{II}+1)(N_{II}+1)$ equations in Ω_{II}.

3.2 Stream function-vorticity formulation

In the stream function-vorticity formulation we write (6) in an equivalent form

$$\bigtriangledown^2\psi=\zeta \tag{14}$$

$$\bigtriangledown^2\zeta=f(r,\theta), \tag{15}$$

where ζ is the vorticity. At the origin, the equations corresponding to (8)-(11) are:

$$\frac{\partial^2\psi}{\partial\theta^2}(0,\theta)=0, \tag{16}$$

$$\frac{\partial\psi}{\partial r}(0,\theta)+\frac{\partial^3\psi}{\partial r\partial\theta^2}(0,\theta)=0, \tag{17}$$

$$\frac{\partial^2\zeta}{\partial\theta^2}(0,\theta)=0, \tag{18}$$

$$\frac{\partial\zeta}{\partial r}(0,\theta)+\frac{\partial^3\zeta}{\partial r\partial\theta^2}(0,\theta)=0 \tag{19}$$

for $0\le\theta<2\pi$.

In the stream function-vorticity formulation we approximate both the the stream function ψ and the vorticity ζ by truncated Chebyshev expansions of the form (5) in each of the two regions. In region I, the governing equation is collocated at the points $(r_i,\theta_j), 1\le i\le M_I-1, 1\le j\le N_I-1$. The boundary conditions, pole conditions and interface conditions are applied as before with the exception of (13) which becomes

$$\zeta^I(r_i^I,\beta)=\zeta^{II}(r_i^I,\beta), \qquad \frac{\partial\zeta^I}{\partial\theta}(r_i^I,\beta)=\frac{\partial\zeta^{II}}{\partial\theta}(r_i^{II},\beta).. \tag{20}$$

3.3 The Navier-Stokes equations

In polar coordinates, two dimensional Navier-Stokes equations in terms of stream function can be written in the following form

$$\frac{\partial^4\psi}{\partial r^4}+\frac{2}{r}\frac{\partial^3\psi}{\partial r^3}+\frac{1}{r^2}\left(2\frac{\partial^4\psi}{\partial r^2\partial\theta^2}-\frac{\partial^2\psi}{\partial r^2}\right)+\frac{1}{r^3}\left(\frac{\partial\psi}{\partial r}-2\frac{\partial^3\psi}{\partial r\partial\theta^2}\right)+\frac{1}{r^4}\left(4\frac{\partial^2\psi}{\partial\theta^2}+\frac{\partial^4\psi}{\partial\theta^4}\right)$$

$$= \frac{\mathrm{R}}{r}\left[\frac{\partial\psi}{\partial\theta}\left(\frac{\partial^3\psi}{\partial r^3}+\frac{1}{r}\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r^2}\left(\frac{\partial^3\psi}{\partial r\partial\theta^2}-\frac{\partial\psi}{\partial r}\right)-\frac{2}{r^3}\frac{\partial^2\psi}{\partial\theta^2}\right)\right.$$
$$\left.-\frac{\partial\psi}{\partial r}\left(\frac{\partial^3\psi}{\partial r^2\partial\theta}+\frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\theta}+\frac{1}{r^2}\frac{\partial^3\psi}{\partial\theta^3}\right)\right], \tag{21}$$

where R is the Reynolds number. In the stream function-vorticity formulation, the Navier-Stokes equations become

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}=\zeta, \tag{22}$$
$$\frac{\partial^2\zeta}{\partial r^2}+\frac{1}{r}\frac{\partial\zeta}{\partial r}+\frac{1}{r^2}\frac{\partial^2\zeta}{\partial\theta^2}=\frac{R}{r}\left(\frac{\partial\psi}{\partial\theta}\frac{\partial\zeta}{\partial r}-\frac{\partial\zeta}{\partial r}\frac{\partial\zeta}{\partial\theta}\right). \tag{23}$$

The schemes described in Sections 3.1 and 3.2 can be applied for the above equations. The difference in this case is that the resulting global systems of equations for the determination of the coefficients in the Chebyshev expansions are nonlinear and therefore a nonlinear solver needs to be used for its solution.

4 Numerical examples

4.1 A second order problem

The technique described in Section 2 was applied to the problem $\bigtriangledown^2\phi(r,\theta)=3\cos\theta$ in the unit cicle Ω, given $\phi=\cos\theta$ on the boundary. The exact solution to this problem is $\phi=r^2\cos\theta$.

The absolute error was calculated on a 9×18 point grid in the circle (9 points in the r-direction and 18 in the θ-direction) and its maximum is presented in Figure 1 for different values of numbers of degrees of freedom ($M_I=M_{II}=8$ and varying $N=N_I=N_{II}$). In this case β was taken to be equal to π. A spectral accuracy can be observed from the figure.

4.2 A fourth order problem

The technique of Section 3 was applied to a problem governed by the NS equations (21)-(23) in the unit circle Ω given $\psi=0$ and $\frac{\partial\psi}{\partial r}=-\frac{1}{2}(1+\cos\theta)$ on the boundary $\partial\Omega$. For the Stokes case ($R=0$) the exact solution of the problem is

$$\psi=\frac{1}{4}(1-r^2)(1+r\cos\theta)$$

This problem was suggested by Kuwahara and Imai [10]. Both the stream function and the stream function vorticity formulations were applied to the problem. The maximum absolute error was examined for $\beta=\pi$. It was observed that for the stream function formulation with $N>14$ (where $N=N_I=N_{II}$) this maximum error started increasing. The maximum error for a typical case is presented in Figure 2. (keeping $M_I=M_{II}=8$ fixed and varying $N=N_I=N_{II}$). This is due

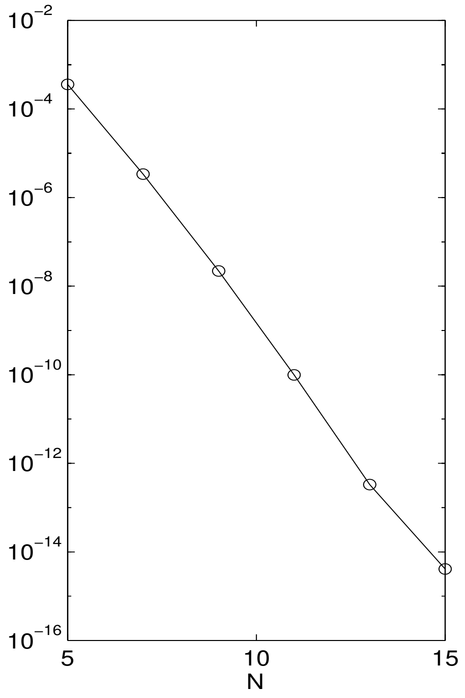

Figure 1: Maximum error for Problem 4.1 (for fixed $M_I = M_{II} = 8$).

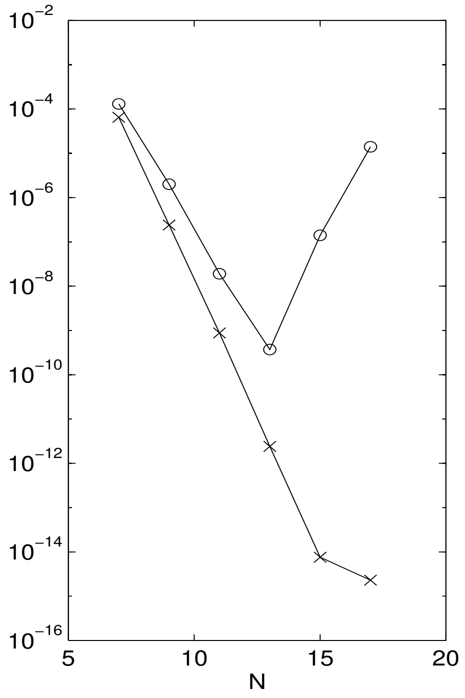

Figure 2: Maximum error for Problem 4.2 (for fixed $M_I = M_{II} = 8$). Circle: stream function results; x-mark: stream function-vorticity results.

to the ill-conditioning of the system of equations resulting from a spectral collocation discretization of a fourth order operator. It is shown by Quarteroni and Valli [13] that the condition number of the Chebyshev collocation matrices for fourth order problems is $O(N^4)$. In fact, the ill -conditioning of the Chebyshev collocation differentiation operator leads to round-off errors of $O(\epsilon N^2)$, where ϵ is the machine accuracy and N is the order of the expansion. In the application of the Chebyshev collocation method to a fourth order problem, this means that the fourth order derivatives which appear in the satisfaction of the differential equation (21) lead to round off errors of $O(\epsilon N^8)$, i.e. we expect a total breakdown of significance for $N = O(100)$ (we assume that $\epsilon = 10^{-16}$). In the present case $N = 14$ which leads to loss of significance of $O(10^{-8})$. In addition to this we must take into account the high order mixed derivatives in the pole conditions (8)–(11).

The question that remains to be answered is whether the deterioration of spectral accuracy in the pure stream function formulation is due to the imposition of the pole conditions. In order to answer this we must first point out that the results obtained in Figures 1 and 2 indicate that there is no such problem for the solution of Poisson problems. We also observed that when the problem is solved in the stream function-vorticity formulation no such problems are present. The deterioration in spectral accuracy only occurs in the case of the pure stream function formulation. As discussed in the last paragragh, the loss of the spectral accuracy is due to the ill-conditioning of the global system. The question is whether the imposition of the pole conditions makes the situation worse. In order to test this we imposed a finite difference scheme for the biharmonic equation in Cartesian coordinates (13-point formula) at the origin instead of the pole conditions. We thus solved the Stokes problem using a mixed finite difference/spectral method with mesh size the distance of the second Gauss-Lobatto point ($i = 1$) from the origin. The results revealed that this made essentially no difference and that the ill-conditioning appeared for the same value of N (see Table 1). This test indicates that the imposition of the pole conditions is not the main reason for the ill- conditioning of the system.

Schultz et al. [14] successfully applied a Chebyshev collocation method to the driven cavity problem in the pure stream function formulation. In order to overcome the underdeterminacy and ill-conditioning difficulties they encountered, they solved an overdetermined linear system at each iteration. More specifically, they removed the ill-conditioning by adding additional collocation points so that the differential equation and boundary conditions were enforced at more grid points than unknowns. This required them to solve the (slightly) overdetermined matrix problem by a QR factorization instead of the usual LU factorization. Following their approach (see also [3], p. 202), we tried to solve an overdetermined system for our stream function formulation. The number of equations of the system was increased by satisfying the differential equation at more points. After a QR factorization of the global system by the NAG routine F04ANF, the least squares solution was calculated using the NAG routine F01ANF. Although in some cases some improvement of the results is observed, the pattern of the decay in the accuracy of the results persists. In Table 1 we list the results obtained using the various techniques described above. It is found that the ill-conditioning cannot be removed in any of these cases. The only case which is free of this poor conditioning is the stream function-vorticity formulation.

The algebraic system resulting from the spectral collocation scheme is nonlinear for non-zero Reynold numbers. Two standard packages were tested for its solution, namely the IMSL routine DNEQNJ [8], which uses the Levenberg-Marquardt algorithm and the NAG routine C05PBF [12] which is based on a modification of the Powell-hybrid method. In both methods the user is required to provide the Jacobian of the system. The performance of the two packages was found to be very similar.

A continuation procedure in R was employed by first solving the Stokes equation, storing the expansion coefficients in approximation (5) and then increasing R by steps of 50. The continuation enabled us to reach values up to $R \sim 1000$. In Figure 3 we present the vorticity along the x-axis of the circle for the values of $R = 0, 64$ and 1024. They are in excellent agreement with the results reported in [10]. At high Reynolds numbers, the flow takes on a boundary layer characteristic. Vorticity gradients are appreciable only in the region near the flow boundary at high Reynolds numbers. It is observed that Batchelor's flow model of constant vorticity is valid for closed streamline flows (see, [1]).

N	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)
7	0.13(-3)	0.15(-2)	0.13(-3)	0.12(-2)	0.90(-3)	0.55(-2)	0.29(-3)	0.65(-4)	0.16(-2)
9	0.20(-5)	0.40(-4)	0.18(-5)	0.60(-4)	0.72(-4)	0.41(-4)	0.79(-4)	0.24(-6)	0.13(-4)
11	0.19(-7)	0.27(-5)	0.22(-7)	0.52(-5)	0.39(-5)	0.63(-5)	0.12(-4)	0.88(-9)	0.91(-7)
13	0.15(-9)	0.13(-6)	0.52(-8)	0.53(-7)	0.58(-7)	0.24(-6)	0.79(-7)	0.24(-11)	0.54(-9)
15	0.83(-8)	0.18(-4)	0.41(-6)	0.24(-6)	0.12(-9)	0.11(-3)	0.11(-8)	0.76(-14)	0.12(-7)
17	0.46(-6)	0.22(-3)	0.55(-3)	0.43(-5)	0.95(-9)	0.66(-2)	0.97(-7)	0.24(-14)	0.17(-5)

Table 1: Maximum error for Problem 4.2 (for fixed $M = M_I = M_{II} = 8$ and different values of $N = N_I = N_{II}$): (a) pure stream function formulation; (b) least squares by adding $N-5$ points (equations) in each element; (c) adding $M-3$ points; (d) adding $(N-4)+(M-3)$ points; (e) adding $(N-4)+(M-3)+(N-4)$ points; (f) adding $(N-4)+(M-3)+(M-2)$ points; (g) adding $(N-4)+(M-3)+(N-3)+(M-2)$ points; (h) stream function-vorticity formulation; (i) stream function formulation with finite difference scheme at center point.

4.3 A moving wall problem

The technique of Section 3 was also applied to a moving wall problem (see [11]). In it, we examine the motion of a liquid inside a cylinder which is generated by the movement of constant speed at part of its surface. The flow is governed by the NS equations and the boundary conditions are given by

$$\begin{aligned} &\psi = 0, \qquad \text{on} \quad \partial\Omega, \\ &\frac{\partial\psi}{\partial r} = -1, \qquad \text{on} \quad \partial\Omega_I := \{(1,\theta) : 0 \le \theta < \beta\}, \\ &\frac{\partial\psi}{\partial r} = 0, \qquad \text{on} \quad \partial\Omega_{II} := \{(1,\theta) : \beta \le \theta < 2\pi\}. \end{aligned}$$

For the Stokes case, the exact solution of this problem is known ([11]) and can be expressed in closed form as

$$\psi = \frac{1-r^2}{2\pi}\left[\lambda + \tan^{-1}\left(\frac{1+r}{1-r}\tan\frac{\beta-\theta}{2}\right) + \tan^{-1}\left(\frac{1+r}{1-r}\tan\frac{\theta}{2}\right)\right]$$

where

$$\lambda = \begin{cases} 0, & \text{if} \quad \beta-\pi < \theta < \pi, \\ \pi, & \text{if} \quad \pi < \theta < \beta+\pi. \end{cases}$$

We first considered the Stokes case. The L_2-norm error for the stream function,

$$\|e\| := \frac{1}{K_1 K_2}\sqrt{\sum_{i=1}^{K_1}\sum_{j=1}^{K_2}(\psi_{ij} - \psi(r_i,\theta_j))^2}$$

was evaluated for different numbers of degrees of freedom. A typical pattern of the behavior of the error norm $\|e\|$ is given in Figure 4 (with $\beta = \pi$) for $M_I = M_{II} = 14$ while varying $N = N_I = N_{II}$.

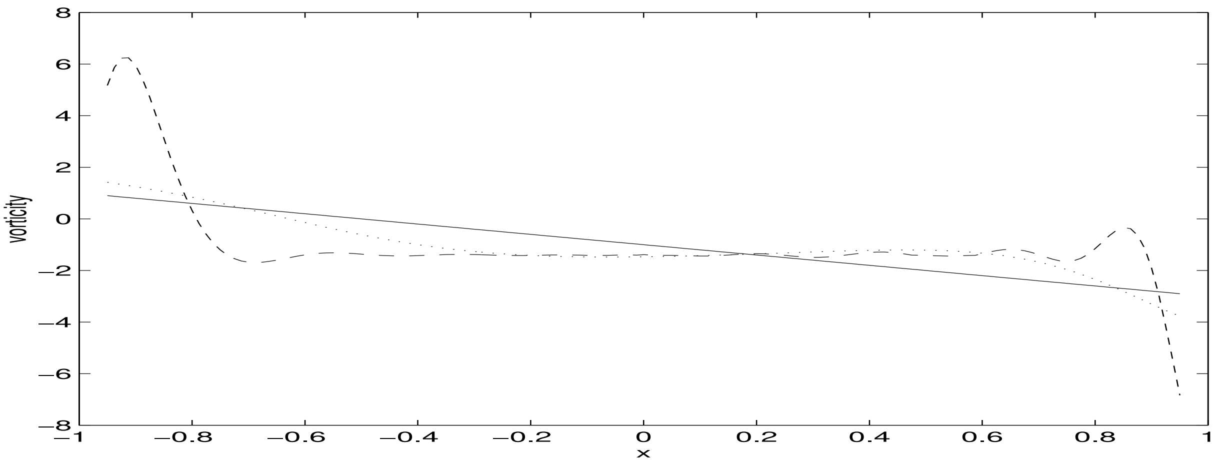

Figure 3: Vorticity distribution on x axis for Problem 4.2. Solid line: R=0; dotted line: R=64; dashed line: R=1024.

The solid line in this figure presents the error with the stream function-vorticity formulation. The slow convergence of the error is due to the presence of boundary singularities at the points (1,0) and $(1, \beta)$. The dashed line in Fig. 4 gives the error with the stream function formulation. Again it is seen that the spectral method with this formulation is unstable.

For the moving wall problem with nonzero Reynolds numbers, we found the stream function formulation gives results for R $\leq$ 250 only. The ill-conditioning for larger Reynolds numbers prevented us from obtaining convergent solutions. However, with the stream function-vorticity formulation, numerical solutions can be obtained for Reynolds numbers up to 1000. In Figure 5 we plot the stream lines for R $= 0, 50, 100, 250, 400$ and 1000. They were obtained by using $N_I = N_{II} = 29, M_I = M_{II} = 14$ (1800 degrees of freedom). It is seen that at high Reynolds numbers, the inertial effect becomes dominant, and the vortex center tends to align with the center of the circle.

5 Concluding remarks

This paper presents a spectral domain decomposition method for the numerical solution of second and fourth order problems in circular geometries. In particular, the method is investigated for the solution of the steady Navier-Stokes equations. The Stokesian case revealed that the stream function formulation leads to ill-conditioning for large numbers of degrees of freedom in contrast to the stream function-vorticity formulation. For the moving wall problem, results with the stream function-vorticity formulation were recorded for large values of the Reynolds number. Using the NS equations in terms of stream function we observed that reliable solutions can be obtained for R $\leq$ 250 only.

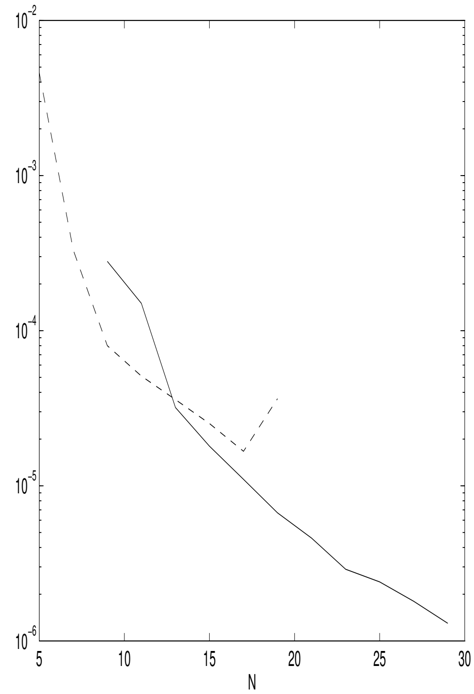

Figure 4: L_2 error for moving wall problem at R $=0$. Solid line: stream function-vorticity formulation; dashed line: stream function formulation.

Acknowledgements: We wish to thank the reviewers for their constructive comments and suggestions. Part of this work was performed while the first author was visiting the Department of Mathematics and Statistics, Simon Fraser University. This research was supported by an NSERC Canada grant.

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