SIAM J. NUMER. ANAL.
Vol. 35, No. 5, pp. 1939–1968, October 1998
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CONVERGENCE ANALYSIS FOR OPERATOR-SPLITTING METHODS APPLIED TO CONSERVATION LAWS WITH STIFF SOURCE TERMS*

TAO TANG†

Abstract. We analyze the order of convergence for operator splitting methods applied to conservation laws with stiff source terms. We suppose that the source term $q(u)$ is dissipative. It is proved that the L^1 error introduced by the time splitting can be bounded by $O(\Delta t\|q(u_0)\|_{L^1})$, which is an improvement of the $O(Q\Delta t)$ upper bound, where Δt is the splitting time step, Q is the Lipschitz constant of q, or $Q = \max_u |q'(u)|$ in case q is smooth. A generic model with a special form of stiff source is also investigated. We propose a nonuniform temporal mesh to eliminate the effect of the initial layer introduced by the stiff source term. Our results are derived by using parabolic regularizations, rather than using Kuznetsov's approximation theory, which has been employed as a standard approach for error analysis to the dimensional- or time-splitting methods. Numerical examples are presented to illustrate the theoretical results.

Key words. conservation laws, splitting methods, stiff source terms, error estimates

AMS subject classifications. 35L65, 65M10, 65M15

PII. S0036142996308927

1. Introduction.

The purpose of this paper is to study operator splitting procedures for multidimensional conservation laws with stiff source terms. Without loss of generality, we consider Cauchy problems for the two-dimensional conservation law

$$u_t + f_1(u)_x + f_2(u)_y = q(u), \qquad (x,t) \in \mathbf{R} \times (0,T], \tag{1.1}$$

$$u(x,y,0) = u_0(x,y), \tag{1.2}$$

where f_1, f_2, and q are smooth functions of u. The operator splitting procedures are defined from local solutions of the corresponding homogeneous equation

$$u_t + f_1(u)_x + f_2(u)_y = 0 \tag{1.3}$$

and the ordinary differential equation (ODE)

$$u_t = q(u). \tag{1.4}$$

Operator splitting methods have been used frequently as part of a computational strategy for solving (1.1)–(1.2). One of the advantages of using the splitting methods is that we can use some successful numerical methods, e.g., TVD, ENO schemes [11, 12, 33], to solve the homogeneous equation (1.3). In particular, LeVeque's CLAWPACK [20], a software package for solving multidimensional conservation laws, makes time splitting more attractive. When $q \equiv 0$, the first-order dimensional-splitting method for solving the homogeneous equation was introduced by Godunov [8] and was modified by Strang [35] to achieve second order accuracy. Numerical and theoretical investigations of their methods were carried out by many authors (see, e.g., [4, 13, 34, 40]). If q is a

*Received by the editors September 9, 1996; accepted for publication (in revised form) October 8, 1997; published electronically August 25, 1998. This work was supported by Natural Sciences and Engineering Research Council of Canada grant OGP0105545.

http://www.siam.org/journals/sinum/35-5/30892.html

†Department of Mathematics, Simon Fraser University, Burnaby, B.C. V5A 1S6, Canada (tangtao@cs.sfu.ca).

nonstiff source term, splitting methods for (1.1)–(1.2) have been studied recently by Tang and Teng [38] and Langseth, Treito, and Winther [18].

In this research, we wish to analyze splitting methods to (1.1) with stiff source terms. There has been extensive research for the stiff problems in recent years (see, e.g., [1, 2, 14, 21] and references therein). For stiff problems, most of the numerical methods are semi-implicit, which are related to operator splitting (explicit method for the homogeneous conservation laws and implicit method for the stiff ODEs). Theoretical analysis for the semi-implicit methods have been given by Chalabi [1], Schroll, Treito and Winther [31], and Schroll and Winther [32]. In this work we will concentrate on time splitting and assume that the solution of (1.3) is found exactly. The convergence results obtained in this paper can be easily extended to operator-splitting methods with (i) time splitting plus dimensional splitting for (1.1) or (ii) fully discretization (say, monotone schemes) for the homogeneous problem (1.3) and backward Euler in time for (1.4). These can be done following section 6.2 of [38] and section 3.2 of [18], respectively. The extension is quite straightforward.

For ease of exposition we shall consider the one-dimensional problem

$$u_t + f(u)_x = q(u), \qquad (x,t) \in \mathbf{R} \times (0,T], \tag{1.5}$$

$$u(x,0) = u_0(x) \tag{1.6}$$

in the subsequent analysis. The extension to the multidimensional equation will be discussed in section 5. According to Pember [28, 29], the conservation law (1.1) is stiff if the time scale introduced by the source term q is small compared with the characteristic speed f' and some appropriate length scale. In particular,

$$Q := \max_{|u| \le \|u_0\|_\infty} |q'(u)| \tag{1.7}$$

may be related to a very small parameter $\delta > 0$. Previous error analysis, e.g., [18], shows that the L^1 error between the entropy solution and the operator splitting solution can be bounded by

$$C\Delta t \left(1 + \max_{|u| \le \|u_0\|_\infty} |q'(u)|\right), \tag{1.8}$$

provided that $u_0 \in BV \cap L^\infty \cap L^1$, where C is a constant independent of Δt and q. That is, the error bound is of the order $O(Q\Delta t)$. A typical relationship between Q and δ is $Q = O(\delta^{-1})$. If this is the case, the splitting error obtained by [18] is $O(\delta^{-1}\Delta t)$. This error bound is unsatisfactory since convergence requires that $\Delta t \sim \delta \ll 1$.

It is observed that a realistic assumption on the source term is $q'(u) \le 0$ for all $u \in \mathbf{R}$. It is indeed the case for many practical problems, e.g., in the models of combustion [5, 25], water waves in presence of the friction force in the bottom [43]. This assumption is also used in many theoretical papers, for example, Chalabi [1], Chen, Levermore, and Liu [2], Liu [23], and Schroll and Winther [32]. In the sense of Chen, Levermore, and Liu [2], $q' \le 0$ means the dissipativity of the source term. Furthermore, as usual, we assume that $u = 0$ is an equilibrium. Hence, throughout this paper we assume that

$$q(0) = 0, \qquad q'(u) \le 0. \tag{1.9}$$

With the above assumptions, we will show in this paper that the error for the operator-splitting method based on backward Euler for (1.4) can be bounded by

$$C\Delta t \left(1 + \int_{\mathbf{R}} |q(u_0(x))| dx\right), \tag{1.10}$$

provided that $u_0 \in BV \cap L^\infty \cap L^1$, where C is a constant independent of Δt and q. It follows from the assumption $q(0) = 0$ that

$$\Delta t \left(1 + \int_{\mathbf{R}} |q(u_0(x))| dx\right) \leq \Delta t \left(1 + \max_{|u| \leq \|u_0\|_\infty} |q'(u)| \bullet \|u_0\|_{L^1(\mathbf{R})}\right) \leq C\Delta t \left(1 + \max_{|u| \leq \|u_0\|_\infty} |q'(u)|\right). \tag{1.11}$$

In other words, (1.10) is an improvement of the error bound (1.8). Moreover, it can be shown that *even if $Q = O(\delta^{-1})$, the error bound* (1.10) *may be δ-free for some classes of source terms q.* This can be regarded as a substantial improvement of the $O(\delta^{-1}\Delta t)$ bound given by (1.8). We will give some examples of such q in section 3.

We will also discuss the initial layer problem which is quite common for stiff conservation laws. A generic form of a stiff conservation law is (1.5) with

$$q(u) = \frac{1}{\delta} g(u), \tag{1.12}$$

where the relaxation time $\delta > 0$ is a very small parameter and the function g satisfies

$$g(0) = 0, \qquad g'(u) \leq -\alpha < 0 \qquad \text{for} \quad u \in \mathbf{R}. \tag{1.13}$$

It will be shown that there exists a small initial layer for the simple model problem (1.5) with (1.12) and (1.13). This model problem has been studied in [15, 21, 32]. In this paper, the efforts will be made to eliminate the effect of the initial layer. We propose to divide $[0, T]$ into two subintervals and then use two different meshes in the two subintervals. It will be shown rigorously that this approach can successfully eliminate the effect of the initial layer in the sense that the L^1 error can be bounded by $O(\Delta t |\log \delta|)$.

Most of the results on error estimate have been based on the work of Kuznetsov [17], who was the first to provide error bounds for numerical schemes for (1.3). Inspired by the work of Kruzkov [16], he established a key approximation inequality. His approach has been extended to obtain error estimates for several numerical schemes to scalar conservation laws; see, e.g., [3, 24, 30]. The error estimates for dimensional-splitting methods and time-splitting methods have also been based on Kuznetsov's approach: the convergence of Godunov and Strang's dimensional-splitting methods are analyzed by Crandall and Majda [4]; the rate of convergence for the dimensional-splitting methods was first obtained by Teng [40]; the rate of convergence for the time-splitting methods was first given by Tang and Teng [38]; a sharper result on the latter rate was recently obtained by Langseth, Treito, and Whinter [18]. Although Kuznetsov's method is useful for dealing with hyperbolic conservation laws with a *general class* of initial functions and a *general class* of flux functions, it seems difficult (or impossible) to provide sharp/optimal error bounds when more practical initial data or flux functions are given. For example, it is well known that for scalar conservation laws without source term, the L^1 convergence rate for viscosity methods and monotone schemes is half. In fact, this result was first obtained by Kuznetsov. However, if the flux function is strictly convex and the entropy solution is piecewise smooth, then it has been shown that the L^1 error bounds for viscosity methods and monotone schemes are of order $O(\Delta t |\log \Delta t|)$; see [39, 41]. Optimal pointwise error estimates are also possible for piecewise smooth solutions [6, 9, 36, 37]. It seems unlikely that Kuznetsov's method can lead to these optimal bounds.

Recently, there has also been some interest in developing other theoretical approaches for error estimates. For example, using a matching method developed by Goodman and Xin [9] and Liu and Xin [22], it is proved that the viscosity methods approaching piecewise smooth solutions with finitely many noninteracting shocks have a local $O(\epsilon)$ error bound away from shocks, where $\epsilon > 0$ is the small viscosity amplitude. Another example is a general convergence theory for one-dimensional convex conservation laws developed by Tadmor and co-authors; see, e.g., [26, 27, 36]. They proved that when measured in $W^{-1,1}$, the convergence rate of the viscous solution is of order ϵ in case of rarefaction-free initial data [26, 36] and is of order $\epsilon|\ln\epsilon|$ in the general case [27]. The basic idea for obtaining their results is to use the parabolic regularizations. Partially inspired by [36], we will establish our error estimates by introducing parabolic regularizations. Since our problem involves a special class of source term function, i.e., (1.9), Kuznetsov's approach may have difficulty in providing the (sharp) error bounds obtained in this paper. On the other hand, the results given by [38, 18], which as mentioned earlier are obtained by using Kuznetsov's theory, can be recovered by employing the method proposed in this work.

The paper is organized as follows: in section 2 we introduce some notation and prove some stability properties for the solutions to (1.5)–(1.6) with nonpositive q'. These results play important roles in the error analysis. Time-splitting methods based on backward Euler for time is studied in section 3. We prove that the L^1 error introduced by the splitting is bounded by (1.10). Precise value for the constant C in (1.10) is also obtained. Brief discussion on time-splitting methods based on forward Euler in time is given. A generic model investigated by Schroll and Winther [32] is reconsidered in section 4. For this model, the splitting error is small only when u_0 is close to equilibrium. This is due to the presence of the initial layer. We propose in section 4 a nonuniform mesh which can resolve the initial layer. In section 5 we discuss the extension of our one-dimensional analysis to several space dimensions. In section 6 we present two numerical examples with the aim of illustrating the theoretical convergence results.

2. Preliminaries. We begin by introducing some definitions and summarizing some properties of solutions to the nonhomogeneous scalar conservation law (1.5)–(1.6). A bounded measurable function, u, is a weak solution of (1.5)–(1.6) if for all $\phi \in C_0^1(\mathbf{R} \times [0,T))$,

$$\int_{\mathbf{R}}\int_0^T \Big(u\phi_t + f(u)\phi_x\Big)dxdt + \int_{\mathbf{R}} u_0(x)\phi(x,0)dx = -\int_{\mathbf{R}}\int_0^T q\phi dxdt. \tag{2.1}$$

In this paper, we will deal with composite functions belonging to BV space. A composite function $p(u(x))$ belongs to BV if

$$|p(u)|_{BV} = \sup_{0<|h|<\infty}\int_{\mathbf{R}} \frac{|p(u(x+h,t)) - p(u(x,t))|}{|h|}dx < \infty.$$

The norm on $L^p(\mathbf{R})$ is denoted by $\|\bullet\|_p$.[1] The L^∞ and L^1 norms for $p(u(x))$ are defined by

$$\|p(u)\|_\infty = \sup_{x\in\mathbf{R}}|p(u(x))|, \qquad \|p(u)\|_1 = \int_{\mathbf{R}}|p(u(x))|dx.$$

[1] It should be noted that $\|\bullet\|_1$ used frequently in this paper stands for the L^1 norm, rather than the norm for the usual Sobolev space H^1.

The following property of the source term is a consequence of the monotonicity of q, i.e., $q' \le 0$, and is fundamental in the analysis below. For all $u, v \in \mathbf{R}$, we have

$$\operatorname{sgn}(u-v)\Big(q(u)-q(v)\Big) = -|q(u)-q(v)|. \tag{2.2}$$

Weak solutions are not uniquely determined by their initial data and additional principles, entropy conditions, are needed to select the appropriate physical solution. A weak solution, u, of (1.5)–(1.6) is an entropy solution if for all $\phi \in C_0^1(\mathbf{R}\times(0,T))$ with $\phi(x,t)\ge 0$ and any $c\in\mathbf{R}$,

$$\int_{\mathbf{R}}\int_0^T \Big(|u-c|\phi_t + \operatorname{sgn}(u-c)(f(u)-f(c))\phi_x + \operatorname{sgn}(u-c)q\phi\Big)dxdt \ge 0. \tag{2.3}$$

There are several different (and equivalent) forms of entropy conditions. It can be shown that entropy solutions of (1.5)–(1.6) can be constructed as the limit of solutions u^ϵ of the parabolic equation

$$u_t^\epsilon + f(u^\epsilon)_x = q(u^\epsilon) + \epsilon u_{xx}^\epsilon, \tag{2.4}$$

$$u^\epsilon(x,0) = u_0(x), \tag{2.5}$$

where $\epsilon > 0$. It follows easily from the maximum principle that (2.4)–(2.5) has at most one solution. It is also true that a solution u^ϵ exists for all t and that as $\epsilon \to 0+$ these solutions converge in the L^1 sense to u, the entropy solution of (1.5)–(1.6).

The error estimates for dimensional-splitting [4, 40] and time-splitting methods [18, 38] obtained before have been based on the entropy condition (2.3). Our estimate in this paper will be based on the parabolic regularization (2.4)–(2.5). Moreover, we need the following propositions.

PROPOSITION 2.1. *If $u_0 \in L^\infty(\mathbf{R})\cap L^1(\mathbf{R})$, $f\in C^1(\mathbf{R})$, and $q\in C^1(\mathbf{R})$ such that $q(0)=0, q'\le 0$, then the nonhomogeneous conservation law* (1.5)–(1.6) *possesses a unique entropy solution $u(x,t)=S(t)u_0$ satisfying*

$$\|S(t)u_0\|_\infty \le \|u_0\|_\infty. \tag{2.6}$$

Moreover, for any $v_0\in L^\infty(\mathbf{R})\cap L^1(\mathbf{R})$, the following inequalities hold:

$$\|S(t)u_0 - S(t)v_0\|_1 \le \|u_0-v_0\|_1, \tag{2.7}$$

$$\int_0^T \Big\|q\Big(S(\tau)u_0\Big) - q\Big(S(\tau)v_0\Big)\Big\|_1 d\tau \le \|u_0-v_0\|_1. \tag{2.8}$$

Proof. The estimates (2.6) and (2.7) have been obtained by Kruzkov [16]. Next we follow Lax [19] and Tang and Teng [39] to prove (2.8). Consider the following parabolic regularizations of (1.5)–(1.6):

$$u_t^\epsilon + f(u^\epsilon)_x = \epsilon u_{xx}^\epsilon + q(u^\epsilon), \qquad u^\epsilon(x,0)=u_0(x),$$
$$v_t^\epsilon + f(v^\epsilon)_x = \epsilon v_{xx}^\epsilon + q(v^\epsilon), \qquad v^\epsilon(x,0)=v_0(x).$$

Let $w := u^\epsilon - v^\epsilon$. It follows from the above two equations that

$$w_t + (f'(\zeta)w)_x = \epsilon w_{xx} + q(u^\epsilon) - q(v^\epsilon), \tag{2.9}$$

where ζ is some intermediate value between u^ϵ and v^ϵ. Assume that w changes signs at $p_1(t)$ and $p_2(t)$ and does not change signs for $x\in(p_1,p_2)$. Since u^ϵ and v^ϵ are

continuous, we have $w = 0$ at $x = p_1$ and $x = p_2$. Multiplying (2.9) by $\operatorname{sgn}(w)$ and integrating the resulting equation over (p_1, p_2) gives

$$\int_{p_1}^{p_2} \operatorname{sgn}(w) w_t dx = \epsilon\Big(w_x(p_2 - 0, t) - w_x(p_1 + 0, t)\Big) \operatorname{sgn}(w) + \int_{p_1}^{p_2} \operatorname{sgn}(u^\epsilon - v^\epsilon)(q(u^\epsilon) - q(v^\epsilon)) dx. \tag{2.10}$$

Since $w(p_1, t) = w(p_2, t) = 0$, we have

$$\frac{d}{dt} \int_{p_1}^{p_2} |w| dx = \int_{p_1}^{p_2} \operatorname{sgn}(w) w_t dx.$$

Since w changes signs at p_1 and p_2, it is easy to check that

$$w_x(p_2 - 0, t) \operatorname{sgn}(w) \le 0, \qquad w_x(p_1 + 0, t) \operatorname{sgn}(w) \ge 0.$$

Moreover, it follows from (2.2) that

$$\int_{p_1}^{p_2} \operatorname{sgn}(u^\epsilon - v^\epsilon)(q(u^\epsilon) - q(v^\epsilon)) dx = -\int_{p_1}^{p_2} |q(u^\epsilon) - q(v^\epsilon)| dx.$$

The inequality (2.10), together with the above three results, yields

$$\frac{d}{dt} \int_{p_1}^{p_2} |u^\epsilon - v^\epsilon| dx \le -\int_{p_1}^{p_2} |q(u^\epsilon) - q(v^\epsilon)| dx. \tag{2.11}$$

A similar inequality holds for any interval in which w does not change signs. Following the same arguments used in Lax [19], we conclude that

$$\frac{d}{dt} \|u^\epsilon - v^\epsilon\|_1 \le -\|q(u^\epsilon) - q(v^\epsilon)\|_1. \tag{2.12}$$

Integrating (2.12) with respect to t over $[0, T]$ and letting $\epsilon \to 0$ yields (2.8). □

Remark. There is an alternative way to prove (2.8). It is to use the splitting method with forward Euler in time, which has been studied in [18, 38]. This technique will be used to prove (2.18) below. □

PROPOSITION 2.2. *If $u_0 \in \mathrm{BV} \cap L^\infty(\mathbf{R}) \cap L^1(\mathbf{R})$, $f \in C^1(\mathbf{R})$, $q \in C^1(\mathbf{R})$ such that $q(0) = 0, q' \le 0$, then the problem* (1.5)–(1.6) *possesses a unique entropy solution $u(x, t) = S(t)u_0$ satisfying*

$$|S(t)u_0|_{\mathrm{BV}} \le |u_0|_{\mathrm{BV}}, \tag{2.13}$$

$$\|S(t)u_0\|_1 \le \|u_0\|_1, \tag{2.14}$$

$$\int_0^T |q(u)|_{BV} d\tau \le |u_0|_{BV}, \tag{2.15}$$

$$\int_0^T \|q(u)\|_1 d\tau \le \|u_0\|_1. \tag{2.16}$$

Moreover, for any $\tau_2 > \tau_1 \ge 0$, we have

$$\|u(\cdot, \tau_2) - u(\cdot, \tau_1)\|_1 \le \Big(C_0 |u_0|_{BV} + Q\|u_0\|_1\Big)(\tau_2 - \tau_1), \tag{2.17}$$

$$\|u(\cdot, \tau_2) - u(\cdot, \tau_1)\|_1 \le C_0 |u_0|_{BV} (\tau_2 - \tau_1) + \int_{\tau_1}^{\tau_2} \|q(u(\cdot, \tau))\|_1 d\tau, \tag{2.18}$$

where the constant Q is defined by (1.7) *and C_0 is defined by*

$$C_0 = \max_{|u|\le\|u_0\|_\infty} |f'(u)|. \tag{2.19}$$

Proof. Setting $v_0(x) = u_0(x+h)$ in (2.7) and (2.8)will lead to 2.13) and (2.15), respectively. Setting $v_0 \equiv 0$ in (2.7) and (2.8) will lead to (2.14) and (2.16), respectively. It remains to check (2.17) and ref2a9). For any given $\tau_2 > \tau_1 \ge 0$, let $k = (\tau_2 - \tau_1)/M$ be the mesh size in the time direction, where M is a fixed integer such that

$$kQ \le 1. \tag{2.20}$$

Let $t_m = \tau_1 + mk$ and $v(\cdot, \tau_1) = u(\cdot, \tau_1)$. Given $v(x, t_m)$, we compute $v(x, t_{m+1})$ by the following forward Euler splitting method. We first compute the intermediate value $\eta(\cdot, t_{m+1})$ by the following equation:

$$\eta(x, t_{m+1}) = v(x, t_m) + kq(v(x, t_m)). \tag{2.21}$$

Then compute $v(x, t_{m+1})$ by solving the following homogeneous hyperbolic equation with initial data $\eta(\cdot, t_{m+1})$:

$$v_t + f(v)_x = 0, \quad t_m < t \le t_{m+1}; \quad v(x, t_m) = \eta(x, t_{m+1}). \tag{2.22}$$

Starting from $m = 0$, the above procedure will give $\{v(x, t_m)\}_{m=0}^M$. It is proved in [18, 38] that as $k \to 0$ (i.e., $M \to \infty$), we have the following result:

$$\sup_{0\le m\le M} \|v(\cdot, t_m) - u(\cdot, t_m)\|_1 \to 0 \qquad \text{as} \quad k \to 0. \tag{2.23}$$

It follows from (2.21) that

$$\|\eta(\cdot, t_{m+1}) - v(\cdot, t_m)\|_1 \le k\|q(v(\cdot, t_m))\|_1. \tag{2.24}$$

Using the standard theory of homogeneous conservation laws for (2.22) gives

$$\|v(\cdot, t_{m+1}) - \eta(\cdot, t_{m+1})\|_1 \le C_0 k |v(\cdot, t_m)|_{BV}. \tag{2.25}$$

Since $q' \le 0$, it follows from (2.21) and (2.22) that

$$|v(\cdot, t_{m+1})|_{BV} \le |\eta(\cdot, t_{m+1})|_{BV} \le |v(\cdot, t_m)|_{BV} \quad \text{for} \quad 0 \le m \le M-1,$$

provided that (2.20) holds. The above result leads to

$$|v(\cdot, t_m)|_{BV} \le |v(\cdot, t_0)|_{BV} = |v(\cdot, \tau_1)|_{BV} = |u(\cdot, \tau_1)|_{BV} \le |u_0|_{BV}$$

for $0 \le m \le M$. This, together with (2.24) and (2.25), yields

$$\|v(\cdot, t_{m+1}) - v(\cdot, t_m)\|_1 \le C_0 k |u_0|_{BV} + k\|q(v(\cdot, t_m))\|_1,$$

which implies that

$$\|v(\cdot, \tau_2) - v(\cdot, \tau_1)\|_1 \le C_0(\tau_2 - \tau_1)|u_0|_{BV} + \sum_{m=0}^{M-1} k\|q(v(\cdot, t_m))\|_1. \tag{2.26}$$

The above result, together with (2.14) and (2.23), leads to (2.17). Using (2.17) and (2.23), it is readily shown that

$$\sum_{m=0}^{M-1} k\|q(v(\cdot,t_m))\|_1 \to \int_{\tau_1}^{\tau_2} \|q(u(\cdot,\tau))\|_1 d\tau \qquad \text{as} \quad k \to 0.$$

Letting $k \to 0$ in (2.26) yields (2.18). □

PROPOSITION 2.3. *Let $u_0 \in \mathrm{BV} \cap L^\infty(\mathbf{R}) \cap L^1(\mathbf{R})$, $f \in C^1(\mathbf{R})$, $q \in C^1(\mathbf{R})$ such that $q(0) = 0, q' \le 0$. Consider the parabolic regularization of* (1.5)–(1.6)

$$u_t^\epsilon + f(u^\epsilon)_x = \epsilon u_{xx}^\epsilon + q(u^\epsilon), \qquad u^\epsilon(x,0) = u_0(x). \tag{2.27}$$

Then the following inequalities hold:

$$\|u^\epsilon(\cdot,t)\|_\infty \le \|u_0\|_\infty, \tag{2.28}$$

$$\|u_x^\epsilon(\cdot,t)\|_1 \le |u_0|_{BV}, \tag{2.29}$$

$$\int_0^T |q(u^\epsilon(\cdot,\tau))|_{BV} d\tau \le |u_0|_{BV}, \tag{2.30}$$

$$\int_0^T \|q(u^\epsilon(\cdot,\tau))\|_\infty d\tau \le 2\|u_0\|_\infty. \tag{2.31}$$

Proof. The proof for (2.28) and (2.29) is simple and will not be given here. The inequality (2.30) follows directly from (2.12). It remains to prove (2.31). Let

$$V(u) = \begin{cases} 0 & \text{if} \quad u \le 0, \\ u & \text{if} \quad u > 0\ . \end{cases}$$

Let $\zeta_\lambda = \frac{1}{\lambda}\zeta(x/\lambda)$ be a standard positive C_0^∞-mollifier with unit mass. We also assume that $\mathrm{supp}\zeta \subset [0,1]$. We regularize $V(\bullet)$ by convolution, $V_\lambda \equiv V * \zeta_\lambda$. It is easy to verify that V_λ satisfies

$$\begin{cases} V_\lambda'(x) \ge 0,\ V_\lambda''(x) \ge 0 \quad \text{for } x \in \mathbf{R}; \qquad V_\lambda \to V, \text{ a.e., as } \lambda \to 0+\ ; \\ \\ V_\lambda(x) = 0,\ V_\lambda'(x) = 0 \quad \text{for } x \le 0; \qquad V_\lambda'(x) = 1 \quad \text{for } x \ge \lambda. \end{cases} \tag{2.32}$$

Straightforward calculation and (2.27) lead to

$$\begin{aligned} &V_\lambda(u^\epsilon)_t + f'(u^\epsilon)V_\lambda(u^\epsilon)_x - \epsilon V_\lambda(u^\epsilon)_{xx} - V_\lambda'(u^\epsilon)q(u^\epsilon) \\ &= -\epsilon V_\lambda''(u^\epsilon)(\epsilon_x)^2 \le 0, \end{aligned} \tag{2.33}$$

where in the last step we have used the fact that V_λ is convex. Since V_λ is a nondecreasing function, we have

$$\sup_x V_\lambda(u^\epsilon(x,t)) = V_\lambda\left(\sup_x u^\epsilon(x,t)\right).$$

In other words, the maximum point for $V_\lambda(u^\epsilon)$ is also the maximum point for u^ϵ. Further, since V_λ is nonnegative, an application of the maximum principle for (2.33) yields

$$\frac{d}{dt}\left[\sup_x V_\lambda(u^\epsilon(x,t))\right] \le V_\lambda'\left(\sup_x u^\epsilon(x,t)\right) q\left(\sup_x u^\epsilon(x,t)\right). \tag{2.34}$$

Let $u^+ = H(u^\epsilon)u^\epsilon$, where $H(\bullet)$ is the Heaviside function. Now we consider three cases. First, if $\sup_x u^\epsilon(x,t) \le 0$, we have

$$\text{right-hand side (RHS) of (2.34)} = 0 = q\left(\sup_x u^+(x,t)\right),$$

where we have used $V_\lambda(x) = 0$ for $x \le 0$, $\sup_x u^+(x,t) = 0$, and $q(0) = 0$. Second, if $0 < \sup_x u^\epsilon(x,t) \le \lambda$, then the RHS of (2.34) is nonpositive since

$$-q\left(\sup_x u^+(x,t)\right) = -q\left(\sup_x u^\epsilon(x,t)\right) \le \max_{0\le u\le\lambda} |q(u)| \le c\lambda,$$

where $c = \max_{0\le u\le\|u_0\|_\infty} |q'(u)|$ is independent of λ, and where we again used the fact that $q(0) = 0$. The above inequality and the fact that the RHS of (2.34) is nonpositive yield

$$\text{RHS of (2.34)} \le q\left(\sup_x u^+(x,t)\right) + c\lambda.$$

The final case is that $\sup_x u^\epsilon(x,t) > \lambda$. In this case, we have $V_\lambda'(\sup_x u^\epsilon(x,t)) = 1$ so that

$$\text{RHS of (2.34)} = q\left(\sup_x u^\epsilon(x,t)\right) = q\left(\sup_x u^+(x,t)\right).$$

Combining (2.34) and the results for the above three cases gives

$$\frac{d}{dt}\left[\sup_x V_\lambda(u^\epsilon(x,t))\right] \le q\left(\sup_x u^+(x,t)\right) + c\lambda, \tag{2.35}$$

which leads to

$$-\int_0^T q\left(\sup_x u^+(x,t)\right)dt \le \sup_x V_\lambda(u^\epsilon(x,0)) + cT\lambda = \sup_x V_\lambda(u_0(x)) + cT\lambda.$$

Letting $\lambda \to 0+$ above yields

$$-\int_0^T q\left(\sup_x u^+(x,t)\right)dt \le \|u_0\|_\infty. \tag{2.36}$$

Similarly, we can show that

$$\int_0^T q\left(\inf_x u^-(x,t)\right)dt \le \|u_0\|_\infty, \tag{2.37}$$

where $u^-(x,t) := (1 - H(u^\epsilon))u^\epsilon$. Since $q' \le 0$, it is observed that

$$\begin{aligned}\|q\left(u^\epsilon(\cdot,t)\right)\|_\infty &= \max\left[q\left(\inf_x u^-(x,t)\right), -q\left(\sup_x u^+(x,t)\right)\right]\\ &\le q\left(\inf_x u^-(x,t)\right) - q\left(\sup_x u^+(x,t)\right).\end{aligned}$$

This, together with (2.36) and (2.37), yields (2.31). □

PROPOSITION 2.4. *If u^ϵ and v^ϵ are continuously differentiable functions satisfying the following nonhomogeneous parabolic equations:*

$$u_t^\epsilon + f(u^\epsilon)_x = \epsilon u_{xx}^\epsilon + p_1(x,t), \qquad \tau_1 < t < \tau_2,$$
$$v_t^\epsilon + f(v^\epsilon)_x = \epsilon v_{xx}^\epsilon + p_2(x,t), \qquad \tau_1 < t < \tau_2,$$

then the following inequality holds:

$$\|u^\epsilon(\cdot,\tau_2) - v^\epsilon(\cdot,\tau_2)\|_1 \le \|u^\epsilon(\cdot,\tau_1) - v^\epsilon(\cdot,\tau_1)\|_1 + \int_{\tau_1}^{\tau_2} \|p_1(\cdot,\tau) - p_2(\cdot,\tau)\|_1 d\tau. \tag{2.38}$$

The above result can be found in [39, 41]. Its proof can be found in [39].

3. Operator splitting. We will consider an operator-splitting procedure for the initial value problem (1.5)–(1.6), where we alternate between solving an homogeneous equation and an ODE. More precisely, let S_1 be the solution operator of the homogeneous problem, i.e., $v(\cdot,t) = S_1(t)v_0$ is the entropy solution of the initial value problem

$$v_t + f(v)_x = 0, \qquad v(x,0) = v_0(x). \tag{3.1}$$

Let S_2 be the *backward* Euler operator for the ordinary differential problem, i.e., $S_2(\Delta t)v_0 = v(\cdot,\Delta t)$ is given by

$$v(\cdot,\Delta t) = v_0 + \Delta t q(v(\cdot,\Delta t)). \tag{3.2}$$

The operator-splitting solution $\{v(x,t_n)\}_{n=0}^N$ is defined by

$$v(\cdot,t_n) = [S_2(\Delta t)S_1(\Delta t)]^n v_0, \tag{3.3}$$

where $t_n = n\Delta t$ and $t_N = T$. Note that this approximate solution is only defined at discrete t-values. We require in this section to solve (3.1) and (3.2) exactly.

For $t \in (t_n, t_{n+1})$, we consider the following procedure: first solve

$$\eta_t + f(\eta)_x = 0, \qquad t \in (t_n, t_{n+1}]; \qquad \eta(x,t_n) = v(x,t_n), \tag{3.4}$$

where $v(x,t_n)$ is defined by (3.3). Then compute $v(x,t_{n+1})$ by the following equation;

$$v(x,t_{n+1}) = \eta(x,t_{n+1}) + \Delta t q(v(x,t_{n+1})). \tag{3.5}$$

The scheme (3.4)–(3.5) defines

$$\eta(\cdot,t_{n+1}) = S_1(\Delta t)v(\cdot,t_n), \qquad v(\cdot,t_{n+1}) = S_2(\Delta t)\eta(\cdot,t_{n+1}).$$

The main purpose of this section is to derive an error bound for the splitting scheme (3.3). To this end, we require the following two lemmas.

LEMMA 3.1. *Assume that $u_0 \in BV \cap L^\infty(\mathbf{R}) \cap L^1(\mathbf{R}), f \in C^2(\mathbf{R}), q \in C^1(\mathbf{R})$ satisfying $q(0) = 0, q' \le 0$. Let u be the entropy solution of* (1.5)–(1.6)*, let η and v be the solutions of* (3.4) *and* (3.5)*, respectively. The following estimate holds:*

$$\sum_{n=0}^{N-1} \left\| u(\cdot,t_{n+1}) - \int_{t_n}^{t_{n+1}} q(u)d\tau - \eta(\cdot,t_{n+1}) \right\|_1 \le \sum_{n=0}^{N-1} \|u(\cdot,t_n) - v(\cdot,t_n)\|_1 + \Delta t |u_0|_{BV}\Big(C_1 + 4C_2\|u_0\|_\infty\Big), \tag{3.6}$$

where the constants C_1 and C_2 are defined by

$$C_j = \max_{|u|\le 3\|u_0\|_\infty} |f^{(j)}(u)|. \tag{3.7}$$

Proof. Consider the following equations:

$$u_t^\epsilon + f(u^\epsilon)_x = q(u^\epsilon) + \epsilon u_{xx}^\epsilon, \quad t > 0; \qquad u^\epsilon(x,0) = u_0(x), \tag{3.8}$$

$$\eta_t^\epsilon + f(\eta^\epsilon)_x = \epsilon \eta_{xx}^\epsilon, \qquad t_n < t \le t_{n+1}; \qquad \eta^\epsilon(x,t_n) = v(x,t_n). \tag{3.9}$$

As $\epsilon \to 0$, the limits for u^ϵ and η^ϵ are u and η, respectively. Let $\omega_\lambda(x) = \frac{1}{\lambda}\omega(x/\lambda)$ be a standard positive C_0^∞-mollifier with unit mass. Let

$$\phi^{\lambda,\epsilon}(x,t) = q(u^\epsilon(\cdot,t)) * \omega_\lambda = \int \omega_\lambda(x-y) q(u^\epsilon(y,t)) dy.$$

It is easy to verify that, for any fixed time $t \in [t_n, t_{n+1}]$,

$$\begin{cases} \|\phi^{\lambda,\epsilon}(\cdot,t)\|_\infty \le \|q(u^\epsilon(\cdot,t))\|_\infty \le Q\|u_0\|_\infty \quad \Big(\text{using (2.28)}\Big), \\ \\ \|\phi_x^{\lambda,\epsilon}(\cdot,t)\|_1 \le |q(u^\epsilon(\cdot,t))|_{BV} \le Q|u_0|_{BV} \quad \Big(\text{using (2.29)}\Big), \end{cases} \tag{3.10}$$

where Q is defined by (1.7). Furthermore, using (2.29) we obtain

$$\|\phi_{xx}^{\lambda,\epsilon}(\cdot,t)\|_1 \le C\lambda^{-1} Q|u_0|_{BV} := \tilde{c}(\lambda), \tag{3.11}$$

where $C = \max_x |\omega'(x)|$ is independent of ϵ. Rewrite (3.8) as

$$\begin{aligned} &u_t^{\lambda,\epsilon} + f(u^{\lambda,\epsilon})_x \\ &= \epsilon u_{xx}^{\lambda,\epsilon} + f(u^{\lambda,\epsilon})_x - f(u^\epsilon)_x + q(u^\epsilon) - \phi^{\lambda,\epsilon}(x,t) + \epsilon \int_{t_n}^t \phi_{xx}^{\lambda,\epsilon}(x,\tau) d\tau, \end{aligned} \tag{3.12}$$

where the function $u^{\lambda,\epsilon}$ is defined by

$$u^{\lambda,\epsilon}(x,t) = u^\epsilon(x,t) - \int_{t_n}^t \phi^{\lambda,\epsilon}(x,\tau) d\tau. \tag{3.13}$$

Using Proposition 2.4 for (3.9) and (3.12) and noting that $u^{\lambda,\epsilon}(\cdot,t_n) = u^\epsilon(\cdot,t_n)$ and $\eta^\epsilon(\cdot,t_n) = v(\cdot,t_n)$, we obtain

$$\begin{aligned} &\|u^{\lambda,\epsilon}(\cdot,t_{n+1}) - \eta^\epsilon(\cdot,t_{n+1})\|_1 \\ &\le \|u^\epsilon(\cdot,t_n) - v(\cdot,t_n)\|_1 + \int_{t_n}^{t_{n+1}} \|f(u^{\lambda,\epsilon})_x - f(u^\epsilon)_x\|_1 dt \\ &\quad + \int_{t_n}^{t_{n+1}} \|q(u^\epsilon) - \phi^{\lambda,\epsilon}\|_1 dt + \epsilon \int_{t_n}^{t_{n+1}} \int_{t_n}^t \|\phi_{xx}^{\lambda,\epsilon}(\cdot,\tau)\|_1 d\tau dt. \end{aligned} \tag{3.14}$$

We first estimate the third term in the above inequality. Observe that

$$\begin{aligned} \Big(f(u^{\lambda,\epsilon}) - f(u^\epsilon)\Big)_x &= -\int_0^1 f'\Big(\theta u^{\lambda,\epsilon} + (1-\theta)u^\epsilon\Big) d\theta \int_{t_n}^t \phi_x^{\lambda,\epsilon} d\tau \\ &\quad - \int_0^1 f''\Big(\theta u^{\lambda,\epsilon} + (1-\theta)u^\epsilon\Big)\Big(\theta u_x^{\lambda,\epsilon} + (1-\theta)u_x^\epsilon\Big) d\theta \int_{t_n}^t \phi^{\lambda,\epsilon} d\tau. \end{aligned} \tag{3.15}$$

Since

$$\theta u^{\lambda,\epsilon} + (1-\theta)u^{\epsilon} = u^{\epsilon} - \theta \int_{t_n}^{t} \phi^{\lambda,\epsilon} d\tau, \qquad t \in [t_n, t_{n+1}],$$

we obtain from Proposition 2.3 and (3.10) that, for $t \in [t_n, t_{n+1}]$ and $\theta \in (0,1)$,

$$\|\theta u^{\lambda,\epsilon}(\cdot,t) + (1-\theta)u^{\epsilon}(\cdot,t)\|_\infty \le \|u_0\|_\infty + \int_{t_n}^{t} \|q(u^{\epsilon}(\cdot,t))\|_\infty d\tau \le 3\|u_0\|_\infty,$$
$$\|\theta u_x^{\lambda,\epsilon}(\cdot,t) + (1-\theta)u_x^{\epsilon}(\cdot,t)\|_1 \le |u_0|_{BV} + \int_{t_n}^{t} |q(u^{\epsilon}(\cdot,t))|_{BV} d\tau \le 2|u_0|_{BV}.$$

The above results, together with (3.10), yield

$$\begin{aligned} &\|f(u^{\lambda,\epsilon})_x - f(u^{\epsilon})_x\|_1 \\ &\le C_1 \int_{t_n}^{t} |q(u^{\epsilon}(\cdot,\tau))|_{BV} d\tau + 2C_2|u_0|_{BV} \int_{t_n}^{t} \|q(u^{\epsilon}(\cdot,\tau))\|_\infty d\tau \end{aligned} \tag{3.16}$$

for $t_n < t < t_{n+1}$, where the constants C_1 and C_2 are defined by (3.7). Combining (3.14) and (3.16) yields

$$\begin{aligned} &\sum_{n=0}^{N-1} \|u^{\lambda,\epsilon}(\cdot,t_{n+1}) - \eta^{\epsilon}(\cdot,t_{n+1})\|_1 \\ &\le \sum_{n=0}^{N-1} \|u^{\epsilon}(\cdot,t_n) - v(\cdot,t_n)\|_1 + C_1\Delta t \int_0^{t_N} |q(u^{\epsilon}(\cdot,\tau))|_{BV} d\tau \\ &\quad + 2C_2\Delta t|u_0|_{BV} \int_0^{t_N} \|q(u^{\epsilon}(\cdot,\tau))\|_\infty d\tau + \int_0^{t_N} \|q(u^{\epsilon}) - \phi^{\lambda,\epsilon}\|_1 dt + \tilde{c}(\lambda)\epsilon, \end{aligned} \tag{3.17}$$

where $\tilde{c}(\lambda)$ is a constant independent of ϵ and for the last term we have used (3.11). Using Proposition 2.3, (3.17) leads to

$$\begin{aligned} &\sum_{n=0}^{N-1} \|u^{\lambda,\epsilon}(\cdot,t_{n+1}) - \eta^{\epsilon}(\cdot,t_{n+1})\|_1 \le \sum_{n=0}^{N-1} \|u^{\epsilon}(\cdot,t_n) - v(\cdot,t_n)\|_1 \\ &\quad + \Delta t|u_0|_{BV}\Big(C_1 + 4C_2\|u_0\|_\infty\Big) + \int_0^{t_N} \|q(u^{\epsilon}(\cdot,t)) - \phi^{\lambda,\epsilon}(\cdot,t)\|_1 dt + \tilde{c}(\lambda)\epsilon. \end{aligned} \tag{3.18}$$

Letting $\epsilon \to 0+$ yields

$$\begin{aligned} &\sum_{n=0}^{N-1} \|u^{\lambda}(\cdot,t_{n+1}) - \eta(\cdot,t_{n+1})\|_1 \le \sum_{n=0}^{N-1} \|u(\cdot,t_n) - v(\cdot,t_n)\|_1 \\ &\quad + \Delta t|u_0|_{BV}\Big(C_1 + 4C_2\|u_0\|_\infty\Big) + \int_0^{t_N} \|q(u(\cdot,t)) - \phi^{\lambda}(\cdot,t)\|_1 dt, \end{aligned} \tag{3.19}$$

where

$$u^{\lambda}(x,t) = u(x,t) - \int_{t_n}^{t} \phi^{\lambda}(x,\tau) d\tau, \qquad \phi^{\lambda}(x,t) = \int \omega_\lambda(x-y) q(u(y,t)) dy.$$

Since $q(u) \in BV$, it is easy to show that $\phi^\lambda(\cdot,t)$ converges to $q(u(\cdot,t))$ strongly in L^1 when $\lambda \to 0+$. Letting $\lambda \to 0+$ in (3.19) yields (3.6). □

Remark. From (3.15), we noticed that $f \in C^2(\mathbf{R})$ is required. This assumption is also used in [18], but not in [38].

LEMMA 3.2. *Assume that* $u_0 \in BV \cap L^\infty(\mathbf{R}) \cap L^1(\mathbf{R}), f \in C^1(\mathbf{R}), q \in C^1(\mathbf{R})$ *satisfying* $q(0) = 0, q' \le 0$. *Let* u *be the entropy solution of* (1.5)–(1.6). *The following estimate holds:*

$$\sum_{n=0}^{N-1} \left\| \int_{t_n}^{t_{n+1}} q(u(\cdot,\tau))d\tau - \Delta t\, q(u(\cdot,t_{n+1})) \right\|_1 \le \Delta t\Big(C_0 |u_0|_{BV} + \|q(u_0)\|_1 \Big), \tag{3.20}$$

where C_0 *is defined by* (2.19).

Proof. We first extend $u(\cdot,t)$ to $-\infty < t < \infty$ by letting

$$U(x,t) = \begin{cases} u(x,t) & \text{if} \quad t \ge 0, \\ u_0(x) & \text{if} \quad t < 0\ . \end{cases}$$

It follows from (2.18) that

$$\|U(\cdot,\tau_2) - U(\cdot,\tau_1)\|_1 \le \Big(C_0|u_0|_{BV} + Q\|u_0\|_1 \Big)(\tau_2 - \tau_1) \quad \text{for} \quad \tau_2 > \tau_1. \tag{3.21}$$

Let $\omega_\lambda(x) = \frac{1}{\lambda}\omega(x/\lambda)$ be a positive C_0^∞-mollifier satisfying

$$\int \omega_\lambda(x)dx = 1; \qquad \omega_\lambda(x) = 0 \quad \text{if} \quad |x| \ge \lambda.$$

Let $\psi^\lambda(x,t) = q(U(x,\cdot)) * \omega$, i.e.,

$$\psi^\lambda(x,t) = \int \omega_\lambda(t-y)q(U(x,y))dy = \int \omega_\lambda(y)q(U(x,t-y))dy. \tag{3.22}$$

Observe that

$$\begin{aligned} &\text{left} - \text{hand side (LHS) of (3.20)} \\ &\le \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \|q(u(\cdot,\tau)) - \psi^\lambda(\cdot,\tau)\|_1 d\tau + \sum_{n=0}^{N-1} \Delta t \|q(u(\cdot,t_{n+1})) - \psi^\lambda(\cdot,t_{n+1})\|_1 \\ &\quad + \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \|\psi^\lambda(\cdot,\tau) - \psi^\lambda(\cdot,t_{n+1})\|_1 d\tau \\ &= \int_0^T \|q(U(\cdot,\tau)) - \psi^\lambda(\cdot,\tau)\|_1 d\tau + \Delta t \sum_{n=0}^{N-1} \|q(U(\cdot,t_{n+1})) - \psi^\lambda(\cdot,t_{n+1})\|_1 \\ &\quad + \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \|\psi^\lambda(\cdot,\tau) - \psi^\lambda(\cdot,t_{n+1})\|_1 \\ &=: J_1 + J_2 + J_3, \end{aligned}$$

where $T = N\Delta t$. Using (3.21) and (3.22) we can show that

$$J_1 \le \tilde{c}\lambda, \qquad J_2 \le \tilde{c}\lambda, \tag{3.23}$$

where $\tilde{c}$ is a constant independent of λ. It is easy to see that ψ^λ is differentiable with respect to x and t. In particular, we have

$$\|\psi_t^\lambda(\cdot,t)\|_1 = \left\| \lim_{h\to 0+} \frac{1}{h} \int \omega_\lambda(y) \Big[q(U(\cdot,t-y+h)) - q(U(\cdot,t-y)) \Big] dy \right\|_1 . \tag{3.24}$$

It follows from Fatou's lemma [7] and (3.21) that

$$\|\psi_t^\lambda(\cdot,t)\|_1 \le Q \lim_{h\to 0+} \frac{1}{h} \int \omega_\lambda(y) \|U(\cdot,t-y+h) - U(\cdot,t-y)\|_1 dy \le \tilde{c}, \tag{3.25}$$

where the constant $\tilde{c}$ is independent of λ. Observe that

$$\begin{aligned} J_3 &= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left\| \int_\tau^{t_{n+1}} \psi_t^\lambda(\cdot,t) dt \right\|_1 d\tau \\ &\le \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \int_{t_n}^{t_{n+1}} \|\psi_t^\lambda(\cdot,t)\|_1 dt d\tau \\ &= \Delta t \int_0^T \|\psi_t^\lambda(\cdot,t)\|_1 dt \\ &=: \Delta t (J_{31} + J_{32}), \end{aligned} \tag{3.26}$$

where

$$J_{31} = \int_0^\lambda \|\psi_t^\lambda(\cdot,t)\|_1 dt\,, \qquad J_{32} = \int_\lambda^T \|\psi_t^\lambda(\cdot,t)\|_1 dt\,.$$

We first estimate J_{32}. For $t \ge \lambda, |y| \le \lambda$, and $h > 0$, we have $t - y \ge 0, t - y + h \ge 0$ which suggest that $U(\cdot,t-y) = u(\cdot,t-y)$ and $U(\cdot,t-y+h) = u(\cdot,t-y+h)$. Noting that $\omega_\lambda(y) = 0$ for all $|y| \ge \lambda$, we obtain from (3.24) that

$$\begin{aligned} J_{32} &\le \lim_{h\to 0+} \frac{1}{h} \int_\lambda^T \int_{-\lambda}^\lambda \omega_\lambda(y) \|q(U(\cdot,t-y+h)) - q(U(\cdot,t-y))\|_1 dy dt \\ &= \lim_{h\to 0+} \frac{1}{h} \int_{-\lambda}^\lambda \omega_\lambda(y) \int_\lambda^T \|q(u(\cdot,t-y+h)) - q(u(\cdot,t-y))\|_1 dt dy. \end{aligned} \tag{3.27}$$

Using the same technique for the proof of (2.8) we can obtain

$$\int_\lambda^T \|q(u(\cdot,t-y+h)) - q(u(\cdot,t-y))\|_1 dt \le \|u(\cdot,\lambda-y+h) - u(\cdot,\lambda-y)\|_1$$

for all $|y| \le \lambda$ and $h > 0$. This, together with (2.18), yields

$$\begin{aligned} &\int_\lambda^T \|q(u(\cdot,t-y+h)) - q(u(\cdot,t-y))\|_1 dt \\ &\le C_0 |u_0|_{BV} h + \int_{\lambda-y}^{\lambda-y+h} \|q(u(\cdot,\tau))\|_1 d\tau \\ &\le C_0 |u_0|_{BV} h + \|q(u_0)\|_1 h + \int_{\lambda-y}^{\lambda-y+h} \|q(u(\cdot,\tau)) - q(u_0)\|_1 d\tau \\ &\le C_0 |u_0|_{BV} h + \|q(u_0)\|_1 h + \tilde{c}(2\lambda + h) h \end{aligned} \tag{3.28}$$

for all $|y| \le \lambda$ and $h > 0$, where the constant $\tilde{c}$ is independent of λ and h and in the last step we have used (2.17) and the fact that $\lambda - y + h \le 2\lambda + h$. Combining (3.27) and (3.28) gives

$$J_{32} \le C_0|u_0|_{BV} + \|q(u_0)\|_1 + 2\tilde{c}\lambda. \tag{3.29}$$

Moreover, it follows from (3.25) that $J_{31} \le \tilde{c}\lambda$. Therefore, we have proved that

$$\text{LHS of (3.20)} \le \Delta t\Big(C_0|u_0|_{BV} + \|q(u_0)\|_1\Big) + \tilde{c}\lambda,$$

where $\tilde{c}$ is a constant independent of λ. Letting $\lambda \to 0$ in the above inequality gives (3.20). □

We are now able to proceed with the construction of a bound on the splitting error using the above two lemmas.

THEOREM 3.3. *Assume that $u_0 \in L^1(\mathbf{R}) \cap BV(\mathbf{R}) \cap L^\infty(\mathbf{R})$,[2] $f \in C^2(\mathbf{R}), q \in C^1(\mathbf{R})$ satisfying $q(0) = 0, q' \le 0$. Let u be the entropy solution of (1.5)–(1.6), S_1 be the solution operator of the homogeneous problem (3.1), and S_2 be the backward Euler operator (3.2). The following estimate holds:*

$$\begin{aligned}\text{(3.30)}\quad &\Big\|u(\cdot, N\Delta t) - [S_2(\Delta t)S_1(\Delta t)]^N v_0\Big\|_1 \\ &\le \|u_0 - v_0\|_1 + \Delta t\,\Big(C_1|u_0|_{BV} + 4C_2|u_0|_{BV}\|u_0\|_\infty + \|q(u_0)\|_1 + C_0|u_0|_{BV}\Big),\end{aligned}$$

where the constants C_0, C_1, and C_2 depend on the function f and are defined by (2.19) and (3.7).

Proof. It follows from (3.5) that

$$\begin{aligned}\text{(3.31)}\quad &\|u(\cdot, t_{n+1}) - \Delta t\, q(u(\cdot, t_{n+1})) - v(\cdot, t_{n+1}) + \Delta t q(v(\cdot, t_{n+1}))\|_1 \\ &\le \left\|u(\cdot, t_{n+1}) - \int_{t_n}^{t_{n+1}} q(u)d\tau - \eta(\cdot, t_{n+1})\right\|_1 + \left\|\int_{t_n}^{t_{n+1}} q(u)d\tau - \Delta t\, q(u(\cdot, t_{n+1}))\right\|_1.\end{aligned}$$

Observe that for any $x \in \mathbf{R}$,

$$\begin{aligned}&\operatorname{sgn}\Big[u(x, t_{n+1}) - \Delta t\, q(u(x, t_{n+1})) - v(x, t_{n+1}) + \Delta t\, q(v(x, t_{n+1}))\Big] \\ &= \operatorname{sgn}\Big[(1 - \Delta t q'(\bullet))(u(x, t_{n+1}) - v(x, t_{n+1}))\Big] \\ &= \operatorname{sgn}\Big[u(x, t_{n+1}) - v(x, t_{n+1})\Big],\end{aligned}$$

where $\bullet$ denotes an intermediate value between u and v and in the last step we have used the fact that $q' \le 0$. It follows from the above result that

$$\begin{aligned}\text{LHS of (3.31)} &= \int_{\mathbf{R}} \operatorname{sgn}(u(x, t_{n+1}) - v(x, t_{n+1}))\Big[u(x, t_{n+1}) - \Delta t\, q(u(x, t_{n+1})) \\ &\qquad - v(x, t_{n+1}) + \Delta t\, q(v(x, t_{n+1}))\Big]dx \\ &= \|u(\cdot, t_{n+1}) - v(\cdot, t_{n+1})\|_1 + \Delta t\, \|q(u(\cdot, t_{n+1})) - q(v(\cdot, t_{n+1}))\|_1, \qquad \text{(3.32)}\end{aligned}$$

[2] In the one-dimensional case, the L^∞ condition for u_0 can be dropped since $BV(\mathbf{R}) \subset L^\infty(\mathbf{R})$, but for the N-dimensional problem ($N > 1$) the $L^\infty(\mathbf{R}^N)$ condition has to be imposed since $BV(\mathbf{R}^N) \not\subset L^\infty(\mathbf{R}^N)$; see [10]. In order to extend our results to multidimensional problems we still list the L^∞ condition for u_0 in the one-dimensional case.

where in the last step we have used (2.2). It follows from (3.31) and (3.32) that

$$\begin{aligned}
&\sum_{n=0}^{N-1}\Big(\|u(\cdot,t_{n+1})-v(\cdot,t_{n+1})\|_1+\Delta t\,\|q(u(\cdot,t_{n+1}))-q(v(\cdot,t_{n+1}))\|_1\Big)\\
\le\quad&\sum_{n=0}^{N-1}\left(\left\|u(\cdot,t_{n+1})-\int_{t_n}^{t_{n+1}}q(u)d\tau-\eta(\cdot,t_{n+1})\right\|_1\right.\\
+\quad&\left.\left\|\int_{t_n}^{t_{n+1}}q(u)d\tau-\Delta t\,q(u(\cdot,t_{n+1}))\right\|_1\right).
\end{aligned}$$

This, together with Lemmas 3.1 and 3.2, yields

$$\begin{aligned}
&\|u(\cdot,t_N)-v(\cdot,t_N)\|_1+\sum_{n=1}^{N}\Delta t\,\|q(u(\cdot,t_n))-q(v(\cdot,t_n))\|_1\\
&\le\|u(\cdot,0)-v(\cdot,0)\|_1+\Delta t\Big(C_1|u_0|_{BV}+4C_2|u_0|_{BV}\|u_0\|_\infty+\|q(u_0)\|_1+C_0|u_0|_{BV}\Big),
\end{aligned}$$

which leads to (3.30). This completes the proof of this theorem. □

Theorem 3.3 implies that if $u_0\in BV\cap L^\infty(\mathbf{R})\cap L^1(\mathbf{R})$, then we have the following error bound:

$$\|u(\cdot,N\Delta t)-[S_2(\Delta t)S_1(\Delta t)]^N u_0\|_1\le C\Big(\|q(u_0)\|_1\Big)\Delta t,$$

where $C(\|q(u_0)\|_1)$ is a constant depending on $\|q(u_0)\|_1$, but not on $Q=\max_u|q'(u)|$. Since $q(0)=0$, it is clear that

$$\|q(u_0)\|_1\le Q\|u_0\|_1.$$

This suggests that if Q is uniformly bounded, in the sense that it is independent of some small parameter, then the splitting error is $O(\Delta t)$. The main point of Theorem 3.3 is that *even if Q is not uniformly bounded, the splitting error may also be free of some small parameter*. This can be seen from the following examples.

Example 3.1.

$$q(u)=\frac{-u/\delta}{\sqrt{1+(u/\delta)^2}},\qquad \delta\ll 1.$$

Example 3.2.

$$q(u)=-\tan^{-1}(u/\delta),\qquad \delta\ll 1.$$

In the above examples, we can easily verify that $Q=O(\delta^{-1})$. We can also verify that for Examples 3.1 and 3.2

$$q(0)=0,\quad q'\le 0,\quad \max_{u\in\mathbf{R}}|q(u)|\le O(1),$$

where $O(1)$ indicates a constant independent of δ. If u_0 has a compact support, then the last inequality above implies that

$$\|q(u_0)\|_1\le O(1). \tag{3.33}$$

Theorem 3.3 suggests that with the above given q, the L^1 convergence rate for the operator-splitting method (3.3) is $O(\Delta t)$, provided u_0 has a compact support.

Remark. From the proof of Theorem 3.3, we also obtain the following result:

$$\sum_{n=1}^{N} \Delta t \, \|q(u(\cdot,t_n)) - q(v(\cdot,t_n))\|_1 \le \|u_0 - v_0\|_1 + C\left(\|q(u_0)\|_1\right)\Delta t. \quad \square$$

Remark. Similar to the proof of Theorem 3.3, we can show that

$$\left\|u(\cdot, N\Delta t) - [S_1(\Delta t)S_2(\Delta t)]^N v_0\right\|_1 \le \|u_0 - v_0\|_1 + C\Big(\|q(u_0)\|_1\Big)\Delta t,$$

provided that $u_0 \in BV \cap L^\infty(\mathbf{R}) \cap L^1(\mathbf{R})$. In other words, the order of splitting procedure (3.4)–(3.5) can be changed by starting with the backward Euler method and then solving the homogeneous equation. $\square$

By simple modifications to Lemmas 3.1 and 3.2 and Theorem 3.3, we can obtain the following estimate.

THEOREM 3.4. *Let the functions u_0, f, and q in* (1.5)–(1.6) *be subject to the conditions stated in Theorem* 3.3. *Let u be the entropy solution of* (1.5)–(1.6), *S_1 be the solution operator of the homogeneous problem* (3.1), *and S_2 be the backward Euler operator* (3.2). *The following estimate holds:*

$$\begin{aligned}(3.34)\quad &\max_{0\le n\le N}\left\|u(\cdot,t_n) - [S_2(\Delta t)S_1(\Delta t)]^n v_0\right\|_1 \\ &\le \|u_0 - v_0\|_1 + \Delta t\left(C_1|u_0|_{BV} + 4C_2|u_0|_{BV}\|u_0\|_\infty + \|q(u_0)\|_1 + C_0|u_0|_{BV}\right),\end{aligned}$$

where $t_n = n\Delta t$ and the constants C_0, C_1, and C_2 are defined by (2.19) *and* (3.7).

The above result is useful if an initial layer is presented. It implies that the initial layer can be resolved if $\|q(u_0)\|_1$ is not large. See section 4 for more details about resolving initial layers.

In practical applications, some important initial data does not belong to L^1, such as piecewise constant initial data for Riemann problems. Simple modification to the proofs in this section gives the following result.

THEOREM 3.5. *Let the functions u_0, f, and q in* (1.5)–(1.6) *be subject to the conditions stated in Theorem* 3.3, *except $u_0 \in L^1(\mathbf{R})$. Let u be the entropy solution of* (1.5)–(1.6), *S_1 be the solution operator of the homogeneous problem* (3.1), *and S_2 be the backward Euler operator* (3.2). *The following estimate holds:*

$$\begin{aligned}(3.35)\quad &\max_{0\le n\le N}\left\|u(\cdot,t_n) - [S_2(\Delta t)S_1(\Delta t)]^n v_0\right\|_{L^1([a,b])} \le \|u_0 - v_0\|_{L^1([a-ct_N,\,b+ct_N])} \\ &+\Delta t\left(C_1|u_0|_{BV} + 4C_2|u_0|_{BV}\|u_0\|_\infty + \|q(u_0)\|_{L^1([a-ct_N,\,b+ct_N])} + C_0|u_0|_{BV}\right),\end{aligned}$$

where $t_n = n\Delta t$, c is a constant independent of Δt, and the constants C_0, C_1, and C_2 are defined by (2.19) *and* (3.7).

A numerical example with Riemann initial data will be given in section 6. The following source term will be used for that numerical experiment.

EXAMPLE 3.3. *Consider*

$$q(u) = e^{-u/\delta} - 1, \qquad 0 < \delta \ll 1,$$

with initial functions u_0 satisfying $u_0(x) \ge 0$ for all $x \in \mathbf{R}$.

It can be shown that $Q = O(\delta^{-1})$ with the nonnegative initial data. However, it can be verified that

$$\|q(u_0)\|_{L^1([a,b])} \le (b-a)\cdot O(1) \qquad \text{if} \quad u_0 \ge 0, \tag{3.36}$$

where $O(1)$ is a number independent of δ.

In the remainder of this section, we will briefly discuss splitting method with forward Euler operator. This approach has been used for conservation laws with nonstiff source terms [18, 34, 38]. Again, let S_1 be the solution operator of the homogeneous problem, as defined by (3.1). Let E be the forward Euler operator for the ordinary differential equation (ODE) problem $v_t = q(v)$, i.e., $v(\cdot,t) = E(t)v_0$ is given by

$$v(\cdot,t) = v_0(\cdot) + \Delta t\, q(v_0(\cdot)). \tag{3.37}$$

The operator-splitting solution is defined by

$$v(\cdot,t_n) = [S_1(\Delta t)E(\Delta t)]^n v_0. \tag{3.38}$$

Given a starting value $v(x,t_n)$, we compute $v(x,t_{n+1})$ using the following steps. First, find the intermediate value $\eta(x,t_{n+1})$ by using a forward Euler method

$$\eta(x,t_{n+1}) = v(x,t_n) + \Delta t q(v(x,t_n)). \tag{3.39}$$

Then solve the following homogeneous equation to find $v(x,t_{n+1})$:

$$v_t + f(v)_x = 0, \quad t \in (t_n, t_{n+1}]; \quad v(x,t_n) = \eta(x,t_{n+1}). \tag{3.40}$$

Employing similar techniques to those we used for splitting method with backward Euler operator, we can show that

$$\begin{aligned} &\|u(\cdot,N\Delta t) - [S_1(\Delta t)E(\Delta t)]^N v_0\|_1 \\ &\quad \le \|u_0 - v_0\|_1 + \Delta t\left(C_1|u_0|_{BV} + 4C_2\|u_0\|_\infty + \|q(u_0)\|_1 + C_0|u_0|_{BV}\right), \end{aligned} \tag{3.41}$$

provided that

$$\Delta t \max_{|u|\le\|u_0\|_\infty} |q'(u)| \le 1, \tag{3.42}$$

where in (3.41) the constants C_0, C_1, and C_2 are defined by (2.19) and (3.7).

It is clear that the main disadvantage of using the forward splitting method (3.39)–(3.40) is that the step size in time has to depend on $\max|q'(u)|$; see (3.42). For conservation laws with stiff source terms, the maximum value for q' is usually very large. Therefore, the constraint (3.42) is in general unacceptable. This suggests that if a stiff source term is involved, then the splitting method with forward Euler method should not be recommended.

4. A generic model and initial layer. According to Pember [28, 29], a conservation law is stiff if the time scale introduced by the source term is small compared to the characteristic speeds of the hyperbolic equation and some appropriate length scale. A generic form of a stiff conservation law is (1.5) with

$$q(u) = \frac{1}{\delta} g(u), \tag{4.1}$$

where $g \in C^1(\mathbf{R}), g(0) = 0, g' \le 0$, g is independent of δ; and the relaxation time $0 < \delta < 1$ is a very small parameter. In this section, we will consider this model problem

$$u_t + f(u)_x = \frac{1}{\delta} g(u), \qquad (x,t) \in \mathbf{R} \times (0,T], \tag{4.2}$$

$$u(x,0) = u_0(x). \tag{4.3}$$

This model problem has been studied numerically and theoretically by several authors (e.g., [15, 21, 32]). The result in Theorem 3.4 suggests that

$$\begin{aligned} &\max_{0\le n\le N} \left\| u(\cdot,t_n) - [S_2(\Delta t)S_1(\Delta t)]^n v_0 \right\|_1 \\ &\le \|u_0 - v_0\|_1 + C\Delta t + \frac{C\Delta t}{\delta}\|u_0\|_1 , \end{aligned} \tag{4.4}$$

provided that $u_0 \in BV \cap L^\infty \cap L^1$, where C is independent of Δt and δ. Therefore, the splitting error is in general related to δ. It should be pointed out that the estimate (4.4) is the optimal one in the sense that the $O(\Delta t\|u_0\|_1/\delta)$ term is not removable. To see this, we consider a simple example.

EXAMPLE 4.1. *Consider the following conservation law with linear flux and linear source:*

$$u_t + u_x = -\frac{u}{\delta}, \qquad u(x,0) = u_0(x), \tag{4.5}$$

where the initial data $u_0 \in L^1(\mathbf{R})$.

The exact solution of (4.5) is $u(x,t) = e^{-t/\delta}u_0(x-t)$. It can be verified that

$$[S_2(\Delta t)S_1(\Delta t)]^n u_0 = \left(1 + \frac{\Delta t}{\delta}\right)^{-n} u_0(x - t_n).$$

The above solutions imply that

$$\begin{aligned} &\max_{0\le n\le N} \left\| u(\cdot,t_n) - [S_2(\Delta t)S_1(\Delta t)]^n u_0 \right\|_1 \\ &= \max_{0\le n\le N} \left(\left(1 + \frac{\Delta t}{\delta}\right)^{-n} - \exp\left(-\frac{n\Delta t}{\delta}\right) \right) \|u_0\|_1. \end{aligned}$$

Unless $\|u_0\|_1$ is proportional to δ, the RHS of the above equation is small only when $\Delta t \ll \delta$. In particular, it cannot be bounded by an δ-free upper bound if $\|u_0\|_1 = O(1)$. This suggests that the last term in (4.4) cannot be removed. The reason for this is that for $\delta \ll 1$, there is a very small initial layer which needs to be resolved accurately. To illustrate the idea clearly, we consider a rather simple model. That is, we follow Schroll and Winther [32] by assuming that the function g in (4.1) decreases *strictly* with respect to u. In this case, zero is the only equilibrium. More precisely, the function g is assumed to satisfy

$$g(0) = 0, \qquad g'(u) < 0, \qquad \text{for} \quad u \in \mathbf{R}. \tag{4.6}$$

With the above assumptions, there exist two positive constants α, β independent of δ such that

$$\alpha = \min_{|u|\le\|u_0\|_\infty} |g'(u)| > 0, \qquad \beta = \max_{|u|\le\|u_0\|_\infty} |g'(u)| > 0. \tag{4.7}$$

The previous sections employed uniform meshes, and it is readily shown that the results are also valid for quasiuniform meshes which satisfy

$$\max_{0\le n\le N-1}(t_{n+1}-t_n)\le C\min_{0\le n\le N-1}(t_{n+1}-t_n), \tag{4.8}$$

where C is independent of N. In the following, we consider an uneven mesh given by

$$t_n=\begin{cases} 4n\delta|\ln\delta|/\alpha N & \text{for } n\le N/2,\\ 2\delta|\ln\delta|/\alpha+(T-2\delta|\ln\delta|/\alpha)(2n/N-1) & \text{for } N/2<n\le N,\end{cases} \tag{4.9}$$

where N is chosen to be an even number. It is easy to verify that $\{t_n\}_{n=0}^N$ is a mesh sequence of $[0,T]$. Let $\Delta t_n=t_n-t_{n-1}$. It can be verified that

$$\Delta t_n=\begin{cases} 4\delta|\ln\delta|/\alpha N & \text{for } n\le N/2,\\ 2(T-2\delta|\ln\delta|/\alpha)/N & \text{for } N/2<n\le N.\end{cases} \tag{4.10}$$

In other words, we divide $[0,T]$ into two subintervals $[0,2\delta|\ln\delta|/\alpha]$ and $(2\delta|\ln\delta|/\alpha,T]$. Two different uniform meshes, with mesh sizes given by (4.10), are used in the subintervals. With the mesh (4.9), the backward Euler operator-splitting procedure for (4.2)–(4.3) is given by

$$v(\cdot,t_m)=\prod_{n=1}^{m}S_2(\Delta t_n)S_1(\Delta t_n)v_0,\qquad 1\le m\le N. \tag{4.11}$$

In order to obtain sharper error bounds than (4.4), we need some additional properties for the solutions of (4.2)–(4.3). Using (4.7), we can modify Proposition 2.2 to get the following results.

LEMMA 4.1. *If $u_0\in BV\cap L^1(\mathbf{R})\cap L^\infty(\mathbf{R})$, $f\in C^1(\mathbf{R})$, and $g\in C^1(\mathbf{R})$ satisfying* (4.6)*, then the nonhomogeneous conservation law* (4.2)–(4.3) *possesses a unique entropy solution $u(x,t)$ satisfying*

$$|u(\cdot,t)|_{BV}\le e^{-\alpha t/\delta}|u_0|_{BV}, \tag{4.12}$$

$$\|u(\cdot,t)\|_1\le e^{-\alpha t/\delta}\|u_0\|_1, \tag{4.13}$$

$$\|u(\cdot,\tau_2)-u(\cdot,\tau_1)\|_1\le C_0\int_{\tau_1}^{\tau_2}|u(\cdot,\tau)|_{BV}d\tau+\frac{\beta}{\delta}\int_{\tau_1}^{\tau_2}\|u(\cdot,\tau)\|_1d\tau,\quad \tau_2\ge\tau_1, \tag{4.14}$$

where C_0 is defined by (2.19) *and α,β are given by* (4.7).

In the proof of Theorem 3.3, the term containing $\|q(u_0)\|_1$ is introduced in the estimate (3.20). We will use Lemma 4.1 to modify (3.20). Observe

$$\begin{aligned}&\sum_{n=0}^{N-1}\left\|\int_{t_n}^{t_{n+1}}q(u(\cdot,\tau))d\tau-\Delta t_nq(u(\cdot,t_{n+1}))\right\|_1\\ &\le\frac{\beta}{\delta}\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\|u(\cdot,\tau)-u(\cdot,t_{n+1})\|_1d\tau\\ &\le\frac{\beta}{\delta}\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\left[C_0\int_{\tau}^{t_{n+1}}|u(\cdot,t)|_{BV}dt+\frac{\beta}{\delta}\int_{\tau}^{t_{n+1}}\|u(\cdot,t)\|_1dt\right]d\tau,\end{aligned} \tag{4.15}$$

where in the last step we have used (4.14). Note that $0 < \delta < 1$. Using (4.12) and (4.13), we obtain from (4.15) that

$$
\begin{aligned}
(4.16)\qquad & \sum_{n=0}^{N-1} \left\| \int_{t_n}^{t_{n+1}} q(u(\cdot,\tau))d\tau - \Delta t_n q(u(\cdot,t_{n+1})) \right\|_1 \\
&\le \frac{C}{\delta^2} \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \int_{\tau}^{t_{n+1}} e^{-\alpha t/\delta} dt d\tau \\
&\le \frac{C}{\delta^2} \sum_{n<N/2} \Delta t_n \int_{t_n}^{t_{n+1}} e^{-\alpha t/\delta} dt + \frac{C}{\delta^2} \sum_{n\ge N/2} (\Delta t_n)^2 e^{-\alpha t_n/\delta} \\
&\le \frac{C}{\delta^2} \frac{2\delta|\ln\delta|}{N} \int_0^{2\delta|\ln\delta|/\alpha} e^{-\alpha t/\delta} dt + \frac{C}{(N\delta)^2} \sum_{n\ge N/2} e^{-2\delta|\ln\delta|/\delta} \\
&\le CN^{-1}|\ln\delta| + CN^{-1},
\end{aligned}
$$

where C is a constant independent of δ and N. Combining the above analysis, we obtain the following conclusions.

THEOREM 4.2. *Assume that $u_0 \in BV \cap L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$, $f \in C^2(\mathbf{R})$, and $g \in C^1(\mathbf{R})$ satisfying $g(0) = 0, g'(0) < 0$. Let u be the entropy solution of* (4.2)–(4.3) *and the mesh $\{t_n\}_{n=0}^N$ be given by* (4.9). *Then the following estimate holds:*

$$
\begin{aligned}
(4.17)\qquad & \left\| u(\cdot, N\Delta t) - \prod_{n=1}^{N} S_2(\Delta t_n) S_1(\Delta t_n) v_0 \right\|_1 \\
&\le \|u_0 - v_0\|_1 + CN^{-1}\Big(|\ln\delta| + 1\Big),
\end{aligned}
$$

where the constant C is independent of N and δ.

As mentioned at the end of section 3, we can modify the above proof to get the following estimates.

THEOREM 4.3. *Let the functions u_0, f, and g in* (4.2)–(4.3) *be subject to the conditions stated in Theorem* 4.2. *Let u be the entropy solution of* (4.2)–(4.3). *Then the following estimates hold:*

$$
\begin{aligned}
(4.18)\qquad & \max_{1\le n\le N} \left\| u(\cdot,t_n) - \prod_{m=1}^{n} S_2(\Delta t_m) S_1(\Delta t_m) u_0 \right\|_1 \\
&\le \begin{cases} C(N\delta)^{-1} & \text{for uniform mesh,} \\ CN^{-1}\Big(|\ln\delta| + 1\Big) & \text{for mesh (4.9),} \end{cases}
\end{aligned}
$$

where C is independent of N and δ.

For numerical verification of the results stated in Theorem 4.3, we reconsider the problem in Example 4.1, with $\delta = 0.001$, $0 < t \le 1$, and with initial data such that $\|u_0\|_1 = 1$. The L^1 error for the operator-splitting methods, given by the LHS of (4.18), with uniform mesh and (4.9) are plotted in Figure 1. It is observed that the error obtained by using mesh (4.9) is linear with respect to N^{-1}. On the other hand, the error associated with uniform mesh is quite large since N^{-1} is not smaller than δ. An example with nonlinear flux will be considered in section 6.

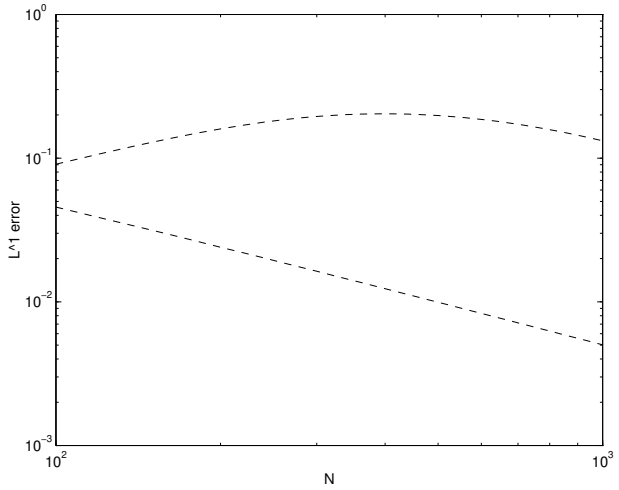

FIG. 1. L^1 *error for operator-splitting method for Example* 4.1. *The upper curve is obtained by using uniform mesh and the lower one is given by mesh* (4.9).

5. Multidimensional equation. We consider the two-dimensional Cauchy problem

$$u_t + f_1(u)_x + f_2(u)_y = q(u); \qquad (x,y) \in \mathbf{R}^2, \quad t \in (0,T], \tag{5.1}$$

$$u(x,y,0) = u_0(x,y). \tag{5.2}$$

We point out that Propositions 2.1–2.4 can be extended to scalar conservation laws with several space dimensions. As an example, we will extend (2.8) to the two-dimensional problem (5.1)–(5.2). Assume that we have another equation

$$v_t + f_1(v)_x + f_2(v)_y = q(v); \qquad (x,y) \in \mathbf{R}^2, \quad t \in (0,T], \tag{5.3}$$

$$v(x,y,0) = v_0(x,y). \tag{5.4}$$

We will prove the following result, which is the extension of (2.8). The standard theory for *vanishing viscosity method*, which is based on the idea of passing to the limit as $\epsilon \to 0+$ in (5.9) or (5.11) below, can be found in Kruzkov [16] and Vol'pert [42].

LEMMA 5.1. *If* u_0 *given by* (5.2) *and* v_0 *given by* (5.4) *belong to* $BV(\mathbf{R}^2) \cap L^1(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2)$, *then the following estimate holds:*

$$\int_0^T \|q(u(\cdot,\cdot,\tau)) - q(v(\cdot,\cdot,\tau))\|_1 d\tau \le \|u_0 - v_0\|_1, \tag{5.5}$$

where u *and* v *are solutions of* (5.1)–(5.2) *and* (5.3)–(5.4), *respectively.*

Proof. Again we consider the parabolic regularization of the following forms:

$$\begin{cases} u_t^\epsilon + f_1(u^\epsilon)_x + f_2(u^\epsilon)_y = q(u^\epsilon) + \epsilon\left(u_{xx}^\epsilon + u_{yy}^\epsilon\right); \quad u^\epsilon(x,y,0) = u_0(x,y), \\ \\ v_t^\epsilon + f_1(v^\epsilon)_x + f_2(v^\epsilon)_y = q(v^\epsilon) + \epsilon\left(v_{xx}^\epsilon + v_{yy}^\epsilon\right); \quad v^\epsilon(x,y,0) = v_0(x,y), \end{cases} \tag{5.6}$$

where $\epsilon > 0$. We will employ the techniques used in the proof of Proposition 2.2 to prove (5.5). Let $k = T/M$ and $t_m = mk$. Let $U^\epsilon(x, y, 0) = u_0(x, y)$. For a given starting value $U^\epsilon(x, y, t_m)$, we will obtain $U^\epsilon(x, y, t_{m+1})$ by the following forward Euler splitting method. The first step is to find the intermediate value $\xi_m(x, y)$ by using the information of the given function $U^\epsilon(x, y, t_m)$

$$\xi_m(x, y) = U^\epsilon(x, y, t_m) + kq(U^\epsilon(x, y, t_m)). \tag{5.7}$$

To make this step stable, we require that

$$k \max_{|u|\le\|u_0\|_\infty} |q'(u)| \le 1. \tag{5.8}$$

The second step is to solve the following homogeneous parabolic equation to obtain $U^\epsilon(x, y, t_{m+1})$:

$$\begin{cases} U^\epsilon_t + f_1(U^\epsilon)_x + f_2(U^\epsilon)_y = \epsilon \left(U^\epsilon_{xx} + U^\epsilon_{yy}\right), & t \in (t_m, t_{m+1}), \\ \\ U^\epsilon(x, y, t_m) = \xi_m(x, y). \end{cases} \tag{5.9}$$

This procedure will generate $U(x, y, t_{m+1})$ for $0 \le m \le M - 1$. Similarly, we can obtain $V^\epsilon(x, y, t_{m+1})$ by first computing

$$\eta_m(x, y) = V^\epsilon(x, y, t_m) + kq(V^\epsilon(x, y, t_m)) \tag{5.10}$$

and then solving the homogeneous parabolic equations

$$\begin{cases} V^\epsilon_t + f_1(V^\epsilon)_x + f_2(V^\epsilon)_y = \epsilon \left(V^\epsilon_{xx} + V^\epsilon_{yy}\right), & t \in (t_m, t_{m+1}), \\ \\ V^\epsilon(x, y, t_m) = \eta_m(x, y) \end{cases} \tag{5.11}$$

for $0 \le m \le M-1$. It can be shown that the above splitting procedures are convergent in the following sense:

$$\sup_{0\le m\le M} \|u^\epsilon(\cdot, \cdot, t_m) - U^\epsilon(\cdot, \cdot, t_m)\|_1 \to 0 \qquad \text{as} \quad k \to 0, \tag{5.12}$$

$$\sup_{0\le m\le M} \|v^\epsilon(\cdot, \cdot, t_m) - V^\epsilon(\cdot, \cdot, t_m)\|_1 \to 0 \qquad \text{as} \quad k \to 0. \tag{5.13}$$

Using the standard theory for homogeneous parabolic equations we obtain from (5.9) and (5.11) that

$$\|U^\epsilon(\cdot, \cdot, t_{m+1}) - V^\epsilon(\cdot, \cdot, t_{m+1})\|_1 \le \|\xi_m(\cdot, \cdot) - \eta_m(\cdot, \cdot)\|_1. \tag{5.14}$$

It follows from (2.2) that

$$\begin{aligned} &\operatorname{sgn}(U^\epsilon - V^\epsilon)(U^\epsilon - V^\epsilon + kq(U^\epsilon) - kq(V^\epsilon)) \\ &= |U^\epsilon - V^\epsilon| - k|q(U^\epsilon) - q(V^\epsilon)| \ge 0, \end{aligned}$$

where in the last step we have used (5.8). The above result implies that $U^\epsilon - V^\epsilon + kq(U^\epsilon) - kq(V^\epsilon)$ is of the same sign as $U^\epsilon - V^\epsilon$. In other words, we have proved that

$$\begin{aligned} &\left|U^\epsilon - V^\epsilon + kq(U^\epsilon) - kq(V^\epsilon)\right| \\ &= \left|U^\epsilon - V^\epsilon\right| - k\left|q(U^\epsilon) - q(V^\epsilon)\right|, \end{aligned}$$

provided (5.8) holds. This fact, together with (5.7) and (5.10), yields

$$\begin{aligned}(5.15)\qquad &\|\xi_m(\cdot,\cdot)-\eta_m(\cdot,\cdot)\|_1\\ &=\|U^\epsilon(\cdot,\cdot,t_m)+kq(U^\epsilon(\cdot,\cdot,t_m))-V^\epsilon(\cdot,\cdot,t_m)-kq(V^\epsilon(\cdot,\cdot,t_m))\|_1\\ &=\|U^\epsilon(\cdot,\cdot,t_m)-V^\epsilon(\cdot,\cdot,t_m)\|_1-k\|q(U^\epsilon(\cdot,\cdot,t_m))-q(V^\epsilon(\cdot,\cdot,t_m))\|_1.\end{aligned}$$

Combining (5.14) and (5.15) we obtain

$$\begin{aligned}(5.16)\qquad &\|U^\epsilon(\cdot,\cdot,t_{m+1})-V^\epsilon(\cdot,\cdot,t_{m+1})\|_1\\ &\le\|U^\epsilon(\cdot,\cdot,t_m)-V^\epsilon(\cdot,\cdot,t_m)\|_1-k\|q(U^\epsilon(\cdot,\cdot,t_m))-q(V^\epsilon(\cdot,\cdot,t_m))\|_1\end{aligned}$$

for $0\le m\le M-1$, provided that k is sufficiently small. It is easy to obtain from the above inequality that

$$(5.17)\qquad \sum_{m=0}^{M-1}k\|q(U^\epsilon(\cdot,\cdot,t_m))-q(V^\epsilon(\cdot,\cdot,t_m))\|_1\le\|u_0-v_0\|_1,$$

where we have used the fact that $U^\epsilon(x,y,0)=u_0(x,y)$ and $V^\epsilon(x,y,0)=v_0(x,y)$. Letting $k\to 0$, it follows from (5.12), (5.13), and (5.17) that

$$(5.18)\qquad \int_0^T\|q(u^\epsilon(\cdot,\cdot,\tau))-q(v^\epsilon(\cdot,\cdot,\tau))\|_1d\tau\le\|u_0-v_0\|_1.$$

The inequality (5.5) is obtained by letting $\epsilon\to 0$ in (5.18). □

After extending Propositions 2.1–2.4, we can establish results similar to those in sections 3 and 4. In particular, Theorems 3.3 and 4.3 can be extended to the two-dimensional problem (5.1)–(5.2). We will omit the detailed results and proofs here.

6. Numerical experiments. The purpose of this section is to verify our theoretical estimates obtained in sections 3 and 4 by two numerical experiments. The most interesting phenomena of nonlinear hyperbolic conservation laws is the existence of shock and rarefaction waves. So an important issue of numerical conservation laws is to find accurate location of the discontinuous curves. With this in mind, we design two numerical examples. The first one has a nonlinear flux and a nonlinear source term. With simple Riemann initial data, the entropy solution contains a single shock wave. Particular attention is given to the numerical shock locations. The second example has nonlinear flux but the source term is linear. With appropriate initial data, this linear source will allow us to find an *analytical form* of entropy solution, which contains both shock and rarefaction waves. A recurrence relation for splitting solutions is also available for this example. Therefore, L^1 errors can be computed easily at each time step.

EXAMPLE 6.1. *We first consider the following Riemann problem:*

$$(6.1)\qquad \begin{cases} u_t+(u^2/2)_x=e^{-u/\delta}-1,\\ \\ u(x,0)=1-H(x),\end{cases}$$

where $H(x)$ is the Heaviside function.

The exact solution for (6.1) is given by

$$(6.2)\qquad u(x,t)=\begin{cases} U_L(t) & x<X(t),\\ U_R(t) & x>X(t),\end{cases}$$

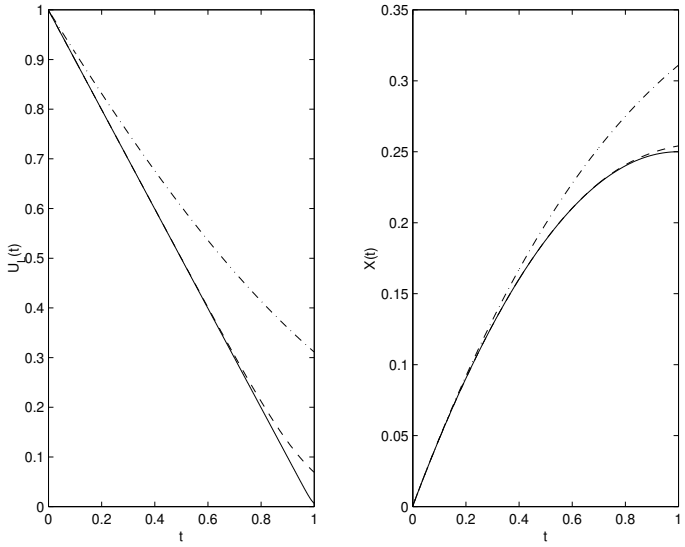

FIG. 2. *$U_L(t)$ and $X(t)$ for $0 \le t \le 1$. The line types are as following: dashdot line for $\delta = 0.5$, dashed line for $\delta = 0.1$, and solid line for $\delta = 0.01$.*

where $U_L(t), U_R(t)$, and $X(t)$ are determined by the following ODEs, respectively:

$$\begin{cases} \dot{U}_L(t) = e^{-U_L/\delta} - 1, & U_L(0) = 1, \\ \dot{U}_R(t) = e^{-U_R/\delta} - 1, & U_R(0) = 0, \\ \dot{X}(t) = \frac{1}{2}(U_L(t) + U_R(t)), & X(0) = 0. \end{cases} \tag{6.3}$$

It is observed that the solution of the second ODE is identically zero. So what we need to find are $U_L(t)$ and $X(t)$ which satisfy the following ODE systems:

$$\begin{cases} \dot{U} = e^{-U/\delta} - 1, & U(0) = 1, \\ \dot{X} = \frac{1}{2}U, & X(0) = 0. \end{cases} \tag{6.4}$$

We solve (6.4) by 4th-order Runge–Kutta method (RK4) with sufficiently small mesh size to get the "exact solution" of $U_L(t)$ and $X(t)$. In Figure 2, we plot the solutions of $U_L(t)$ and $X(t)$ for several values of δ.

On the numerical side, at each time level our procedure consists of two steps. First, an exact solution is found for the following Riemann problem:

$$u_t + (u^2/2)_x = 0, \qquad u(x, t_n) = \begin{cases} U_L^n & x < X_n, \\ U_R^n & x > X_n, \end{cases} \tag{6.5}$$

where $t_n = n\Delta t$ and the values of X_n are determined by

$$X_{n+1} = X_n + \frac{1}{2}\Delta t U_L^n, \qquad X_0 = 0. \tag{6.6}$$

Second, we use the backward Euler method to update the values of U_L^n and U_R^n. This completes the two splitting steps at each time level. It can be shown that $U_R^n \equiv 0$.

TABLE 1
Maximum errors for $U_L(t)$ and $X(t)$, $0 \le t \le 1$.

		$\Delta t = 1/10$	$\Delta t = 1/20$	$\Delta t = 1/40$
$\delta = 0.5$	$\max_n \lvert U_L(t_n) - U_L^n \rvert$	1.4 e-2	7.1 e-3	3.6 e-3
	$\max_n \lvert X(t_n) - X_n \rvert$	2.1 e-2	1.0 e-2	5.2 e-3
$\delta = 0.1$	$\max_n \lvert U_L(t_n) - U_L^n \rvert$	1.6 e-2	8.3 e-3	4.2 e-3
	$\max_n \lvert X(t_n) - X_n \rvert$	2.4 e-2	1.2 e-2	6.2 e-3
$\delta = 0.01$	$\max_n \lvert U_L(t_n) - U_L^n \rvert$	1.1 e-2	6.5 e-3	3.4 e-3
	$\max_n \lvert X(t_n) - X^n \rvert$	2.5 e-2	1.3 e-2	6.2 e-3

For this simple Riemann problem, it is shown by [15] that

$$\begin{aligned} &\max_{0\le n\le N} \Big\| u(\cdot, t_n) - [S_2(\Delta t) S_1(\Delta t)]^n v_0 \Big\|_{L^1([a,b])} \\ &\le C(a,b)\Big(\max_{0\le n\le N} |U_L(t_n) - U_L^n| + \max_{0\le n\le N} |X(t_n) - X_n| \Big). \end{aligned} \tag{6.7}$$

In Table 1 we list the errors for $\max_{0\le n\le N} |U_L(t_n) - U_L^n|$ and $\max_{0\le n\le N} |X(t_n) - X_n|$ with $\Delta t = 1/N$ for several values of δ. From the results in Table 1 and (6.7) we conclude that the splitting errors in $L^1([a,b])$ can be bounded by $C\Delta t$, where C is independent of δ. This is in good agreement with Theorem 3.5.

As mentioned earlier, we are interested in test problems which contain both a shock and a rarefaction. In order to make comparisons between the analytical solution and numerical solution possible, the exact solutions need to be found easily. The last example contains only a shock, but the exact solution can be found quite easily by solving a system of ODEs. When a problem contains both a shock and a rarefaction, it seems difficult to find the exact solution if both the flux and the source are nonlinear. Based on this observation, we will consider the following example.

EXAMPLE 6.2. *Consider*

$$\begin{cases} u_t + (u^2/2)_x = -u/\delta, \\ u(x,0) = \begin{cases} -1 & -1 < x < 0, \\ 0 & \text{otherwise.} \end{cases} \end{cases} \tag{6.8}$$

The source term of this example is linear, but the solution contains both a shock and a rarefaction. The exact solution of (6.8) is

$$u(x,t) = \begin{cases} 0 & x \le -1, \\ \frac{x+1}{\delta(e^{t/\delta}-1)} & -1 < x \le \gamma(t), \\ e^{-t/\delta} & \gamma(t) < x \le X(t), \\ 0 & x > X(t), \end{cases}$$

where the functions γ and X are defined by

$$\gamma(t) = \delta(1 - e^{-t/\delta}) - 1, \qquad X(t) = \frac{\delta}{2}(1 - e^{-t/\delta}). \tag{6.9}$$

On the other hand, it can be verified that

$$S_1(\Delta t_n) \prod_{m=1}^{n} S_2(\Delta t_m) S_1(\Delta t_m) u_0 = \begin{cases} 0 & x \le -1, \\ (x+1)\mathcal{A}_{n+1} & -1 < x \le \gamma_{n+1}\,, \\ \mathcal{B}_{n+1} & \gamma_{n+1} < x < X_{n+1}\,, \\ 0 & x > X_{n+1}\,; \end{cases}$$

TABLE 2
Maximum L^1 errors for Example 6.2: uniform mesh $\Delta t = 1/N$.

	$N = 20$	$N = 40$	$N = 80$	$N = 160$
$\delta = 0.1$	7.8 e-2	4.2 e-2	2.2 e-2	1.1 e-2
$\delta = 0.01$	1.6 e-1	2.0 e-1	1.6 e-1	9.3 e-2
$\delta = 0.001$	1.9 e-2	3.8 e-2	7.4 e-2	1.4 e-1

$$S_2(\Delta t_n)S_1(\Delta t_n)\prod_{m=1}^{n} S_2(\Delta t_m)S_1(\Delta t_m)u_0 = \begin{cases} 0 & x \le -1, \\ (x+1)\alpha_{n+1} & -1 < x \le \gamma_{n+1}\,, \\ \beta_{n+1} & \gamma_{n+1} < x < X_{n+1}\,, \\ 0 & x > X_{n+1}\,, \end{cases}$$

where $\Delta t_n = t_n - t_{n-1}$ and we have the following formulas to compute the parameters $\alpha_{n+1}, \beta_{n+1}$, etc.

- **(i)** Initial data:

$$\begin{cases} \alpha_1 = \Big(\Delta t_1(1 + \Delta t_1/\delta)\Big)^{-1}, \\ \beta_1 = (1 + \Delta t_1/\delta)^{-1}, \\ \gamma_1 = -1 + \Delta t_1, \quad X_1 = \frac{1}{2}\Delta t_1\,. \end{cases} \tag{6.10}$$

- **(ii)** With the operator S_1, i.e., the solver for homogeneous conservation laws, we have for $1 \le n \le N-1$

$$\begin{cases} \mathcal{A}_{n+1} = (\Delta t_{n+1} + 1/\alpha_n)^{-1}\,, \quad \mathcal{B}_{n+1} = \beta_n\,, \\ \gamma_{n+1} = -1 + (\Delta t_{n+1} + 1/\alpha_n)\beta_n, \\ X_{n+1} = X_n + \frac{1}{2}\beta_n \Delta t_{n+1}\,. \end{cases} \tag{6.11}$$

- **(iii)** With the operator S_2, i.e., the backward Euler solver, we have for $1 \le n \le N-1$

$$\begin{cases} \alpha_{n+1} = \mathcal{A}_{n+1}/(1 + \Delta t_{n+1}/\delta), \\ \beta_{n+1} = \mathcal{B}_{n+1}/(1 + \Delta t_{n+1}/\delta). \end{cases} \tag{6.12}$$

In Tables 2 and 3 we list the maximum L^1 errors

$$\max_{1\le n\le N} \left\| u(\cdot, t_n) - \prod_{m=1}^{n} S_2(\Delta t_m)S_1(\Delta t_m)u_0 \right\|_1$$

for a uniform mesh, $\Delta t = 1/N$, and for the nonuniform mesh (4.9), respectively. In (4.9), $\alpha = 1$ and $T = 1$. For the uniform mesh, first-order convergence is observed for $\delta = 0.1$, but not for $\delta = 0.001$. It is seen in Table 3 that errors with the nonuniform mesh are proportional to $N^{-1}|\ln \delta|$, which is in good agreement with Theorem 4.3. For small values of δ, it is observed that the nonuniform mesh is more suitable than the uniform one.

We also plot the right boundary of the rarefaction wave, $\gamma(t)$, and the shock location curve, $X(t)$, given by (6.9), in Figure 3. The parameters we used for this

TABLE 3
Maximum L^1 errors for Example 6.2: nonuniform mesh (4.9).

	$N = 20$	$N = 40$	$N = 80$	$N = 160$
$\delta = 0.1$	7.3 e-2	3.9 e-2	2.0 e-2	1.1 e-2
$\delta = 0.01$	1.2 e-1	7.1 e-2	3.8 e-2	2.0 e-2
$\delta = 0.001$	1.7 e-1	9.8 e-2	5.5 e-2	2.9 e-2

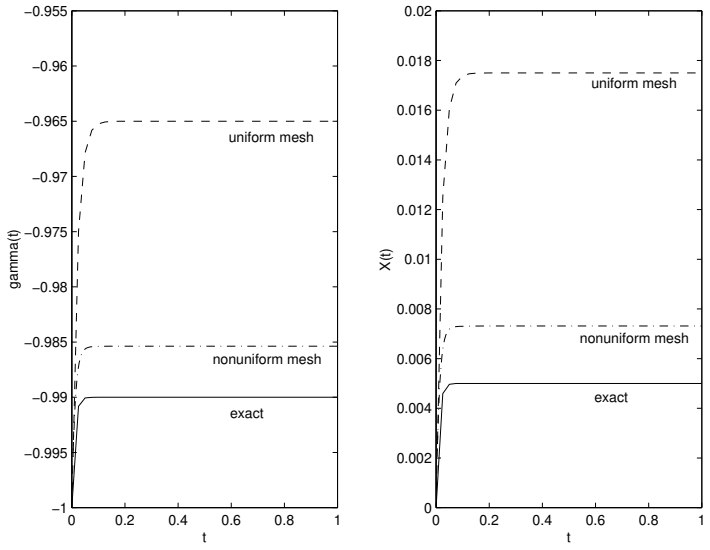

FIG. 3. *The location functions for rarefaction and shock waves, $\gamma(t)$ and $X(t)$, for Example 6.2, with $\delta = 0.01$ and $N = 40$.*

figure are $\delta = 0.01$ and $N = 40$. It is clear that the nonuniform mesh produces more accurate results than the uniform mesh does.

Finally, we point out that the computational complexity of both meshes are essentially the same for a given value of N (the number of time steps required for the uniform mesh, given by $\Delta t = T/N$, is N, and the number of steps for the nonuniform mesh, given by (4.9), is also N). The only extra time for the latter mesh is to find the grid distribution by using (4.9) and to evaluate the meshsizes by using (4.10).

7. Concluding remarks. In this work, we have analyzed time-splitting methods applied to hyperbolic conservation laws. The solution properties obtained in section 2 are important in the error analysis. In a separate work, we will consider full discretizations for the nonhomogeneous equation with stiff source terms. We will analyze the order of convergence of the fully discretized schemes. It will be shown that the schemes possess a discrete analogy of the stability properties given in section 2.

This work concentrates on time splitting based on Euler-type methods. As expected, the error introduced by the splitting is at most linear with respect to the time step. A frequently used splitting method for inhomogeneous hyperbolic equations is Strang's splitting [35]. As before, if we call the stiff ODE operators $S_2(t)$ and the homogeneous convection operator $S_1(t)$, then the Strang's splitting takes the form

$$u(\cdot, t_{n+1}) \approx S_2\left(\frac{\Delta t}{2}\right) S_1(\Delta t) S_2\left(\frac{\Delta t}{2}\right) u(\cdot, t_n).$$

Consider the simple model (4.2)–(4.3). It can be shown that the above scheme is a second-order splitting for $\delta = O(1)$, as long as the exact solution is sufficiently *smooth*. A natural question is, can Strang's splitting be second order as $\delta \to 0$ while holding Δt fixed? In [14], Jin considered a prototypical relaxation model

$$h_t + w_x = 0,$$
$$w_t + p(h)_x = -\frac{1}{\delta}(w - f(h)), \quad p'(h) > 0.$$

By asymptotic analysis, he claimed that in the regime $\delta \to 0$ and $\delta/\Delta t \to 0$ the Strang's splitting is only a first-order approach to the equilibrium equation. He also designed a second-order Runge–Kutta-type splitting method that possesses the discrete analogy of the continuous asymptotic limit, which is able to capture the correct physical behaviors with high order accuracy, even if the initial layer is not numerically resolved. The convergence of Jin's second-order splitting scheme was proved by Chalabi [1]. It is interesting to provide rigorous analysis for the convergence order of Strang's second-order splitting and Jin's second-order splitting.

Acknowledgments. Part of this work was done during the author's visit to ETH Zurich and the Chinese University of Hong Kong. The hospitality of Professors Rolf Jeltsch at ETH and Raymond Chan at CUHK is gratefully acknowledged. Thanks also to the referee for valuable suggestions which in particular led to a new section, section 6, and to the editor for useful comments which led to an improved presentation of this paper.

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