ON MIXED ERROR ESTIMATES FOR ELLIPTIC OBSTACLE PROBLEMS

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Abstract. In this paper, we establish sharp error estimates of residual type for finite element approximation to elliptic obstacle problems. The estimates are of mixed nature, which are neither of a pure *a priori* form nor of a pure *a posteriori* form but instead they are combined by an *a priori* part and an *a posteriori* part. The key ingredient in our derivation for the mixed error estimates is the use of a new interpolator which enables us to eliminate inactive data from the error estimators. One application of our mixed error estimates is to construct reliable and efficient *a posteriori* error indicators useful in mesh-refinements and adaptive grid generations. In particular, by approximating the *a priori* part with some *a posteriori* quantities we can successfully track the free boundary of elliptic obstacle problems.

Key words. finite element approximation, elliptic obstacle, sharp *a posteriori* error estimates

AMS subject classification. 65N15, 65N30, 41A05, 41A29, 41A36

Contents

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1. Introduction

Variational inequalities represent an important class of nonlinear physical models. In particular, variational inequalities have been used to study and compute various free or moving boundary problems, and have been an important subject of studies since 60s. Obstacle problem is among the most studied variational inequalities due to its importance in the theory and applications. Some of the progress in this area can be found in, for example, [4, 8, 9, 11, 13], and the references cited therein. Finite element approximation of variational inequalities is among important topics in variational inequality theory, and has been extensively studied. Some earlier work has been summarized in [6] and [12].

In recent years adaptive algorithms for the finite element approximation have been extensively investigated. At the heart of any adaptive finite element method is an error monitor or indicator which is used as a guide as to show how the adjustment might be accomplished most efficiently. The literature directly relevant to our work can be found in [1, 2, 3, 7, 19, 20, 21].

Adaptive finite element approximation is among the most important means to boost accuracy and efficiency of finite element discretization. There has been some work on h-version adaptive finite element computation of the variational inequalities (particularly of static obstacle problems) in the last decade, see, e.g., [21] using local solution, and [14] using gradient recovery. Recently there is a renewed interest in deriving *a posteriori* error indicators of residual type, and in developing efficient adaptive schemes for finite element approximation of the variational inequalities, see [5, 10, 18] where various residual type estimates are derived. For example, the following estimate was shown in [18] for a class of variational inequalities, using the standard Lagrange interpolator,

$$|u - u_h|_{1,\Omega}^2 \leq C\Big(\eta_1^2 + \eta_2^2 + \epsilon\Big), \tag{1.1}$$

where

$$\eta_1^2 = \sum_\tau h_\tau^2 \int_\tau (f + \mathrm{div}(A\nabla u_h))^2, \quad \eta_2^2 = \sum_{l\cap\partial\Omega=\emptyset} h_l \int_l [(A\nabla u_h)\cdot \mathbf{n}]^2,$$

h_τ is the size of an element τ, h_l is the size of an element edge l, u is the solution of the variational inequality, u_h is the finite element solution, ϵ is a (uncomputable) higher order term for smooth data, and $[(A\nabla u_h)\cdot\mathbf{n}]$ denotes the jump of A-normal derivative on an interior edge l, defined by

$$[(A\nabla u_h)\cdot \mathbf{n}]_l = (A\nabla u_h|_{\tau_l^1} - A\nabla u_h|_{\tau_l^2})\cdot \mathbf{n}, \quad l = \bar\tau_l^1 \cap \bar\tau_l^2,$$

where $\mathbf{n}$ is the outer normal vector of τ_l^1. For the obstacle problem, a positivity-preserving and H^1 stable interpolator was introduced in [5]. By applying this interpolator it is easy to show that the higher order term ϵ can actually be removed from the above estimate for the obstacle problem, though it is not clear if this is true for the other problems studied in [18]. Furthermore in [5], *a posteriori* estimates were also derived for the Lipschitz obstacle case. However from what is to be discussed below, these estimates may be pessimistic for many cases frequently seen in applications, and thus may not be very useful in constructing efficient error indicators in adaptive finite element schemes.

The purpose of this paper is to derive sharp mixed error estimates for finite element approximation of the following elliptic obstacle problem:

$$\begin{cases} -\mathrm{div}(A\nabla u) \geq f, \\ u \geq \phi, \\ (-\mathrm{div}(A\nabla u) - f)(u-\phi) = 0 \end{cases} \tag{1.2}$$

where A is a given matrix function, f and ϕ are given functions, and Ω is a bounded open set in $\mathbf{R}^2$. The estimate is mixed in the sense that it consists of an *a priori* part and an *a posteriori* part. We then are able to derive some *a posteriori* error estimators, which seem useful in mesh refinement of finite element approximation for the obstacle problem. We assume that the solution of (1.2) is continuous. This implies that the noncoincidence set of the obstacle problem (1.2) $\Omega^+ = \{x \in \Omega \ : \ u(x) > \phi(x)\}$ is open, and the coincidence set $\Omega^- = \{x \in \Omega \ : \ u(x) = \phi(x)\}$ is closed. The boundary of the coincidence set in Ω is referred to as *free boundary*.

It is an essential observation for obtaining sharp error estimates for the obstacle problem that the values of f inside the coincidence set make little influence on the solution u, and so do these of obstacle ϕ outside the set. Based on this observation, one may attempt to think that it should be possible to remove the data from the existing estimates of residual type (see e.g. [5, 18]) and then to obtain sharper estimates. However, they (i.e: $f|_{\Omega^-}$ and $\phi|_{\Omega^+}$) do affect the finite element solution u_h. In fact, the influence may be propagated over the whole domain through very dense free boundary. Fortunately, well-posted physical models do not often exhibit such an ill-property. Indeed we are able to show that it is possible to obtain sharper mixed error estimates, which only depend on the values of f in a small neighborhood of Ω^+, and on these of ϕ near Ω^-, under some mild assumptions that hold in many practical applications. This will enable us to derive sharp *a posteriori* error indicators, which are useful in tracking the free boundaries numerically.

As a matter of fact, this idea was explored for the case of a zero obstacle in [18] where the following sharper error estimate was derived for static obstacle problem:

$$c\Big(\hat{\eta}_1^2 + \hat{\eta}_2^2 + \epsilon_1\Big) \leq |u-u_h|_{1,\Omega}^2 \leq C\Big(\hat{\eta}_1^2 + \hat{\eta}_2^2 + \epsilon_2\Big), \tag{1.3}$$

where

$$\hat{\eta}_1^2 = \sum_\tau h_\tau^2 \int_\tau (f(1-\chi_{u_h}^h))^2, \qquad \hat{\eta}_2^2 = \sum_{l\cap\partial\Omega=\emptyset} h_l \int_l [(A\nabla u_h)\cdot \mathbf{n}]^2,$$

$\chi_{u_h}^h = h^\alpha/(h^\alpha + u_h)$ ($\alpha > 0$), and ϵ_1 and ϵ_2 are higher order terms when the solution is in $H^2(\Omega)$ and the free boundary consists of a finite numbers of piecewise smooth curves. In [18] and [16], the error indicator $\eta_1^2 + \eta_2^2$ has been applied in both the h-version and the r-version (mesh motion) adaptive finite element schemes for static obstacle problems. Numerical experiments confirmed that this error indicator can guide the meshes to track the free boundary efficiently, which is in contrast with the indicators based on (1.1).

One of the important conclusions from the existing theoretical analysis and numerical experiment is that in general the maximum approximation error is concentrated around free boundaries (i.e., the boundary of the coincidence set), see [16]. Thus it is essential to use sharper *a posteriori* error estimators as error indicators in adaptive finite element schemes, in order to refine the meshes for obstacle problems efficiently.

It is now realized that the uncomputable term ϵ_2 in (1.3) can be removed by employing the positivity-preserving interpolator introduced in [5]. We shall first deal with this problem, which also serves as a brief introduction to the more complicated problem: to obtain sharp mixed error estimates for the case with *non-zero obstacles*. In [18], sharp error estimates were only established for the zero obstacle case. The general case was handled by using the transformation $u^* = u - \phi$. However, the resulting estimates may be no longer sharp in general. It is much more involving to derive sharp error estimates for the non-zero obstacle case. One of the important tricks in obtaining our results is to use a new interpolator. Roughly speaking, we choose

$$v_h = \begin{cases} \pi_h u, & \text{inside } \Omega^-, \\ \Pi_h u, & \text{outside } \Omega^-, \end{cases} \tag{1.4}$$

where π_h is the standard Lagrange interpolation operator and Π_h is the positivity-preserving one used in [5]. This approach is different with the previous methods which use either $v_h = \pi_h u$ (see e.g. [18]) or use $v_h = \Pi_h u$ (see e.g. [5]). It will be demonstrated in this paper that useful mixed error estimates can be obtained by using the interpolator (1.4).

The paper is organized as follows: In Section 2 we describe finite element approximation of the elliptic obstacle problem (1.2). In Section 3, we derive sharp mixed error estimates for the problem with a zero obstacle. Both upper bound and lower bound are established. Then the results are generalized to the case of nonzero obstacle in Section 4.

2. Elliptic obstacle problem and finite element approximation

Let Ω be a bounded open set in $\mathbf{R}^2$ with a Lipschitz boundary $\partial\Omega$. We adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{m,q,\Omega}$ and semi-norm $|\cdot|_{m,q,\Omega}$. We denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ with norm $\|\cdot\|_{m,\Omega}$ and semi-norm $|\cdot|_{m,\Omega}$. We set $H_0^1(\Omega) \equiv \{v \in H^1(\Omega) \ : \ v|_{\partial\Omega} = 0\}$. In addition c or C denotes a general positive constant independent of functions and the mesh size.

We first review some known results about variational inequalities and in particular, the obstacle problem (1.2). Let $f \in L^2(\Omega), g \in H^1(\Omega)$ and

$$V_g = \{v \in H^1(\Omega) \ : \ v - g \in H_0^1(\Omega)\}.$$

Let

$$\begin{aligned} a(u,v) &= \int_\Omega (A\nabla u)\cdot\nabla v, \qquad && \forall\, u, v \in H^1(\Omega), \\ (f,v) &= \int_\Omega fv, && \forall\, f, v \in L^2(\Omega), \end{aligned}$$

where

$$A(\cdot) = (a_{i,j}(\cdot))_{2\times 2} \in (L^\infty(\Omega))^{2\times 2}.$$

Assume that there is a constant $c > 0$ such that for any vector $\xi \in \mathbf{R}^2$

$$(A\xi)\cdot\xi \geq c|\xi|^2. \tag{2.1}$$

Let K be a closed convex set in V_g. Then the following inequality is commonly referred to as variational inequality: For given f, g, find $u \in K$ such that

$$a(u, v-u) \geq (f, v-u), \qquad \forall v \in K. \tag{2.2}$$

It is well known that the variational inequality (2.2) has a unique solution in K, see [13] for example. In particular, let $\phi \in H^1(\Omega)$ be such that $(g-\phi)|_{\partial\Omega} \geq 0$. Then the following variational inequality is a weak formulation of the obstacle problem (1.2) with the boundary condition $u = g$ on $\partial\Omega$: For given f, ϕ, g, find $u \in K_g(\phi)$ such that

$$a(u, v-u) \geq (f, v-u), \qquad \forall v \in K_g(\phi), \tag{2.3}$$

where

$$K_g(\phi) = \{v \in V_g \ : \ v \geq \phi\}.$$

It is well known that the variational inequality (1.2) has a unique solution $u \in K_g(\phi)$, and furthermore that the solution is regular, say in $H^2(\Omega)$, if the data are so. For instance, $u \in W^{2,p}(\Omega)$ for any $p > 1$ if $A \in (C^1(\bar\Omega))^{2\times 2}, \phi \in C^2(\bar\Omega), f \in C^\alpha(\bar\Omega), g \in C^{2+\alpha}(\bar\Omega), \partial\Omega \in C^{2+\alpha}$ for an $\alpha > 0$. Furthermore if $u \in H^2(\Omega)$, then (2.3) is equivalent to (1.2) (see e.g. [11]).

In this section we study finite element approximation of the elliptic obstacle problem. For ease of exposition we only consider triangular elements in this paper. It does not seem to have any essential difficulty to generalize the main results to the 3d-simplex case. Let Ω^h be a polygonal approximation to Ω with boundary $\partial\Omega^h$. Let T^h be a partitioning of Ω^h into disjoint regular triangular τ, so that $\bar\Omega^h = \cup_{\tau\in T^h}\bar\tau$. Each element has at most one edge on $\partial\Omega^h$, and $\bar\tau$ and $\bar\tau'$ have either only one common vertex or a whole edge if τ

and $\tau' \in T^h$. We further require that $P_i \in \partial\Omega^h$ implies that $P_i \in \partial\Omega$, where $\mathbf{P} := \{P_i\}$ is the vertex set associated with the triangulation T^h. Let $\mathbf{P_0} := \{P_i\}_{i=1}^J = \mathbf{P}\backslash\partial\Omega$.

Associated with T^h is a finite dimensional subspace V^h of $C^0(\bar{\Omega}^h)$, such that $v_h|_\tau$ are affine functions for all $v_h \in V^h$ and $\tau \in T^h$. For ease of exposition we will assume that $\Omega^h = \Omega$. Let

$$V_g^h = \{v_h \in V^h \ : \ v_h(P_i) = g(P_i), \qquad \forall\, P_i \in (\mathbf{P} \cap \partial\Omega)\}.$$

Let $\pi_h : C^0(\bar{\Omega}^h)$ to V^h, denote the standard Lagrange interpolation operator such that for any $v \in C^0(\bar{\Omega}^h)$, $\pi_h v(P_i) = v(P_i)$ for all $P_i \in \mathbf{P}$.

We now consider the finite element approximation for (2.3). We only consider conforming linear Lagrange elements here. Let $K_g^h(\phi)$ be a closed convex set in V_g^h such that:

$$K_g^h(\phi) = \{v_h \in V_g^h \ : \ v_h(P_i) \geq \phi(P_i), \qquad \forall\, P_i \in \mathbf{P}\}.$$

Then a possible finite element approximation of (2.3) reads: For given f and g, find $u_h \in K_g^h(\phi)$ such that

$$a(u_h, v_h - u_h) \geq (f, v_h - u_h), \qquad \forall\, v_h \in K_g^h(\phi). \tag{2.4}$$

It is known that there is a unique solution to this approximation scheme. We shall assume that $g = 0$ so that $K_g^h(\phi)$ is contained in $K_g(\phi)$. For the general g, one may use the transformation $u - g$.

3. Sharp error estimates for zero obstacles

In this section, we derive sharp mixed error estimates for the finite element approximation to the elliptic obstacle problem (1.2), where the obstacle function $\phi(x) \equiv 0$. As mentioned in Section 1, it is impossible to replace f in (1.1) with $f|_{\Omega^+}$. What we shall do is to replace f with $f|_{\Omega_h^{+b}}$, where Ω_h^{+b} is a small neighborhood of the noncoincidence set, to be described below. When the free boundary consists of a finite numbers of piecewise smooth curves, meas$\,(\Omega_h^{+b}\backslash\Omega^+) \to 0$ as the maximum diameter of the finite elements tends to zero.

3.1. **Upper bounds.** In this subsection, we derive upper bounds of the approximation error. We define, noting Ω^+ is open and Ω^- is close,

$$\begin{aligned} &\Omega_h^+ = \{\cup\bar{\tau} \ : \ \bar{\tau} \subset \Omega^+, \ \tau \in T^h\}, \qquad &&\Omega_h^- = \{\cup\bar{\tau} \ : \ \tau \subset \Omega^-, \ \tau \in T^h\}, \\ &\Omega_h^b = \Omega\backslash(\Omega_h^+ \cup \Omega_h^-), &&\Omega_h^{+b} = \Omega_h^+ \cup \Omega_h^b. \end{aligned}$$

For a given set Q, we use the notations:

$$\bar{\bar{Q}} = \{\cup\bar{\tau} \ : \ \bar{\tau} \cap \bar{Q} \neq \emptyset, \ \tau \in T^h\}, \qquad \widetilde{Q} = \bar{\bar{Q}}\backslash\partial\bar{\bar{Q}}.$$

We denote by χ_Q the characteristic function of Q. Let h_τ denote the maximum diameter of the element τ in T^h and h_l be the maximum diameter of the edge l. Let ∂T^h be the set of all edges not on $\partial\Omega^h$:

$$\partial T^h = \{l = \partial\tau : \tau \in T^h,\ l \cap \partial\Omega = \emptyset\}.$$

We need the following lemmas in deriving error estimates of residual type.

Lemma 3.1 ([6]). *Let π_h be the standard Lagrange interpolation operator. Then for $m = 0, 1$, and $1 < q \le \infty$,*

$$\|v - \pi_h v\|_{m,q,\tau} \le C h_\tau^{2-m} |v|_{2,q,\tau}, \qquad \forall\, v \in W^{2,q}(\Omega). \tag{3.1}$$

Lemma 3.2 ([15]). *For all $v \in W^{1,q}(\Omega), 1 \le q \le \infty$,*

$$\|v\|_{0,q,\partial\tau} \le C\Big(h_\tau^{-1/q}\|v\|_{0,q,\tau} + h_\tau^{1-1/q}|v|_{1,q,\tau}\Big). \tag{3.2}$$

We also need the following positivity-preserving and H^1-stable interpolation operator $\Pi_h : L^1(\Omega) \to V_0^h$ defined by

$$(\Pi_h v)(x) = \sum_{i=1}^{J} \left(\frac{1}{\text{meas}\,(B_i)} \int_{B_i} v\right) \varphi_i(x), \tag{3.3}$$

where $\varphi_i(x)$ is the canonical basis function associated with the interior nodes $\{P_i\}_{i=1}^J$ of mesh T^h, i.e. $\varphi_i(P_j) = \delta_{ij}$, B_i is the largest ball inside of $\Delta_i = \{\cup\bar{\tau}\ :\ P_i \text{ is a vertex of } \tau\}$. The operator Π_h was introduced in [5], whose basic properties are provided below.

Lemma 3.3 ([5]). *If $v \in H_0^1(\Omega)$, then*

$$\|v - \Pi_h v\|_{0,\tau} \le C h_\tau |v|_{1,\tilde{\tau}},\ \ \tau \in T^h, \tag{3.4}$$

$$\|v - \Pi_h v\|_{0,l} \le C h_l^{1/2} |v|_{1,\tilde{l}},\ \ l \in \partial T^h. \tag{3.5}$$

Lemma 3.4 ([5]). *Let $v_h \in V_0^h, \tau \in T^h$, and $l \in \partial T^h$. Then*

$$\|v_h - \Pi_h v_h\|_{0,\tau} \le C \sum_{l \subset \tilde{\tau}} h_l^{3/2} \|[(\nabla v_h)\cdot \mathbf{n}]\|_{0,l}, \tag{3.6}$$

$$\|v_h - \Pi_h v_h\|_{0,l} \le C \sum_{l' \subset \tilde{l}} h_{l'} \|[(\nabla v_h)\cdot \mathbf{n}]\|_{0,l'}. \tag{3.7}$$

Having the above two lemmas, we are now in the position of stating our upper error bound:

Theorem 3.1. *Assume that A is a constant matrix, $f \in L^2(\Omega)$, $g \equiv 0$, and $\phi \equiv 0$. Let u and u_h be the solutions of* (2.3) *and* (2.4) *respectively. Then*

$$|u - u_h|_{1,\Omega}^2 \le C\Big(\eta_1^2 + \eta_2^2\Big), \tag{3.8}$$

where C is a positive constant depending on the minimum angle of T^h,

$$\eta_1^2 = \sum_{\tau\in T^h} h_\tau^2 \int_\tau f^2 \chi_{\Omega_h^{+b}}, \qquad \eta_2^2 = \sum_{l\in\partial T^h} h_l \int_l [(A\nabla u_h)\cdot \mathbf{n}]^2.$$

Proof. At this stage, we prove a slightly weaker estimate, where η_1 in (3.8) is replaced by $\hat\eta_1$

$$\hat\eta_1^2 = \sum_{\tau\in T^h} h_\tau^2 \int_\tau f^2 \chi_{\widetilde\Omega_h^{+b}}.$$

Roughly speaking, we slightly enlarge Ω_h^{+b} by adding $\mathcal{O}(h)$ elements around the free boundary, due to the use of the interpolator in [5, 18]. This makes no difference in constructing error indicators for practical computation as to be seen later. The result (3.8) can be proved in the same way by using the new interpolation operator introduced in Section 4.

Let $e = u - u_h$. For any $v_h \in K_0^h(0)$, it follows from (2.3) and (2.4) that

$$\begin{aligned} c|e|_{1,\Omega}^2 &\le a(u, u-u_h) - a(u_h, u-u_h) \\ &\le (f, u-u_h) - a(u_h, u-u_h) + a(u_h, v_h - u_h) - (f, v_h - u_h) \\ &= (f, u - v_h) + a(u_h, v_h - u). \end{aligned}$$

By taking $v_h = \Pi_h u$ and noting that $\operatorname{supp} u \subset \Omega^+$ and $\operatorname{supp} \Pi_h u \subset \widetilde\Omega_h^{+b}$, we have

$$(f, u - \Pi_h u) = (f\chi_{\widetilde\Omega_h^{+b}}, u - \Pi_h u) = (f\chi_{\widetilde\Omega_h^{+b}}, (I-\Pi_h)(u-u_h) + (I-\Pi_h)u_h).$$

It follows from (3.4) with $v = u - u_h$ and (3.6) with $v_h = u_h$ that

$$\begin{aligned} \|(I-\Pi_h)(u-u_h)\|_{0,\tau} &\le C h_\tau |u - u_h|_{1,\tilde\tau}, \\ \|(I-\Pi_h)u_h\|_{0,\tau} &\le C \sum_{l\subset\tilde\tau} h_l^{3/2} \|[(\nabla u_h)\cdot\mathbf{n}]\|_{0,l}, \end{aligned}$$

as in [5]. Thus, by (2.1)

$$(f, u - \Pi_h u) \le C\Big(\hat\eta_1^2 + \eta_2^2\Big) + \frac{c}{2}|e|_{1,\Omega}^2. \tag{3.9}$$

Next, it follows from (3.5), (3.7), and the Green's formula that

$$\begin{aligned} & a(u_h, v_h - u) \\ &= \sum_{\tau\in T^h} \int_\tau (A\nabla u_h)\nabla(v_h - u) = \sum_{l\in\partial T^h} \int_l [(A\nabla u_h)\cdot\mathbf{n}](v_h - u) \\ &\le \sum_{l\in\partial T^h} \|[(A\nabla u_h)\cdot\mathbf{n}]\|_{0,l}(\|(I-\Pi_h)(u-u_h)\|_{0,l} + \|(I-\Pi_h)u_h\|_{0,l}) \\ &\le \sum_{l\in\partial T^h} \|[(A\nabla u_h)\cdot\mathbf{n}]\|_{0,l}(h_l^{1/2}|u-u_h|_{1,\tilde l} + \sum_{l'\subset\tilde l} h_{l'} \|[(A\nabla u_h)\cdot\mathbf{n}]\|_{0,l'}) \\ &\le C\eta_2^2 + \frac{c}{4}|e|_{1,\Omega}^2. \end{aligned} \tag{3.10}$$

Then the desired result follows from (3.9) and (3.10). □

Remark 3.1. It follows from the above proof that one simply needs to replace η_1^2 by

$$\eta_1^2 = \sum_{\tau\in T^h} h_\tau^2 \int_\tau (f + \mathrm{div}(A\nabla u_h))^2 \chi_{\Omega_h^{+b}}$$

in Theorem 3.1 if A is a non-constant smooth matrix.

3.2. **Lower bounds.** We begin with the following lemmas for the bubble functions, the proof of which can be found in [2, 19].

Lemma 3.5. *Let τ_l^1, τ_l^2 be two elements in T^h with a common edge $l = \bar{\tau}_l^1 \cap \bar{\tau}_l^2$. For any constants B_τ and D_l, there exist polynomials w_τ in $H_0^1(\tau)$ and w_l in $H_0^1(\tau_l^1 \cup \tau_l^2)$ such that for $m = 0, 1$,*

$$\begin{aligned} &\int_\tau B_\tau w_\tau = h_\tau^2 \int_\tau B_\tau^2, \qquad |w_\tau|^2_{m,\tau} \le C h_\tau^{2(1-m)+2} \int_\tau B_\tau^2\,, \\ &\int_\tau D_l w_l = h_l \int_l D_l^2, \qquad |w_l|^2_{m,\tau_l^1\cup\tau_l^2} \le C h_l^{2(1-m)+1} \int_l D_l^2\,. \end{aligned}$$

To give local lower bounds, we rewrite the error estimators in (3.8) of Theorem 3.1 as

$$\begin{aligned} &\eta_1^2 = \sum_{\tau\in T^h} \eta_\tau^2, \qquad \eta_\tau^2 = \begin{cases} h_\tau^2 \displaystyle\int_\tau f^2, & \forall\, \tau \in \Omega_h^{+b} = \Omega_h^+ \cup \Omega_h^b, \\ 0, & \forall\, \tau \in \Omega_h^-, \end{cases} \\ &\eta_2^2 = \sum_{l\in\partial T^h} \eta_l^2, \qquad \eta_l^2 = h_l \int_l [(A\nabla u_h)\cdot \mathbf{n}]^2. \end{aligned}$$

Theorem 3.2. *Assume that all the conditions stated in Theorem 3.1 hold and that $l = \tau_l^1 \cap \tau_l^2$ $(l \in \partial T^h)$. Then*

$$\eta_{\tau_l^1}^2 + \eta_{\tau_l^2}^2 + \eta_l^2 \le \begin{cases} C\Big(|u-u_h|^2_{1,\tau_l^1\cup\tau_l^2} + h_l^2 \|f-\bar{f}\|^2_{0,\tau_l^1\cup\tau_l^2}\Big), & \forall(\tau_l^1\cup\tau_l^2) \subset \Omega_h^+, \\ C|u-u_h|^2_{1,\tau_l^1\cup\tau_l^2}, & \forall(\tau_l^1\cup\tau_l^2) \subset \Omega_h^-, \end{cases}$$

where $\bar{f}|_\tau = \int_\tau f/|\tau|$ and C is a positive constant depending on the matrix A and those in Lemma 3.5, but independent of h_τ and h_l..

Proof. For any $\tau \in \Omega_h^+$, let w_τ be defined as in Lemma 3.5 with $B_\tau = \bar{f}|_\tau$. By (2.3),

$$\begin{aligned}
h_\tau^2 \int_\tau f^2 &\le 2h_\tau^2 \int_\tau (\bar{f}^2 + (f-\bar{f})^2) \\
&= 2\int_\tau (f + \bar{f} - f) w_\tau + 2h_\tau^2 \int_\tau (f-\bar{f})^2 \\
&= 2\int_\tau (A\nabla(u-u_h)) \cdot \nabla w_\tau + 2\int_\tau (\bar{f} - f) w_\tau + 2h_\tau^2 \int_\tau (f-\bar{f})^2 \\
&\le C|u-u_h|_{1,\tau}^2 + \delta(|w_\tau|_{1,\tau}^2 + h_\tau^{-2}\|w_\tau\|_{0,\tau}^2) + Ch_\tau^2 \int_\tau (f-\bar{f})^2 \\
&\le C|u-u_h|_{1,\tau}^2 + C\delta h_\tau^2 \int_\tau f^2 + Ch_\tau^2 \|f-\bar{f}\|_{0,\tau}^2.
\end{aligned}$$

The last inequality implies

$$\eta_\tau^2 = h_\tau^2 \|f\|_{0,\tau}^2 \le C|u-u_h|_{1,\tau}^2 + Ch_\tau^2\|f-\bar{f}\|_{0,\tau}^2, \qquad \text{if } \tau \subset \Omega_h^+. \tag{3.11}$$

To estimate η_l, we define w_l as in Lemma 3.5 with $D_l = [(A\nabla u_h)\cdot \mathbf{n}]_l$. It follows from (2.3) and the Green's formula that if $(\tau_l^1 \cup \tau_l^2) \subset \Omega_h^+$,

$$\begin{aligned}
h_l \int_l [(A\nabla u_h)\cdot\mathbf{n}]^2 &= \int_l [(A\nabla u_h)\cdot\mathbf{n}] w_l = \int_{\tau_l^1\cup\tau_l^2} (A\nabla u_h)\cdot\nabla w_l \\
&= \int_{\tau_l^1\cup\tau_l^2} A\nabla(u_h-u)\cdot\nabla w_l + \int_{\tau_l^1\cup\tau_l^2} f w_l \\
&\le C|u-u_h|_{1,\tau_l^1\cup\tau_l^2}^2 + \delta(|w_l|_{1,\tau_l^1\cup\tau_l^2}^2 + h_l^{-2}\|w_l\|_{0,\tau_l^1\cup\tau_l^2}^2) + Ch_l^2\|f\|_{0,\tau_l^1\cup\tau_l^2}^2 \\
&\le C|u-u_h|_{1,\tau_l^1\cup\tau_l^2}^2 + C\delta h_l \int_l [(A\nabla u_h)\cdot\mathbf{n}]^2 + Ch_l^2\|f\|_{0,\tau_l^1\cup\tau_l^2}^2,
\end{aligned} \tag{3.12}$$

and if $(\tau_l^1 \cup \tau_l^2) \subset \Omega_h^-$,

$$\begin{aligned}
h_l \int_l [(A\nabla u_h)\cdot\mathbf{n}]^2 &= \int_{\tau_l^1\cup\tau_l^2} A\nabla(u_h-u)\cdot\nabla w_l \\
&\le C|u-u_h|_{1,\tau_l^1\cup\tau_l^2}^2 + C\delta h_l \int_l [(A\nabla u_h)\cdot\mathbf{n}]^2.
\end{aligned}$$

Thus we obtain that

$$\eta_l^2 \le \begin{cases} C\Big(|u-u_h|_{1,\tau_l^1\cup\tau_l^2}^2 + \eta_{\tau_l^1}^2 + \eta_{\tau_l^2}^2\Big), & \forall(\tau_l^1\cup\tau_l^2)\subset\Omega_h^+, \\ C|u-u_h|_{1,\tau_l^1\cup\tau_l^2}^2, & \forall(\tau_l^1\cup\tau_l^2)\subset\Omega_h^-. \end{cases} \tag{3.13}$$

Then the desired result follows from (3.11) and (3.13). □

It is noted that there is no extra regularity requirement for either u or the free boundary in the above two theorems.

3.3. **Sharp *a posteriori* error indicators.** It follows from Theorem 3.2 that the mixed estimator $\eta_1^2 + \eta_2^2$ is efficient on Ω_h^- and Ω_h^+. Thus one can reasonably expect that it should be efficient globally, at least when the free boundary is not too dense – if it is dense in a two dimensional domain then it is doubtful that whether or not the standard finite element method can produce accurate numerical results. Indeed this expectation can be confirmed. In the following we show the above mixed estimator is equivalent on discretization error when $u \in H^2(\Omega)$, and the free boundary consists of a finite numbers of piecewise smooth curves. Such assumptions are met in many practical applications. In such cases, we have

$$a(u,v) = (f\chi_{\Omega^+}, v), \qquad \forall\, v \in H_0^1(\Omega). \tag{3.14}$$

Then, similar to (3.12), we get for any $l \in \partial T^h$,

$$\begin{aligned} &h_l \int_l [(A\nabla u_h)\cdot \mathbf{n}]^2 \\ &\le C|u-u_h|^2_{1,\tau_l^1\cup\tau_l^2} + Ch_l^2\|f-\bar f\|^2_{0,(\tau_l^1\cup\tau_l^2)\cap\Omega_h^+} + Ch_l^2\|f\|^2_{0,(\tau_l^1\cup\tau_l^2)\cap\Omega^+}. \end{aligned} \tag{3.15}$$

Thus, we have from Theorem 3.2 and the above inequality that for any $l \in \partial T^h$,

$$\begin{aligned} &\eta^2_{\tau_l^1} + \eta^2_{\tau_l^2} + \eta_l^2 \\ &\le C\Big(|u-u_h|^2_{1,\tau_l^1\cup\tau_l^2} + h_l^2\|f-\bar f\|^2_{0,(\tau_l^1\cup\tau_l^2)\cap\Omega_h^+} + h_l^2\|f\|^2_{0,(\tau_l^1\cup\tau_l^2)\cap\Omega_h^b}\Big). \end{aligned}$$

Therefore,

$$\begin{aligned} &c\Big(\eta_1^2 + \eta_2^2 - \sum_{\tau\in T^h} h_\tau^2 \int_\tau (f-\bar f)^2\chi_{\Omega_h^+} - \sum_{\tau\in T^h} h_\tau^2 \int_\tau f^2\chi_{\Omega_h^b}\Big) \\ &\le |u-u_h|^2_{1,\Omega} \le C\Big(\eta_1^2+\eta_2^2\Big). \end{aligned} \tag{3.16}$$

It follows that if the free boundary $\partial\Omega^+$ is regular, say if the free boundary consists of a finite numbers piecewise smooth curves, or if the total length of the free boundary is finite, then meas (Ω_h^b) is of the order h. Thus the mixed error estimator $\eta_1^2 + \eta_2^2$ in (3.8) is reliable and efficient globally.

However, the above error estimator cannot be directly applied in mesh refinement since the characteristic function $\chi_{\Omega_h^{+b}}$ is uncomputable – we usually do not know the location of the free boundary. Nevertheless one can approximate it with some *a posteriori* quantities, thus obtain some *a posteriori* error indicators, which can then be used in adaptive finite element method.

One can simply let $\chi_{\Omega_h^{+b}} \equiv 1$ in η_1. This gives us just the *a posteriori* error estimator (1.1), see [5]. Another *a posteriori* approximator was suggested in [18]. The basic idea is to approximate $\chi_{\Omega_h^{+b}}$ by the following function $\chi^h_{\Omega^+}$: For $\alpha > 0$, let

$$\chi^h_{\Omega^+} = \frac{u_h}{h^\alpha + u_h}.$$

Then, in practical computations we replace η_1^2 by

$$\widetilde{\eta}_1^2 = \sum_{\tau\in T^h} h_\tau^2 \int_\tau f^2\chi^h_{\Omega^+}.$$

Let us show that this approximator is more reasonable. To demonstrate this, we divide Ω into three parts:

$$\Omega_h^-, \quad \Omega^{\alpha/2} := \{x \in \Omega^+ : u_h < h^{\alpha/2}\}, \text{ and } \Omega_h^{+b} \backslash \Omega^{\alpha/2}.$$

Then, for $\tau \subset \Omega_h^-$,

$$\|\chi_{\Omega^+}^h - \chi_{\Omega_h^{+b}}\|_{0,\infty,\tau} = \left\| \frac{u_h - u}{h^\alpha + u_h - u} \right\|_{0,\infty,\tau} \leq \min\{1, h^{-\alpha}\|u_h - u\|_{0,\infty,\tau}\},$$

and for $\tau \subset \Omega_h^{+b} \backslash \Omega^{\alpha/2}$,

$$\|\chi_{\Omega^+}^h - \chi_{\Omega_h^{+b}}\|_{0,\infty,\tau} = \left\| \frac{h^\alpha}{h^\alpha + u_h} \right\|_{0,\infty,\tau} \leq h^{\alpha/2}.$$

Therefore, if the error $\|u_h - u\|_{0,\infty,\tau}$ for $\tau \subset \Omega_h^-$ (where $u \equiv 0$) is of the order h^β with $\beta > \alpha$, then the difference caused by this approximation is a high-order small local quantity for all $\tau \subset \Omega_h^- \cup (\Omega_h^{+b} \backslash \Omega^{\alpha/2})$. It has been shown in [18] that if $\|u_h - u\|_{0,\infty,\Omega} = \mathcal{O}(h^\beta)$ $(\beta > \alpha)$, then $\text{meas}\,(\Omega^{\alpha/2}) \to 0$, as $h \to 0$. The numerical experiments carried out in [18] and [16] showed that the *a posteriori* error indicator $\widetilde{\eta}_1^2 + \eta_2^2$ leads to efficient mesh refinement schemes.

We should emphasize that it seems not reasonable, from a computational point of view, to simply replace $\chi_{\Omega^+}^h$ by the characteristic function of the numerical contact set, since this approximation is numerically unstable.

4. Sharp error estimates for nonzero obstacles

In this section, we consider the case of nonzero obstacles. We wish to extend the sharp mixed error estimates for the zero obstacle case to more general cases.

To begin with, let us briefly review some related techniques in handling the nonzero obstacles, see [16, 18]. If $\phi \in H^2(\Omega)$, one may set $u^* = u - \phi$. Then the problem is reduced to the case of zero obstacles. Let $u_h^* \in K_{g-\phi}^h(0)$ be the finite approximate to u^* such that

$$a(u_h^*, v_h - u_h^*) \geq (f + \text{div}(A\nabla\phi), v_h - u_h^*), \qquad \forall\, v_h \in K_{g-\phi}^h(0). \tag{4.1}$$

As discussed in Section 3, the error indicators for $|u^* - u_h^*|_{1,\Omega}^2$ should be

$$\begin{aligned} \eta_1^2 &= \sum_{\tau \in T^h} h_\tau^2 \int_\tau (f + \text{div}(A\nabla\phi))^2 \chi_{\Omega_h^{+b}}, \\ \eta_2^2 &= \sum_{l \in \partial T^h} h_l \int_l [(A\nabla u_h^*) \cdot \mathbf{n}]^2. \end{aligned} \tag{4.2}$$

However, as mentioned before these error indicators may not be sharp for this case, as $\text{div}(A\nabla\phi)$ now appears on the whole Ω^+. Also, the method does not work if ϕ is not in $H^2(\Omega)$. Thus here we directly consider the error $|u - u_h|_{1,\Omega}^2$. Similar to the observations in the last section we expect that the values of f outside a small neighborhood of Ω^+

and the values of ϕ outside that of Ω^- play no role in our error estimates, as long as the free boundary is regular.

The general nonzero case is very complicated, see some examples and discussions in [5]. Here we are interested in the situation where the obstacle $\phi \in W^{1,p}(\Omega)$ $(p > 2)$ is a piecewise smooth function; that is, there exist bounded open sets $\{\Omega_i\}_{i=1}^M$ such that

$$\phi \in H^2(\Omega_i),\ i = 1, 2, \cdots, M,\ \text{meas}\,(\Omega \backslash \cup_{i=1}^M \Omega_i) = 0. \tag{4.3}$$

Furthermore, we assume that the solution $u \in H^2(\Omega)$ and the free boundary has Lebesgue measure zero. Again these assumptions are not very restrictive in applications, see [11].

The key difficulty in deriving sharp mixed error estimates for a nonzero obstacle case is as follows. In residual estimation, one needs to take a good approximation of u in $K_g^h(\phi)$ as the test function. If $\phi = 0$, then one can just take $v_h = \Pi_h u$ as before. However in general $\Pi_h u \notin K_g^h(\phi)$ as $\Pi_h \phi \ngeq \phi_h$. Thus in [5], $v_h = \pi_h(\max\{\Pi_h u, \phi_h\})$. Unfortunately, the test function in this case is required to satisfy $v_h \le u_h$ in the area where the inactive data are to be removed. Thus the selection $v_h = \pi_h(\max\{\Pi_h u, \phi_h\})$ will not produce sharp estimates. We will introduce a new interpolator to overcome this difficulty. More precisely, we introduce a new hybrid interpolation operator, which is a patchwork of the positivity-preserving interpolation operator Π_h introduced in [5] and the standard Lagrange interpolation operator π_h. In this way, we are able to obtain sharp mixed error estimates for the non-zero obstacle case. The key observation of introducing this interpolator is the following: Since u is known inside the coincidence set Ω^- it is unnecessary to require H^1-stability for the interpolator inside Ω^-.

Under our regularity assumptions, we have

$$-\text{div}(A\nabla u) = f\chi_{\Omega^+} - (\text{div}(A\nabla(\phi|_{\Omega^-})))\chi_{\Omega^-}, \qquad \text{a.e. in } \Omega. \tag{4.4}$$

We define

$$f_\phi|_{\Omega_i} = \text{div}(A\nabla(\phi|_{\Omega_i})), \qquad i = 1, 2, \cdots, M.$$

We note that f_ϕ is well defined since $\phi \in H^2(\Omega_i)$ and $\text{meas}\,(\Omega \backslash \cup_{i=1}^M \Omega_i) = 0$. Then the following weak formula of (4.4) holds:

$$a(u, v) = (f\chi_{\Omega^+} - f_\phi\chi_{\Omega^-}, v), \qquad \forall\, v \in H_0^1(\Omega). \tag{4.5}$$

We define the unstable discrete numerical coincidence vertex set Ω_P^- by

$$\Omega_P^- = \{P_i \in \mathbf{P_0} \cap \Omega_h^{+b}\ :\ u_h(P_i) = \phi(P_i)\}.$$

where $\mathbf{P_0} := \{P_i\}_{i=1}^J = \{P_i\} \backslash \partial\Omega$ is defined in Section 2.

4.1. **Upper bounds.** We introduce an interpolation operator $\widetilde{\Pi}_h$ defined by

$$(\widetilde{\Pi}_h v)(x) = \sum_{P_i \in \Omega_h^- \setminus \partial\Omega} v(P_i)\varphi_i(x) + \sum_{P_i \in (\Omega_h^-)^c \setminus \partial\Omega} (\Pi_h v)(P_i)\varphi_i(x) \tag{4.6}$$

where $\varphi_i(x)$ is the canonical basis function associated with the interior nodes $P_i \in \mathbf{P_0}$. This interpolation function is a patchwork of $\pi_h v$ and $\Pi_h v$. We shall take

$$\tilde{u}_h := \pi_h(\max\{\widetilde{\Pi}_h u, \pi_h \phi\}) \tag{4.7}$$

as the test function in our analysis. The following two lemmas provide some basic properties for the operator $\widetilde{\Pi}_h$ given in (4.6).

Lemma 4.1. *Let $\phi \in W^{1,p}(\Omega)$ $(p > 2)$ and $v \in H_0^1(\Omega)$ such that*

$$v \geq \phi \quad \text{a.e. in } \Omega, \qquad v = \phi \quad \text{in } \Omega^-.$$

Let T^h be a regular partitioning of Ω as described in Section 2 so that we can define the finite element spaces as before. We have for any $v_h \in V_0^h$ and $\tau \subset T^h$,

$$\|\widetilde{\Pi}_h v - v\|_{0,\tau} \leq \begin{cases} \|\phi - \pi_h\phi\|_{0,\tau}, & \text{if } \tau \subset \Omega_h^-, \\ C\big(\|\phi - \pi_h\phi\|_{0,\tau'} + h_\tau |v - v_h|_{1,\tilde{\tau}\cup\tilde{\tau}'}\big) & \\ \quad +C \displaystyle\sum_{l \subset \tilde{\tau}\cup\tilde{\tau}'} h_l^{3/2} \|[(\nabla v_h)\cdot \mathbf{n}]\|_{0,l}, & \text{if } \tau \subset \Omega_h^b, \\ Ch_\tau |v - v_h|_{1,\tilde{\tau}} + C\displaystyle\sum_{l\subset\tilde{\tau}} h_l^{3/2}\|[(\nabla v_h)\cdot\mathbf{n}]\|_{0,l}, & \text{if } \tau \subset \Omega_h^+, \end{cases} \tag{4.8}$$

where $\tau' \subset (\tilde{\tau} \cap \Omega_h^-)$, $\tau' \in T^h$, and C is a positive constant independent of $h, v,$ and ϕ .

Proof. For $\tau \subset \Omega_h^-$, the result comes immediately from the fact that $\widetilde{\Pi}_h v|_\tau = \pi_h\phi|_\tau$ and $v|_\tau \equiv \phi|_\tau$. For $\tau \subset \Omega_h^b$, we have

$$\|\widetilde{\Pi}_h v - v\|_{0,\tau} \leq \|\widetilde{\Pi}_h v - \Pi_h v\|_{0,\tau} + \|\Pi_h v - v\|_{0,\tau}.$$

The second term on the right side of the above inequality is estimated by replacing v with $v - v_h$ in (3.4) and using (3.6):

$$\begin{aligned} \|\Pi_h v - v\|_{0,\tau} &\leq \|(\Pi_h - I)(v - v_h)\|_{0,\tau} + \|(\Pi_h - I)v_h\|_{0,\tau} \\ &\leq Ch_\tau|v - v_h|_{1,\tilde{\tau}} + C\sum_{l\subset\tilde{\tau}} h_l^{3/2}\|[(\nabla v_h)\cdot\mathbf{n}]\|_{0,l}. \end{aligned} \tag{4.9}$$

The first term needs to be considered at the each vertex P_i of τ individually. Assume that $P_i \in \partial\Omega_h^- \backslash \partial\Omega$. Let $\tau' \subset (\tilde{P}_i \cap \Omega_h^-)$. Then

$$
\begin{aligned}
&|(\widetilde{\Pi}_h v - \Pi_h v)(P_i)| = |(\pi_h \phi - \Pi_h v)(P_i)| \\
&\le C h_\tau^{-1} \|\pi_h \phi - \Pi_h v\|_{0,\tau'} \\
&\le C h_\tau^{-1} (\|\pi_h \phi - \phi\|_{0,\tau'} + \|v - \Pi_h v\|_{0,\tau'}) \\
&\le C h_\tau^{-1} (\|\phi - \pi_h \phi\|_{0,\tau'} + h_\tau |v - v_h|_{1,\tilde{\tau}'} + \sum_{l \subset \tilde{\tau}'} h_l^{3/2} \|[(\nabla v_h) \cdot \mathbf{n}]\|_{0,l}).
\end{aligned}
\tag{4.10}
$$

Now assume that $P_i \in (\Omega_h^-)^c \backslash \partial\Omega$. Then $(\widetilde{\Pi}_h v - \Pi_h v)(P_i) = 0$. Thus,

$$
\|\widetilde{\Pi}_h v - \Pi_h v\|_{0,\tau} \le \sum_{P_i \in (\bar{\tau} \cap \partial\Omega_h^-) \backslash \partial\Omega} |(\widetilde{\Pi}_h v - \Pi_h v)(P_i)| \|\varphi_i\|_{0,\tau},
$$

and the desired result follows from (4.10) and the inequality $\|\varphi_i\|_{0,\tau} \le C h_\tau$. Finally, for $\tau \subset \Omega_h^+$, $\|\widetilde{\Pi}_h v - v\|_{0,\tau} = \|\Pi_h v - v\|_{0,\tau}$. This term has been estimated in (4.9). □

Lemma 4.2. *Assume that all the conditions stated in Lemma* 4.1 *hold. We have for any* $v_h \in V_0^h$ *and* $l \in \partial T^h$,

$$
\|\widetilde{\Pi}_h v - v\|_{0,l} \le
\begin{cases}
C\Big(h_\tau^{-1/2} \|\phi - \pi_h \phi\|_{0,\tau} + h_\tau^{1/2} |\phi - \pi_h \phi|_{1,\tau}\Big), & \text{if } l \subset \Omega_h^-, \\
C\Big(h_\tau^{-1/2} \|\phi - \pi_h \phi\|_{0,\tau} + h_\tau^{1/2} |v - v_h|_{1,\tilde{l} \cup \tilde{\tau}}\Big) & \\
\quad + C \displaystyle\sum_{l' \subset (\tilde{l} \cup \tilde{\tau})} h_{l'} \|[(\nabla v_h) \cdot \mathbf{n}]\|_{0,l'}, & \text{if } l \subset \Omega_h^b, \\
C h_l^{1/2} |v - v_h|_{1,\tilde{l}} + C \displaystyle\sum_{l' \subset \tilde{l}} h_{l'} \|[(\nabla v_h) \cdot \mathbf{n}]\|_{0,l'}, & \text{if } l \subset \Omega_h^+,
\end{cases}
\tag{4.11}
$$

where $\tau \subset (\tilde{l} \cap \Omega_h^-)$, $\tau \in T^h$, $l' \in \partial T^h$, *and* C *is a positive constant independent of* h, v, *and* ϕ .

Proof. For $l \subset \Omega_h^-$, let $\tau \subset \Omega_h^-$ with $l = \partial\tau$. Then we have from (3.2) that

$$
\|\widetilde{\Pi}_h v - v\|_{0,l} = \|\pi_h \phi - \phi\|_{0,l} \le C\Big(h_\tau^{-1/2} \|\phi - \pi_h \phi\|_{0,\tau} + h_\tau^{1/2} |\phi - \pi_h \phi|_{1,\tau}\Big).
$$

When $l \subset \Omega_h^b$, we have

$$
\|\widetilde{\Pi}_h v - v\|_{0,l} \le \|\widetilde{\Pi}_h v - \Pi_h v\|_{0,l} + \|\Pi_h v - v\|_{0,l}.
$$

The second term on the right side of the above inequality is estimated by replacing v with $v - v_h$ in (3.5) and by using (3.7):

$$
\begin{aligned}
\|\Pi_h v - v\|_{0,l} &\le \|(\Pi_h - I)(v - v_h)\|_{0,l} + \|(\Pi_h - I) v_h\|_{0,l} \\
&\le C h_l^{1/2} |v - v_h|_{1,\tilde{l}} + C \sum_{l' \subset \tilde{l}} h_{l'} \|[(\nabla v_h) \cdot \mathbf{n}]\|_{0,l'}.
\end{aligned}
\tag{4.12}
$$

The first term needs to be considered for each vertex P_i of l. First assume that $P_i \in \partial\Omega_h^-\backslash\partial\Omega$. Let $\tau \subset (\tilde{P}_i \cap \Omega_h^-)$ so that $v|_\tau \equiv \phi|_\tau$. Then

$$\begin{aligned}
&|(\widetilde{\Pi}_h v - \Pi_h v)(P_i)| = |(\pi_h \phi - \Pi_h v)(P_i)| \\
&\leq C h_\tau^{-1} \|\pi_h \phi - \Pi_h v\|_{0,\tau} \\
&\leq C h_\tau^{-1} (\|\pi_h \phi - \phi\|_{0,\tau} + \|v - \Pi_h u\|_{0,\tau}) \\
&\leq C h_\tau^{-1} (\|\phi - \pi_h \phi\|_{0,\tau} + h_\tau |v - v_h|_{1,\tilde{\tau}} + \sum_{l' \subset \tilde{\tau}} h_{l'}^{3/2} \|[(\nabla v_h) \cdot \mathbf{n}]\|_{0,l'}).
\end{aligned} \tag{4.13}$$

Now assume that $P_i \in (\Omega_h^-)^c\backslash\partial\Omega$. Then $(\widetilde{\Pi}_h v - \Pi_h v)(P_i) = 0$. Thus,

$$\|\widetilde{\Pi}_h v - \Pi_h v\|_{0,l} \leq \sum_{P_i \in (\bar{\tau} \cap \partial\Omega_h^-)\backslash\partial\Omega} |(\widetilde{\Pi}_h v - \Pi_h v)(P_i)| \|\varphi_i\|_{0,l},$$

and the desired result follows from (4.13) and the inequality $\|\varphi_i\|_{0,l} \leq C h_l^{1/2}$. Finally, for $l \subset \Omega_h^+$, $\|\widetilde{\Pi}_h v - v\|_{0,l} = \|\Pi_h v - v\|_{0,l}$. This term has again been estimated in (4.12). Thus the proof of this lemma is complete. □

We are now ready to obtain the following upper bounds for non-zero obstacle function ϕ.

Theorem 4.1. *Let u and u_h be the solutions of* (2.3) *and* (2.4) *respectively. Assume that A is a constant matrix, $f \in L^2(\Omega)$, $g \equiv 0$, $\phi \in W^{1,p}(\Omega)$ $(p > 2)$ satisfying* (4.3), *$u \in H^2(\Omega)$ and the free boundary has Lebesgue measure zero so that* (4.5) *holds. Then*

$$|u - u_h|_{1,\Omega}^2 \leq C \sum_{i=1}^{4} \eta_i^2, \tag{4.14}$$

where C is a positive constant depending on the minimum angle of T^h,

$$\begin{aligned}
\eta_1^2 &= \sum_{\tau \in T^h} h_\tau^2 (\|f \chi_{\Omega_h^{+b}}\|_{0,\tau}^2 + \|f_\phi \chi_{\Omega^-}\|_{0,\tau}^2), \\
\eta_2^2 &= \sum_{l \in \partial T^h} h_l \|[(A \nabla u_h) \cdot \mathbf{n}]\|_{0,l}^2, \\
\eta_3^2 &= \sum_{\tau \in T^h} (h_\tau^{-2} \|(\phi - \pi_h \phi) \chi_{\Omega^-}\|_{0,\tau}^2 + |(\phi - \pi_h \phi) \chi_{\Omega_h^-}|_{1,\tau}^2), \\
\eta_4^2 &= \sum_{l \subset \tilde{\Omega}_P^-} h_l \|[(A \nabla (\pi_h \phi)) \cdot \mathbf{n}]\|_{0,l}^2 + \sum_{\tau \subset \tilde{\Omega}_P^-} h_\tau^{-2} \|\phi - \pi_h \phi\|_{0,\tau}^2.
\end{aligned}$$

Proof. By using the approximation scheme (2.4), we have that for any $v_h \in K_0^h(\phi)$,

$$\begin{aligned}
c|u-u_h|_{1,\Omega}^2 \leq\ & a(u,u-u_h) - a(u_h,u-u_h) + a(u_h,v_h-u_h) - (f,v_h-u_h) \\
=\ & a(u,u-u_h) + a(u_h,v_h-u) - (f,v_h-u_h) \\
=\ & \Big\{a(u,u-u_h) - (f,\widetilde{\Pi}_h u - u_h)\Big\} + a(u_h,\widetilde{\Pi}_h u - u) \\
& + \Big\{a(u_h,v_h-\widetilde{\Pi}_h u) - (f,v_h-\widetilde{\Pi}_h u)\Big\} \\
:=\ & I_1 + I_2 + I_3.
\end{aligned}$$

It follows from the equation (4.5) that

$$\begin{aligned}
I_1 &= a(u,u-u_h) - (f,\widetilde{\Pi}_h u - u_h) \\
&= (f\chi_{\Omega^+} - f_\phi\chi_{\Omega^-}, u-u_h) - (f,\widetilde{\Pi}_h u - u_h) \\
&= (f\chi_{\Omega^+} - f_\phi\chi_{\Omega^-}, u-\widetilde{\Pi}_h u) - ((f+f_\phi)\chi_{\Omega^-},\widetilde{\Pi}_h u - u_h) := I_{11} + I_{12}.
\end{aligned} \tag{4.15}$$

We now estimate I_{11} and I_{12} separately. Using (4.8) of Lemma 4.1 gives

$$\begin{aligned}
I_{11} &= (f\chi_{\Omega^+} - f_\phi\chi_{\Omega^-}, u-\widetilde{\Pi}_h u) \leq \sum_{\tau\in T^h} \|f\chi_{\Omega^+} - f_\phi\chi_{\Omega^-}\|_{0,\tau}\|u-\widetilde{\Pi}_h u\|_{0,\tau} \\
&\leq C\Big(\eta_1^2+\eta_2^2+\eta_3^2\Big) + \delta|u-u_h|_{1,\Omega}^2.
\end{aligned}$$

It follows from our regularity assumptions and (1.2)

$$\tilde{f} := f + f_\phi \leq 0, \qquad \forall\, x\in\Omega^-.$$

With the following observations

$$\begin{aligned}
& u_h - \widetilde{\Pi}_h u = u_h - \pi_h u \geq 0, && \forall\, x\in\Omega_h^-, \\
& u - \max\{\phi,u_h\} = \phi - \max\{\phi,u_h\} \leq 0, && \forall\, x\in\Omega^-, \\
& 0 \leq \max\{\phi,u_h\} - u_h \leq |\phi - \pi_h\phi|, && \forall\, x\in\Omega\,,
\end{aligned}$$

we obtain the following estimates for I_{12}

$$\begin{aligned}
I_{12} &= (\tilde{f}\chi_{\Omega^-}, u_h - \widetilde{\Pi}_h u) \\
&= (\tilde{f}\chi_{\Omega_h^-}, u_h - \pi_h u) + (\tilde{f}\chi_{(\Omega^-\backslash\Omega_h^-)}, u_h - \widetilde{\Pi}_h u) \\
&\leq (\tilde{f}\chi_{(\Omega^-\backslash\Omega_h^-)}, u_h - \max\{\phi,u_h\} + u - \widetilde{\Pi}_h u) \\
&\leq \|\tilde{f}\|_{0,\Omega^-\backslash\Omega_h^-}\Big(\|\phi-\pi_h\phi\|_{0,\Omega^-\backslash\Omega_h^-} + \|u-\widetilde{\Pi}_h u\|_{0,\Omega^-\backslash\Omega_h^-}\Big) \\
&\leq C\Big(\eta_1^2+\eta_2^2+\eta_3^2\Big) + \delta|u-u_h|_{1,\Omega}^2.
\end{aligned}$$

We now turn to the estimate of I_2. By applying the Green's formula and (4.11) we obtain

$$
\begin{aligned}
I_2 = a(u_h, \widetilde{\Pi}_h u - u) &= \sum_{l\in\partial T^h} \int_l [(A\nabla u_h)\cdot \mathbf{n}](\widetilde{\Pi}_h u - u) \\
&\le \sum_{l\in\partial T^h} \|[(A\nabla u_h)\cdot\mathbf{n}]\|_{0,l}\|\widetilde{\Pi}_h u - u\|_{0,l} \\
&\le C\Big(\eta_1^2+\eta_2^2+\eta_3^2\Big) + \delta|u-u_h|^2_{1,\Omega}.
\end{aligned}
$$

To estimate I_3, we define $R_h(v) := a(u_h, v) - (f, v)$. Let $v_h = \tilde{u}_h$ with $\tilde{u}_h$ defined by (4.7). Then

$$
v_h - \widetilde{\Pi}_h u = \tilde{u}_h - \widetilde{\Pi}_h u = \sum_{P_i\in\mathbf{P_0}} (\tilde{u}_h - \widetilde{\Pi}_h u)(P_i)\varphi_i(x).
$$

It follows from the definitions of $\tilde{u}_h$ and $\widetilde{\Pi}_h u$ that

$$
\tilde{u}_h(P_i) = \pi_h u(P_i) = \widetilde{\Pi}_h u(P_i), \qquad \forall\, P_i \in \Omega_h^-.
$$

Moreover, by the definition of R_h and the numerical scheme (2.4), we have

$$
R_h(\varphi_i) = a(u_h,\varphi_i) - (f,\varphi_i) = 0, \qquad \forall\, P_i \in \{P_i\in\mathbf{P_0} \;:\; u_h(P_i) > \phi(P_i)\}.
$$

The above results lead to

$$
\begin{aligned}
I_3 &= a(u_h, \tilde{u}_h - \widetilde{\Pi}_h u) - (f, \tilde{u}_h - \widetilde{\Pi}_h u) = R_h(\tilde{u}_h - \widetilde{\Pi}_h u) \\
&= \sum_{P_i\in\Omega_P^-} (\tilde{u}_h - \widetilde{\Pi}_h u)(P_i) R_h(\varphi_i).
\end{aligned}
\tag{4.16}
$$

For $R_h(\varphi_i)$, we have

$$
\begin{aligned}
|R_h(\varphi_i)| &= \sum_{l\subset\tilde{P}_i} \int_l |[(A\nabla u_h)\cdot\mathbf{n}]\varphi_i| + |(f,\varphi_i)| \\
&\le \sum_{l\subset\tilde{P}_i} \|[(A\nabla u_h)\cdot\mathbf{n}]\|_{0,l}\|\varphi_i\|_{0,l} + \|f\|_{0,\tilde{P}_i}\|\varphi_i\|_{0,\tilde{P}_i} \\
&\le C\sum_{l\subset\tilde{P}_i} h_l^{1/2}\|[(A\nabla u_h)\cdot\mathbf{n}]\|_{0,l} + C\sum_{\tau\subset\tilde{P}_i} h_\tau\|f\|_{0,\tau}.
\end{aligned}
$$

For $(\tilde{u}_h - \widetilde{\Pi}_h u)(P_i)$, we note $\Pi_h u \ge \Pi_h\phi$ since $u \ge \phi$. Then, for $P_i \in (\Omega_h^-)^c$,

$$
\begin{aligned}
0 &\le (\tilde{u}_h - \widetilde{\Pi}_h u)(P_i) = \max\{\Pi_h u(P_i), \phi(P_i)\} - \Pi_h u(P_i) \\
&\le |(\pi_h\phi - \Pi_h\phi)(P_i)| \le |(I-\Pi_h)\pi_h\phi(P_i)| + |\Pi_h(\pi_h - I)\phi(P_i)| \\
&\le C\sum_{l\subset\tilde{P}_i} h_l^{1/2}\|[(A\nabla(\pi_h\phi))\cdot\mathbf{n}]\|_{0,l} + C\sum_{\tau\subset\tilde{P}_i} h_\tau^{-1}\|\phi - \pi_h\phi\|_{0,\tau}.
\end{aligned}
$$

The above two results, together with (4.16), yield

$$
I_3 \le C\Big(\eta_1^2+\eta_2^2+\eta_4^2\Big).
$$

Combining the estimates for I_{11}, I_{12}, I_2 and I_3 leads to the desired estimate (4.14). ☐

4.2. **Lower bounds.** In this subsection, we will provide a lower bound for the error estimators, as given in the following theorem.

Theorem 4.2. *Assume that all the conditions stated in Theorem* 4.1 *hold. Then*

$$\sum_{i=1}^{2}\eta_i^2 \le C|u-u_h|_{1,\Omega}^2 + C\sum_{\tau\in T^h} h_\tau^2\|F-\bar F\|_{0,\tau}^2 + \sum_{\tau\in T^h} h_\tau^2\int_\tau f^2\chi_{(\Omega_h^b\backslash\Omega^+)}, \tag{4.17}$$

where η_i are defined in Theorem 4.1*,*

$$F = f\chi_{\Omega^+} - f_\phi\chi_{\Omega^-}, \qquad \bar F = \int_\tau F/|\tau|,$$

and C is a positive constant depending on the matrix A and those in Lemma 3.5*, but independent of the mesh diameter of h_τ.*

Proof. For any $\tau\in T^h$, let w_τ be defined as in Lemma 3.5 with $B_\tau = \bar F|_\tau$. Then,

$$\begin{aligned}
h_\tau^2\int_\tau F^2 &\le 2h_\tau^2\int_\tau(\bar F^2 + (F-\bar F)^2)\\
&= 2\int_\tau w_\tau(F+\bar F - F) + 2h_\tau^2\int_\tau (F-\bar F)^2\\
&= 2\int_\tau (A\nabla(u-u_h))\cdot\nabla w_\tau + 2\int_\tau w_\tau(\bar F - F) + 2h_\tau^2\int_\tau(F-\bar F)^2\\
&\le C|u-u_h|_{1,\tau}^2 + \delta(|w_\tau|_{1,\tau}^2 + h_\tau^{-2}\|w_\tau\|_{0,\tau}^2) + Ch_\tau^2\int_\tau(F-\bar F)^2\\
&\le C|u-u_h|_{1,\tau}^2 + C\delta h_\tau^2\int_\tau F^2 + Ch_\tau^2\|F-\bar F\|_{0,\tau}^2.
\end{aligned}$$

As a consequence, we have

$$h_\tau^2\|F\|_{0,\tau}^2 \le C|u-u_h|_{1,\tau}^2 + Ch_\tau^2\|F-\bar F\|_{0,\tau}^2, \qquad \forall\,\tau\in T^h. \tag{4.18}$$

By noting that

$$\eta_1^2 = \sum_{\tau\in T^h} h_\tau^2\|F\|_{0,\tau}^2 + \sum_{\tau\in T^h} h_\tau^2\int_\tau f^2\chi_{(\Omega_h^b\backslash\Omega^+)},$$

we have

$$\eta_1^2 \le C|u-u_h|_{1,\Omega}^2 + C\sum_{\tau\in T^h} h_\tau^2\|F-\bar F\|_{0,\tau}^2 + \sum_{\tau\in T^h} h_\tau^2\int_\tau f^2\chi_{(\Omega_h^b\backslash\Omega^+)}. \tag{4.19}$$

For η_2, we define w_l as in Lemma 3.5 with $D_l = [(A\nabla u_h)\cdot\mathbf{n}]_l$. By the Green's formula,

$$\begin{aligned}
h_l\int_l[(A\nabla u_h)\cdot\mathbf{n}]^2 &= \int_l[(A\nabla u_h)\cdot\mathbf{n}]w_l = \int_{\tau_l^1\cup\tau_l^2}(A\nabla u_h)\cdot\nabla w_l\\
&= \int_{\tau_l^1\cup\tau_l^2} A\nabla(u_h-u)\cdot\nabla w_l + \int_{\tau_l^1\cup\tau_l^2} w_l F\\
&\le C|u-u_h|_{1,\tau_l^1\cup\tau_l^2}^2 + \delta\Big(|w_l|_{1,\tau_l^1\cup\tau_l^2}^2 + h_l^{-2}\|w_l\|_{0,\tau_l^1\cup\tau_l^2}^2\Big) + Ch_l^2\|F\|_{0,\tau_l^1\cup\tau_l^2}^2\\
&\le C|u-u_h|_{1,\tau_l^1\cup\tau_l^2}^2 + C\delta h_l\int_l[(A\nabla u_h)\cdot\mathbf{n}]^2 + Ch_l^2\|F\|_{0,\tau_l^1\cup\tau_l^2}^2.
\end{aligned} \tag{4.20}$$

Then it follows from (4.18) and (4.19) that

$$h_l \int_l [(A\nabla u_h)\cdot \mathbf{n}]^2 \le C|u-u_h|^2_{1,\tau_l^1\cup\tau_l^2} + Ch_l^2\|F-\bar{F}\|^2_{0,\tau_l^1\cup\tau_l^2}, \quad \forall\, l\in \partial T^h, \tag{4.21}$$

$$\eta_2^2 \le C|u-u_h|^2_{1,\Omega} + C\sum_{\tau\in T^h} h_\tau^2\|F-\bar{F}\|^2_{0,\tau} + C\sum_{\tau\in T^h} h_\tau^2 \int_\tau f^2\chi_{(\Omega_h^b\backslash\Omega^+)}. \tag{4.22}$$

The desired result follows from the estimates for η_1 and η_2, namely (4.19) and (4.22). $\square$

We note that, for all $\tau \in \Omega_h^+\backslash\tilde{\Omega}_P^-$, the local indicators in (4.14) become

$$(\eta_1^2)_\tau = h_\tau^2\int_\tau f^2\chi_{\Omega_h^{+b}}, \qquad (\eta_2^2)_\tau = \sum_{l=\partial\tau} h_l \int_l [(A\nabla u_h)\cdot\mathbf{n}]^2,$$

which are exactly the same as in the case of zero obstacle (3.8). For $\tau\in\Omega_h^-$, the local indicators in (4.14) use the *true* information of $u\equiv\phi$, and this should also be reasonable. The size of the remaining domains (meas (Ω_h^b) and meas $(\tilde{\Omega}_P^-)$) would be small under the conditions as discussed at the end of Section 3. Therefore we can see that the mixed error estimate given in Theorem 4.1 is quite sharp. Again most of the inactive data has been removed from the estimators.

4.3. **Sharp *a posteriori* error indicators.** In computation, as in Section 3, we can approximate the characteristic functions χ_{Ω^+} and χ_{Ω^-} by the following χ_+^h and χ_-^h respectively. For $\alpha^+,\alpha^- > 0$, let

$$\chi_+^h = \frac{u_h-\pi_h\phi}{h^{\alpha^+}+u_h-\pi_h\phi}, \qquad \chi_-^h = \frac{h^{\alpha^-}}{h^{\alpha^-}+u_h-\pi_h\phi}.$$

Then we may replace η_i^2 by $\tilde{\eta}_i^2$ (i =1,2,3) such that

$$\begin{aligned}
\tilde{\eta}_1^2 &= \sum_{\tau\in T^h} h_\tau^2\Big(\|f\chi_+^h\|^2_{0,\tau} + \|f_\phi\chi_-^h\|^2_{0,\tau}\Big),\\
\tilde{\eta}_2^2 &= \sum_{l\in\partial T^h} h_l\|[(A\nabla u_h)\cdot\mathbf{n}]\|^2_{0,l},\\
\tilde{\eta}_3^2 &= \sum_{\tau\in T^h}\Big(h_\tau^{-2}\|(\phi-\pi_h\phi)\chi_-^h\|^2_{0,\tau} + |(\phi-\pi_h\phi)\chi_-^h|^2_{1,\tau}\Big).
\end{aligned}$$

These approximations can be analyzed similarly as in Section 3. For η_4, we note that the second term has been included in $\tilde{\eta}_3^2$ (when $u_h\approx\phi$, $\chi_-^h\approx 1$). For the first term, if $l=\tau_l^1\cap\tau_l^2$ and $\phi\in H^2(\tau_l^1\cup\tau_l^2)$, then similar to (4.20),

$$h_l\|[(A\nabla(\pi_h\phi))\cdot\mathbf{n}]\|^2_{0,l} \le C|\phi-\pi_h\phi|^2_{1,\tau_l^1\cup\tau_l^2} + Ch_l^2\|f_\phi\|^2_{0,\tau_l^1\cup\tau_l^2}.$$

These two terms have been included in $\tilde{\eta}_1^2$ and $\tilde{\eta}_3^2$. Thus we only need to use the first term of η_4 in those $l\in\tilde{\Omega}_P^-$ where $\phi\notin H^2(\tau_l^1\cup\tau_l^2)$.

5. Conclusion

In [17], an efficient solver for mesh-redistribution based on harmonic mapping was introduced. The key idea is to construct the harmonic map between the physical space and a parameter space by an *iteration* procedure. Each iteration step is to move the mesh *closer* to the harmonic map. Our recent numerical study has extended the moving mesh methods to solve variational inequalities and optimal control problems. In our work, computational meshes are constructed by combining harmonic mapping and sharper *a posteriori* error estimators which are normally used in h-version adaptive finite element approximation. These estimators depend solely on the discrete solution and data and impose no constraints between consecutive time steps. Selecting appropriate error *monitors* is one of the most important issues that have to be addressed in implementing the moving mesh methods. It seems that the commonly used gradient based monitors are not suitable in solving the variational inequalities and optimal control problems [16]. It was observed that the error monitors used in [16], which are based on *a posteriori* error indicators, are suitable for the zero obstacles only. As discussed in Section 1, straightforward extension based on $u^* = u - \phi$ to the cases of non-zero obstacles does not seem appropriate.

The main motivation of the present theoretical work is to demonstrate that the error monitors used in [17] can be extended to non-zero obstacles in a more reasonable fashion. In this work, we obtained some sharp error estimators suitable for the non-zero obstacle problems. These estimators are combined with some *a priori* and *a posteriori* error estimators. By approximating the *a priori* part with some *a posteriori* quantities, we can derive reliable and efficient *a posteriori* error indicators. Using these *a posteriori* error indicators and the algorithms developed in [16], it is expected that we can track free boundaries for the non-zero obstacles very efficiently.

Acknowledgment. This work was supported in part by Hong Kong Baptist University, Hong Kong Research Grants Council, and the Birish EPSRC. We thank Professor Zhimin Chen of the Chinese Academy of Sciences for some useful discussions.

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