ON THREE-POINT SECOND-ORDER ACCURATE CONSERVATION DIFFERENCE SCHEMES*1)

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§1. Introduction

In this paper we study 3-point 2nd-order accurate conservative TSC difference schemes. The TSC schemes are named after three computational features:

(A) They are 3-point schemes. Only three points are needed to determine a point on the next time level. So they can fit conveniently the boundary conditions for initial boundary value problems.

(B) They have 2nd-order accuracy for smooth solutions. So in the smooth parts of the solution, we can get a better numerical result.

(C) They are in conservation form. By the Lax-Wendroff theorem[10], if the computed solution of the TSC schemes converges boundedly a.e. to u, then u is a weak solution of Eq. (2.1).

In this paper we prove that the TSC scheme is not TVNI (total variation non-increasing). We prove that the 3-point s-order accuracy ($s \ge 2$) linear difference scheme is linearly $l_p(1 \le p \le +\infty,\ p \ne 2)$ unstable. So a linear TSC scheme is linearly $l_p(1 \le p \le +\infty,\ p \ne 2)$ unstable. In addition, a rigorous proof of the nonlinear l_2 instabilities for the two-step Richtmyer[6] scheme is given. At last, a successful modification to the Lax-Wendroff scheme, the Richtmyer scheme and the MacCormack scheme is got. The modified schemes retain their computational features (A), (B) and (C) mentioned above, with an additional property that the limit solutions satisfy the entropy condition for all the convex smooth flux functions f. They are l_2 stable in the sense of Definition 3.2 when we choose $\theta(s) \equiv 1$ in the modified schemes (3.7), (3.9) and (3.10).

§2. The stability results for TSC schemes

The simplest mathematical models of inviscid compressible fluid dynamics are given by solutions, u, of the scalar convex conservation law

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial f(u)}{\partial x} \equiv \dfrac{\partial u}{\partial t} + a(u)\dfrac{\partial u}{\partial x} = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{2.1}$$

where f is a smooth convex function of u.

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We shall discuss numerical approximations to weak solutions of (2.1) which are obtained by $(2k+1)$-point explicit schemes in conservation form,

$$u_j^{n+1} = u_j^n - \lambda(g_{j+\frac{1}{2}}^n - g_{j-\frac{1}{2}}^n), \tag{2.2a}$$

where

$$g_{j+\frac{1}{2}}^n = g(u_{j-k+1}^n, \cdots, u_{j+k}^n). \tag{2.2b}$$

Here $u_j^n = \bar{u}(j\Delta x, n\Delta t)$ and g is a continuous numerical flux function. We require that the numerical flux function be consistent with the flux function $f(u)$ in the following sense

$$g(u, \cdots, u) = f(u). \tag{2.2c}$$

The class of schemes we shall discuss can be written in the form

$$u_j^{n+1} = u_j^n + d_{+,j+\frac{1}{2}}^{(2k+1)} \Delta_+ u_j^n - d_{-,j-\frac{1}{2}}^{(2k+1)} \Delta_+ u_{j-1}^n, \tag{2.3a}$$

where

$$d_{+,j+\frac{1}{2}}^{(2k+1)} = \begin{cases} \lambda \dfrac{f_j - g_{j+\frac{1}{2}}}{\Delta_+ u_j}, & \text{if } \Delta_+ u_j \neq 0, \\ -\lambda a(u_j), & \text{if } \Delta_+ u_j = 0 \end{cases} \tag{2.3b}$$

and

$$d_{-,j-\frac{1}{2}}^{(2k+1)} = \begin{cases} \lambda \dfrac{f_j - g_{j-\frac{1}{2}}}{\Delta_+ u_{j-1}}, & \text{if } \Delta_+ u_{j-1} \neq 0, \\ \lambda \cdot a(u_j), & \text{if } \Delta_+ u_{j-1} = 0. \end{cases} \tag{2.3c}$$

Here and throughout this paper, we use the standard notation

$$\Delta_+ u_j = u_{j+1} - u_j;\ f_j = f(u_j);\ \lambda = \Delta t/\Delta x.$$

Scheme (2.3) includes the TSC scheme and Harten's TVNI scheme[5].

We say that the finite difference scheme (2.2) is total variation nonincreasing TVNI if for all nonnegative integers n, we have

$$TV(u^{n+1}) \leq TV(u^n), \tag{2.4a}$$

where

$$TV(u^n) \equiv \sum_{j=-\infty}^{\infty} |\Delta_+ u_j^n|. \tag{2.4b}$$

Theorem 2.1. *If the scheme (2.1) is* TVNI, *then we have the following inequalities,*

$$\frac{f(a) - g(b, a, \cdots, a)}{b - a} \leq 0, \tag{2.5a}$$

$$\frac{f(a) - g(a, \cdots, a, b)}{b - a} \geq 0 \tag{2.5b}$$

for all real numbers a and b, $a \neq b$.

Proof. If $b > a$, we assume

$$u_j^n = \begin{cases} a, & j < J, \\ b, & j \geq J, \end{cases} \tag{2.6}$$

where J is an arbitrarily fixed integer. If $u_{J-k}^{n+1} < a$, then

$$TV(u^{n+1}) \geq |u_{J-k}^{n+1} - u_{J+2k+2}^{n+1}| = |u_{J-k}^{n+1} - b| > b - a = TV(u^n),$$

which contradicts the assumption that the scheme is TVNI. So we have

$$u_{J-k}^{n+1} \geq a. \tag{2.7}$$

But

$$\begin{aligned} u_{J-k}^{n+1} &= u_{J-k}^n - \lambda[g(u_{J-2k+1}^n, \cdots, u_J^n) - g(u_{J-2k}^n, \cdots, u_{J-1}^n)] \\ &= a - \lambda[g(a, \cdots, a, b) - g(a, \cdots, a)] \\ &= a - \lambda[g(a, \cdots, a, b) - f(a)]. \end{aligned}$$

By using inequality (2.7) and $b > a$, we have

$$\frac{f(a) - g(a, \cdots, a, b)}{b - a} \geq 0 \quad \text{for } b > a.$$

If $b < a$, we still assume u_j^n satisfies (2.6). It is easy to show that $u_{J-k}^{n+1} \leq a$, and

$$u_{J-k}^{n+1} = a - \lambda[g(a, \cdots, a, b) - f(a)].$$

By assumption $b < a$, we can get

$$\frac{f(a) - g(a, \cdots, a, b)}{b - a} \geq 0 \quad \text{for } b < a.$$

Thus inequality (2.5b) is proved. Inequality (2.5a) can be proved in the same way.

Corollary 2.2. The Lax-Wendroff scheme, the Richtmyer scheme and the MacCormack scheme are not TVNI.

Proof. It is easy to learn that the numerical flux function of the MacCormack scheme is

$$g(b, a) = \frac{1}{2}[f(a) + f(b - \lambda(f(a) - f(b)))].$$

Then, if $a \neq b$, we have

$$\begin{aligned} & f(a) - g(b, a) \\ & \quad = \frac{1}{2}[f(a) - f(b - \lambda(f(a) - f(b)))] \\ & \quad = \frac{1}{2}(a - b)\left(1 + \lambda\frac{f(a) - f(b)}{a - b}\right)\int_0^1 f'(sa + (1 - s)(b - \lambda(f(a) - f(b))))ds. \end{aligned}$$

In the process of our numerical computation, the Courant-Friedrichs-Lewy condition is valid, i.e.,

$$\max_{0 \leq s \leq 1} \lambda \cdot |f'(sa + (1 - s)b| \leq 1.$$

Then, if we choose function f satisfying $f'(u)<0$, $f''(u)\neq 0$ for all $u\in R^1$, we have

$$1+\lambda\frac{f(a)-f(b)}{a-b}>0,$$

and

$$\int_0^1 f'(sa+(1-s)(b-\lambda(f(a)-f(b))))ds<0.$$

Hence, (2.5a) is not valid for the real numbers a and b, $a\neq b$. By Theorem 2.1, the MacCormack scheme is not TVNI. We can show similarly that the Lax-Wendroff scheme and the Richtmyer scheme are not TVNI.

The following important corollary is immediate from Theorem 2.1.

Corollary 2.3. *If the scheme (2.3) is TVNI, then* $d^{(3)}_{+,j+\frac12}\geq 0$, $d^{(3)}_{-,j+\frac12}\geq 0$ *for all integers* j *and* $u_j\neq u_{j+1}$.

Theorem 2.4. *If* $d^{(2k+1)}_{+,j+\frac12}\cdot d^{(2k+1)}_{-,j+\frac12}\geq 0$ *for all integers* j *and* $u_j\neq u_{u+1}$, *then the scheme (2.3) is only of first-order accuracy.*

Proof. We choose the flux function f which is a smooth one, and the initial value $u_0(x)\in C^\infty(R^1)\cap L^\infty(R^1)$, $u_0'(x)>0$ for all $x\in R^1$.

First we consider the following TSC scheme.

$$u_j^{n+1}=u_j^n-\lambda[\bar g(u_j^n,u_{j+1}^n)-\bar g(u_{j-1}^n,u_j^n)]$$

with

$$\bar g(a,b)=\begin{cases}\frac12[f(a)+f(b)]-\frac{\lambda}{2}(f(b)-f(a))^2/(b-a), & \text{if } a\neq b,\\ f(a), & \text{if } a=b,\end{cases}$$

Clearly, a numerical flux function of a second order accurate scheme $g_{j+\frac12}$ has to satisfy[5]

$$g^0_{j+\frac12}-\bar g(u_j^0,u_{j+1}^0)=O(h^2),$$

where $h=\Delta x$, $u_j^0=u_0(jh)$. Assume that the scheme (2.3) is of second order accuracy. Then we have

$$u_j^1=u_j^0-\lambda(g^0_{j+\frac12}-g^0_{j-\frac12})$$

$$d^{(2k+1)}_{+,j+\frac12}=\lambda\frac{f_j^0-g^0_{j+\frac12}}{\Delta_+u_j^0}=\lambda\frac{f_j^0-\bar g(u_j^0,u_{j+1}^0)}{\Delta_+u_j^0}+O(h)=-\frac12\theta_j+\frac12\theta_j^2+O(h),$$

where $\theta_j=\lambda\dfrac{\Delta_+f(u_j^0)}{\Delta_+u_j^0}$. In the same way, we have

$$d^{(2k+1)}_{-,j+\frac12}=\frac12\theta_j+\frac12\theta_j^2+O(h).$$

For an arbitrary fixed $x \in \mathbb{R}^1$, let $j = \left[\frac{x}{h}\right]$. Here $[r]$ denotes the integral part of the real number r. Thus

$$j \cdot h \to x, \ u_j^0 \to u_0(x) \quad \text{as } h \to 0,$$

$$\begin{aligned} d^{(2k+1)}_{+,j+\frac{1}{2}} \cdot d^{(2k+1)}_{-,j+\frac{1}{2}} &\to \left[-\frac{1}{2}\phi(x) + \frac{1}{2}\phi^2(x)\right] \cdot \left[\frac{1}{2}\phi(x) + \frac{1}{2}\phi^2(x)\right] \\ &= \frac{1}{4}\phi^2(x)[\phi^2(x) - 1] \quad \text{as } h \to 0, \end{aligned}$$

where $\phi(x) = \lambda a(u_0(x))$.

By the assumption $d^{(2k+1)}_{+,j+\frac{1}{2}} \cdot d^{(2k+1)}_{-,j+\frac{1}{2}} \geq 0$ for all integers j and $u_j \neq u_{j+1}$, we get

$$\phi^2(x)[\phi^2(x) - 1] \geq 0. \tag{2.8}$$

Since x is arbitrary, (2.8) is valid for all $x \in \mathbb{R}^1$. But in the process of our numerical computation, we always require that $|\phi(x)| \leq 1$. Combining inequality (2.8), we deduce that $\phi^2(x)(\phi^2(x) - 1) = 0$ for all $x \in \mathbb{R}^1$. So we can easily find a flux function f and an initial function $u_0(x)$ which satisfy the requirements mentioned at the very beginning of the proof, such that this equality is violated for all $x \in \mathbb{R}^1$. This implies that the scheme (2.3) satisfying the assumption of the theorem is not of 2nd-order accuracy and the proof of the theorem is thereby completed.

Using Corollary 2.3 and Theorem 2.4, we have

Theorem 2.5. *The* TSC *scheme is not* TVNI.

Corollary 2.6. The 3-point TVNI scheme in conservation form is only of first-order accuracy.

Theorem 2.7. *The 3-point s-order accurate ($s \geq 2$) linear difference scheme is linearly $l_p(1 \leq p \leq +\infty, p \neq 2)$ unstable.*

Proof. Consider the following scheme

$$u_j^{n+1} = c_{-1}u_{j-1}^n + c_0 u_j^n + c_1 u_{j+1}^n, \tag{2.9}$$

where c_r is constant, $r = -1, 0, 1$. If it is of s-order accuracy ($s \geq 2$) and linearly $l_p(1 \leq p \leq +\infty, p \neq 2)$ stable, then the amplification factor $e(\xi)$[9] has to be in the form $e(\xi) = \exp(i\lambda\xi + O(\xi^4))$. When we use the scheme (2.9) to approximate the solution of the equation $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$. Thus we have the following equality

$$c_{-1}\exp(-i\xi) + c_0 + c_1\exp(i\xi) = \exp(i\lambda\xi + O(\xi^4)).$$

Hence we get the inequalities

$$c_{-1} + c_0 + c_1 = 1, \ -c_{-1} + c_1 = \lambda, \ c_{-1} + c_1 = \lambda^2, \ -c_{-1} + c_1 = \lambda^3.$$

From these equalities we get $c_{-1} = c_0 = 0$, $c_1 = 1$ and $\lambda = 1$. Then $u_j^{n+1} = u_{j+1}^n$, $\lambda = 1$, which implies that the numerical solution is the exact solution of equation $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$. But this is a trivial case. The theorem is thereby proved.

Corollary 2.8. *The Lax-Wendroff scheme, the Richtmyer scheme and the MacCormack scheme are linearly $l_p(1 \le p \le +\infty, p \ne 2)$ unstable.*

There are some nonlinearly l_2 unstable TSC schemes. Majda and Osher[2] have given a rigorous proof of the nonlinear l_2 instabilities for the Lax-Wendroff difference scheme. Besides, numerical examples[3] illustrated that the MacCormack scheme is nonlinearly unstable. We shall give a rigorous proof of the nonlinear l_2 instabilities for the Richtmyer scheme.

For the special function $f(u) = \frac{1}{2}u^2$ the Richtmyer scheme has the form:

$$\begin{aligned} u_j^{n+1} &= u_j^n - \frac{\lambda_n}{8}\Delta_-[(u_{j+1}^n + u_j^n)^2] \\ &+\frac{\lambda_n^2}{8}\Delta_-[(u_{j+1}^n + u_j^n)\Delta_+(u_j^n)^2] - \frac{\lambda_n^2}{32}\Delta_-[\Delta_+(u_j^n)^2]^2 \\ &\equiv G(\lambda_n)u_j^n, \end{aligned} \tag{2.10a}$$

$$u_j^0 = g_i,\ j = 1, 2, \cdots, N;\ N \ne 1, \tag{2.10b}$$

where g_i and u_j are periodic so that $u_{j+N}^n = u_j^n$ and $g_{j+N} = g_j$. At the n-th time level, the CFL condition for linearized stability imposes the restriction that (most practical calculations adops this strategy)

$$\lambda_n : \max_j |u_j^n| \le 1,\ \lambda_n = \frac{k_n}{h}(\lambda_n \max_j |a(u_j^n)| = \mu \le 1)$$

where μ is a fixed constant.

For the sake of simplicity, we shall only define stability below for the trivial state $u_j \equiv 0$. Let $||u||_{h,p}$ denote the l_p norm, defined by $||u||_{h,p} = \left(\sum_{j=1}^{N} h \cdot |u_j|^p\right)^{\frac{1}{p}}$, $p \ge 1$.

Definition[2]**.** *The scheme (2.10) is stable in l_p sense at the state $u_j \equiv 0$ if given any $T > 0$, there exist δ_0, h_0 and $C(T)$ so that for $||g||_{h,p} \le \delta_0$ and $h < h_0$ we can find a sequence $\{\lambda_l(h)\}_{l=0}^r$ satisfying*

1) $$\lambda_l \max_j |u_j^l| = \mu \le 1,\ \mu = \text{const.}, \tag{2.11a}$$

2) $$\sum_{l=0}^{r} K_l = T_1,\ \lambda_l = \frac{K_l}{h},\ 0 < T_1 < T \tag{2.11b}$$

for which the following estimate is valid:

$$\left\|\prod_{l=0}^{r} G(\lambda_l)g\right\|_{h,p} \le C(T)||g||_{h,p}. \tag{2.12}$$

Theorem 2.9. *For the special case $f(u) = \frac{1}{2}u^2$, the Richtmyer scheme is nonlinearly unstable in l_p, $1 \le p \le +\infty$, in a neighborhood of $u_j \equiv 0$, in the sense defined above.*

Proof. Given any δ_0, h_0 and $C(T)$ (without loss of generality, we assume that $C(T) > 1$ and $T_1 = T$ for any $T > 0$), choose a fixed positive positive number a so that

$$a \cdot \left(\frac{2}{3}\right)^{\frac{1}{p}} < \delta_0$$

and consider $h < h_0$ so that if $Nh = 1,\ N = 3\widetilde{N}$. Define the initial state ae^0, where

$$\begin{aligned} e^0_{3j} &= 0,\ j = 0, 1, 2, \cdots, \widetilde{N}, \\ e^0_{3j+1} &= 1,\ j = 0, 1, 2, \cdots, \widetilde{N}, \\ e^0_{3j+2} &= -1,\ j = 0, 1, 2, \cdots, \widetilde{N}. \end{aligned} \tag{2.13}$$

Then $||ae^0||_{h,p} \leq \delta_0$. Consider any state of the form be^0 with $b > 0$. From (2.10) and (2.13), we have

$$G(\lambda_l)be^0 = g(b, \lambda_l)be^0, \tag{2.14a}$$

where

$$g(b, \lambda_l) = 1 + \frac{\lambda_l \cdot b}{8} - \frac{\lambda_l^2 b^2}{8} + \frac{\lambda_l^3 \cdot b^2}{32}. \tag{2.14b}$$

From (2.11a), we have

$$g(b, \lambda_l) \geq 1 + \frac{\lambda_l \cdot b}{32} \geq 1 + \frac{\lambda_l \tilde{b}}{32}, \quad \text{for } 0 \leq \tilde{b} \leq b. \tag{2.15}$$

Assuming the stability estimate in (2.12) is valid, we use (2.13), (2.14) and (2.15) to obtain for $h \leq h_0$

$$\prod_{l=0}^{r} \left(1 + \frac{\lambda_l a}{32}\right) \leq \frac{\left\|\prod_{l=0}^{r} G(\lambda_l)ae^0\right\|_{h,p}}{||ae^0||_{h,p}} \leq C(T).$$

By a simple inequality, we have

$$1 + \sum_{l=0}^{r} \frac{\lambda_l \cdot a}{32} \leq C(T),$$

that is

$$1 + \frac{a \cdot T}{32} \cdot \frac{1}{h} \leq C(T), \tag{2.16}$$

where a is a fixed positive constant determined by $a \cdot \left(\frac{2}{3}\right)^{\frac{1}{p}} \leq \delta_0$. So letting $h \to 0$ in (2.16), we arrive at a contradiction. Thus the proof is completed.

From the results mentioned above, we see that TSC schemes yield no good results for stability. It seems that l_2 stability in the sense of Definition 3.2 is the best result of stability for the TSC scheme. Besides, more seriously, many TSC schemes can produce nonphysical solutions[4,8,11], and some can produce a solution of the conservation law (2.1) which violates the entropy condition even when f is convex and the exact solution is a continuous one. Thus it is quite necessary to add an appropriate artificial viscosity to these TSC schemes.

§3. The Main Theorems

Throughout this section, we assume that the flux function $f(u)$ is a smooth convex function of u.

Now we introduce a class of modified scheme:

$$\begin{aligned} u_j^{n+1} &= u_j^n - \frac{\lambda_n}{2}\Delta_0 f(u_j^n) + \frac{\lambda_n^2}{2}\Delta_-\left[\frac{\Delta_+ f(u_j^n)}{\Delta_+ u_j^n}\cdot\Delta_+ f(u_j^n)\right] \\ &\quad +\lambda_n\Delta_-[h_j^n\Delta_+u_j^n] + \lambda_n\Delta_-[C\cdot\theta(s)\cdot|\Delta_+a(u_j^n)|\cdot\Delta_+u_j^n] \end{aligned}\tag{3.1a}$$

where

$$\Delta_0 u_j = u_{j+1} - u_{j-1},\ \Delta_- u_j = u_j - u_{j-1},\ \Delta_+ u_j = u_{j+1} - u_j,$$

$$a(u) = f'(u),$$

$$\frac{\Delta_+ f(u_j^n)}{\Delta_+ u_j^n} = a(u_j^n), \quad \text{if } \Delta_+ u_j^n = 0,$$

and C is an appropriate constant. $\theta(x)$ is a function defined by

$$\theta(x) = \begin{cases} 0, & |x| < 1, \\ 1, & |x| \ge 1, \end{cases}$$

and $s = \dfrac{|\Delta_+ u_j^n|}{h^\alpha}$, α is a constant satisfying $\dfrac{1}{3} < \alpha \le 1$, and $h = \Delta x$ the special step-length. h_j^n satisfy

1) $$h_j^n = h(u_j^n, u_{j+1}^n);\tag{3.1b}$$
2) $$\Delta_-[h_j^n\cdot\Delta_+u_j^n] = O(h^3) \quad \text{for smooth solution } u;\tag{3.1c}$$
3) $$|h_j^n| \le \beta|\Delta_+ a(u_j^n)| \quad \text{where } \beta \text{ is a fixed constant.}\tag{3.1d}$$

We define the fixed constant R to be the Courant-Friedrichs-Lewy number

$$\lambda_n \cdot \max_j |a(u_j^n)| = R \le 1.\tag{3.2}$$

It is easy to find that the scheme (3.1) has the following properties

1) 3-point scheme;
2) 2nd-order accurate for smooth solution;
3) in conservation form.

Definition 3.1. *(Entropy inequality). If the numerical solution of a finite-difference scheme $\{u_j^n\}$ converges boundedly a.e. to u, then u is a weak solution of*

(2.1) which satisfies the entropy inequality for any $\rho \in C_0^1,\ \rho \geq 0,$

$$-\iint \left[\frac{\partial \rho}{\partial t} \cdot U(u) + \frac{\partial \rho}{\partial x} \cdot F(u)\right] dx \cdot dt \leq 0, \tag{3.3a}$$

$$U(u) = \frac{1}{2}u^2,\ F(u) = \int_0^u s \cdot a(s)\, ds. \tag{3.3b}$$

Definition 3.2. *(l_2 stability). A finite-difference scheme is l_2 stable if $\{u_j^n\}$ satisfies*

$$\frac{1}{h}\|u^{n+1}\|_h^2 \leq \frac{1}{h}\|u^n\|^2, \tag{3.4a}$$

where

$$\|u^n\|_h \equiv \left[\sum_j |u_j^n|^2 \cdot h\right]^{\frac{1}{2}} \tag{3.4b}$$

for all nonnegative integers n.

Theorem 3.3. *Suppose $\{u_j^n\}$ is determined by the scheme (3.1). Then*

(i) there exists a constant $R_0 \leq 1$. When $R \leq R_0$ and $C = C(R)$ is a proper positive constant, the scheme (3.1) satisfies the entropy inequality in the sense of Definition 3.1;

(ii) $f\theta(s) \equiv 1$, then there exists a constant $R_0 \leq 1$. When $R \leq R_0$ and $C = C(R)$ is a proper positive constant, the scheme (3.1) satisfies l_2 stability in the sense of Definition 3.2.

Remark. To ensure the validity of Theorem 3.3, we can choose only a small R_0 at present to give a theoretical proof. Perhaps it is merely a theoretical problem — maybe by a better method, we can prove that a larger R_0 could be obtained. Numerical examples indicate that R_0 can be chosen larger that 0.4.

We shall postpone the proof of the theorem to the end of this section.

We now show that the Lax-Wendroff scheme, the Richtmyer scheme and the MacCormack scheme can be written in the following form

$$u_j^{n+1} = u_j^n - \frac{\lambda_n}{2}\Delta_0 f(u_j^n) + \frac{\lambda_n^2}{2}\Delta_-\left[\frac{\Delta_+ f(u_j^n)}{\Delta_+ u_j^n} \cdot \Delta_+ f(u_j^n)\right] + \lambda_n \Delta_-[h_j^n \cdot \Delta_+ u_j^n], \tag{3.5}$$

where h_j^n satisfy (3.1b), (3.1c) and (3.1d).

Throughout this paper, we denote

$$w_j^n \equiv \lambda_n \Delta_-[C\theta(s) \cdot |\Delta_+ a(u_j^n)| \Delta_+ u_j^n]. \tag{3.6}$$

Theorem 3.4. *The Richtmyer scheme can be written in the form of (3.5). So the conclusions of Theorem 3.3 are valid for the scheme if we add an artificial* *viscosity w_j^n to it. That is, the scheme*

$$\begin{cases} \tilde{u}_{j+\frac{1}{2}} = \frac{1}{2}(u_{j+1}^n + u_j^n) - \frac{\lambda_n}{2}[f(u_{j+1}^n) - f(u_j^n)], \\ u_j^{n+1} = u_j^n - \lambda_n[f(\tilde{u}_{j+\frac{1}{2}}) - f(\tilde{u}_{j-\frac{1}{2}})] + w_j^n, \end{cases} \tag{3.7}$$

has the conclusions of Theorem 3.3.

Proof. The Richtmyer scheme has the form

$$\begin{cases} \tilde{u}_{j+\frac{1}{2}} = \frac{1}{2}(u_{j+1}^n + u_j^n) - \frac{\lambda_n}{2}[f(u_{j+1}^n) - f(u_j^n)], \\ u_j^{n+1} = u_j^n - \lambda_n[f(\tilde{u}_{j+\frac{1}{2}}) - f(\tilde{u}_{j-\frac{1}{2}})]. \end{cases} \tag{3.8}$$

We shall drop the superscript n wherever convenient. Let

$$I_j \equiv [\min(u_j, u_{j+1}), \max(u_j, u_{j+1})]$$

Since $\lambda \max_j |a(u_j)| = R \le 1$, it is very easy to verify that $\tilde{u}_{j+\frac{1}{2}} \in I_j;\ u_j + \theta(\tilde{u}_{j+\frac{1}{2}} - u_j) \in I_j;\ u_j + \theta\Delta_+ u_j \in I_j$ for all $\theta \in [0,1]$. Now we give the definition of the divided differences as follows[7]:

$$f(x_0, x_1) \equiv \begin{cases} \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}, & \text{if } x_0 \ne x_1, \\ f'(x_0) & \text{if } x_0 = x_1; \end{cases}$$

$$f(x_0, x_1, x_2) = \begin{cases} \dfrac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}, & \text{if } x_0 \ne x_2, \\ \dfrac{f(x_0, x_1) - f'(x_0)}{x_1 - x_0} & \text{if } x_0 = x_2,\ x_0 \ne x_1, \\ \dfrac{1}{2} f''(x_0) & \text{if } x_0 = x_1 = x_2. \end{cases}$$

Let

$$\begin{aligned} \bar{h} &\equiv (\tilde{u}_{j+\frac{1}{2}} - u_{j+1}) f(\tilde{u}_{j+\frac{1}{2}}, u_j, u_{j+1}) = f(\tilde{u}_{j+\frac{1}{2}}, u_j) - f(u_j, u_{j+1}) \\ &= \int_0^1 [a(u_j + \theta(\tilde{u}_{j+\frac{1}{2}} - u_j)) - a(u_j + \theta(u_{j+1} - u_j))] d\theta. \end{aligned}$$

Since f is a convex function, we have

$$\begin{aligned} |\bar{h}_j| &\le \int_0^1 |a(u_j + \theta(u_{j+\frac{1}{2}} - u_j)) - a(u_j + \theta(u_{j+1} - u_j))| d\theta \\ &\le \int_0^1 |\Delta_+ a(u_j)| d\theta = |\Delta_+ a(u_j)|. \end{aligned}$$

By using Newton's interpolation formula[7], we have

$$\begin{aligned} f(\tilde{u}_{j+\frac{1}{2}}) &= f(u_{j+1}) + (\tilde{u}_{j+\frac{1}{2}} - u_{j+1}) \frac{\Delta_+ f(u_j)}{\Delta_+ u_j} + (\tilde{u}_{j+\frac{1}{2}} - u_j)\bar{h}_j, \\ f(\tilde{u}_{j-\frac{1}{2}}) &= f(u_{j-1}) + (\tilde{u}_{j-\frac{1}{2}} - u_{j-1}) \frac{\Delta_+ f(u_{j-1})}{\Delta_+ u_{j-1}} + (\tilde{u}_{j-\frac{1}{2}} - u_{j-1})\bar{h}_{j-1}, \end{aligned}$$

and from (3.8), we get

$$\begin{aligned}\tilde{u}_{j+\frac{1}{2}} - u_{j+1} &= -\frac{1}{2}\Delta_+ u_j - \frac{\lambda}{2}\Delta_+ f(u_j),\\ \tilde{u}_{j-\frac{1}{2}} - u_{j-1} &= \frac{1}{2}\Delta_+ u_{j-1} - \frac{\lambda}{2}\Delta_+ f(u_{j-1}),\end{aligned}$$

Then the scheme (3.7) can be written in the following form

$$\begin{aligned}u_j^{n+1} =& u_j^n - \frac{\lambda_n}{2}\Delta_0 f(u_j^n) + \frac{\lambda_n^2}{2}\Delta_-\left[\frac{\Delta_+ f(u_j^n)}{\Delta_+ u_j^n}\Delta_+ f(u_j^n)\right]\\ &+ \lambda_n \Delta_-\left[\frac{u_j^n - \tilde{u}_{j+\frac{1}{2}}}{\Delta^+ u_j^n}\bar{h}_j^n \Delta_+ u_j^n\right] + w_j^n.\end{aligned}$$

Let

$$h_j^n = \frac{u_j^n - \tilde{u}_{j+\frac{1}{2}}}{\Delta_+ u_j}\bar{h}_j^n,$$

Then

$$|h_j^n| \le |\bar{h}_j^n| \le |\Delta_+ a(u_j^n)|,$$

and the requirements (3.1b) and (3.1c) are satisfied obviously. So based on Theorem 3.3 this theorem is proved.

Theorem 3.5. *If f is a strickly convex function and $a(u)$ satisfies Lipschitz's condition or $a(u) \le 0$, the MacCormack scheme can be written in the form of (3.5). So the scheme also has the conclusions of Theorem 3.3 if we add an artificial viscosity w_j^n to it. That is, the scheme*

$$\begin{cases}\tilde{u}_j = u_j^n - \lambda_n \Delta_+ f(u_j^n),\\ u_j^{n+1} = \frac{1}{2}(u_j^n + \tilde{u}_j) - \frac{\lambda}{2}\Delta_+ f(\tilde{u}_{j-1}) + w_j^n\end{cases} \tag{3.9}$$

has the conclusions of Theorem 3.3. In particular, the inviscid Burgers equation can obtain the stable physical numerical solution by using the scheme (3.9).

Proof. We prove this theorem in the same manner as before. By the modified MacCormack scheme (3.9), we can obtain

$$\begin{aligned}u_j^{n+1} =& u_j^n - \frac{\lambda_n}{2}\Delta_0 f(u_j^n) + \frac{\lambda_n^2}{2}\Delta_-\left[\frac{\Delta_+ f(u_j^n)}{\Delta_+ u_j^n}\cdot \Delta_+ f(u_j^n)\right]\\ &+ \frac{\lambda_n^2}{2}\Delta_-[f(\tilde{u}_j, u_j^n, u_{j+1}^n)\cdot \Delta_+ f(u_j^n)\cdot(\tilde{u}_j - u_{j+1}^n)] + w_j^n.\end{aligned}$$

Let

$$h_j \equiv \frac{\lambda}{2}\frac{\Delta_+ f(u_j)}{\Delta_+ u_j}\cdot \bar{h}_j, \quad \bar{h}_j \equiv f(\tilde{u}_j, u_j, u_{j+1})\cdot(\tilde{u}_j - u_{j+1}).$$

Then

$$\bar{h}_j = f(\widetilde{u}_j, u_j) - f(u_j, u_{j+1}) = \int_0^1 [a(u_j + \theta\cdot(\widetilde{u}_j - u_j)) - a(u_j + \theta\cdot\Delta_+ u_j)]d\theta.$$

(i) If $a(u) \le 0$, it is very easy to verify that

$$\tilde{u}_j \in [\min(u_j, u_{j+1}), \max(u_j, u_{j+1})],$$
$$|\bar{h}_j| \le |\Delta_+ a(u_j)|; \quad |h_j| \le \frac{R}{2}|\Delta_+ a(u_j)|.$$

(ii) If $a(u)$ satisfies Lipschitz's condition, then there exists a constant $L > 0$ such that

$$|a(x) - a(y)| \le L \cdot |x - y|, \quad \text{for all } x,\ y \in \mathbb{R}^1.$$

Since f is a strictly convex function, there exists a constant $\delta > 0$ such that

$$|a'(x)| \ge \delta, \quad \text{for all } x \in \mathbb{R}^1.$$

We can easily obtain that

$$\begin{aligned} |\bar{h}_j| &\le |a(\tilde{u}_j) - a(u_j)| + |a(u_j) - a(u_{j+1})| \le L \cdot |\tilde{u}_j - u_j| + |\Delta_+ a(u_j)| \\ &\le L \cdot R|\Delta_+ u_j| + |\Delta_+ a(u_j)| \le \left(1 + \frac{L \cdot R}{\delta}\right) \cdot |\Delta_+ a(u_j)|. \end{aligned}$$

Then

$$|h_j| \le \frac{R}{2}\left(1 + \frac{L \cdot R}{\delta}\right)$$

and the requirements (3.1b) and (3.1c) are satisfied obviously in both cases. In particular, for the inviscid Burgers equation, $L = 1$, $\delta = 1$. Thus, based on Theorem 3.3, we complete the proof.

But, unfortunately, at present we are not able to give the best results for the modified MacCormack scheme (3.9). That is, in Theorem 3.5, we cannot remove the restriction on $a(u)$.

Theorem 3.6. *The Lax-Wendroff scheme can be written in the form of (3.5). So the scheme has the conclusions of Theorem 3.3 if we add an artificial viscosity w_j^n to it. That is, the scheme*

$$u_j^{n+1} = u_j^n - \frac{\lambda_n}{2}\Delta_0 f(u_j^n) + \frac{\lambda_n^2}{2}\Delta_-\left[a\left(\frac{u_j^n + u_{j+1}^n}{2}\right)\Delta_+ f(u_j^n)\right] + w_j^n \tag{3.10}$$

has the conclusions of Theorem 3.3.

Proof. We merely need to notice that

$$h_j = \frac{\lambda}{2}\left[a\left(\frac{u_j + u_{j+1}}{2}\right) - \frac{\Delta_+ f(u_j)}{\Delta_+ u_j}\right] \cdot \frac{\Delta_+ f(u_j)}{\Delta_+ u_j},$$
$$|h_j| \le \frac{R}{2}|\Delta_+ a(u_j)|.$$

Then the proof can be finished easily.

Remark. Our modified Lax-Wendroff scheme (3.10) is slightly different from the modified Lax-Wendroff scheme presented by Majda and Osher[1]

Finally, we prove Theorem 3.3.

Proof. We shall drop the superscript in u_j^n wherever convenient. Suppose $\{u_j^n\}$ is a sequence of numbers of period N (i.e. $u_j^n = u_{j+N}^n$). Let

$$\begin{aligned} A(u_j^n) &= u_j^n - \frac{\lambda_n}{2}\Delta_0 f(u_j^n) + \frac{\lambda_n^2}{2}\Delta_-\left[\frac{\Delta_+ f(u_j^n)}{\Delta_+ u_j^n}\cdot\Delta_+ f(u_j^n)\right] \\ &\quad + C\lambda_n\Delta_-[\,|\Delta_+ a(u_j^n)|\cdot\Delta_+ u_j^n]. \end{aligned}$$

We shall prove conclusion (ii) of Theorem 3.3. The proof of conlcusion (i) is similar and hence is omitted.

If $\theta(s)\equiv 1$, the scheme (3.1) can be written in the following form

$$u_j^{n+1} = A(u_j^n) + \lambda_n\cdot\Delta_-[h_j^n\cdot\Delta_+ u_j^n]. \tag{3.11}$$

Then

$$\begin{aligned} \frac{1}{2k_n}\|u^{n+1}\|_h^2 &= \frac{1}{2k_n}\|A(u^n)\|_h^2 + \sum_{j=1}^N A(u_j^n)\Delta_-[h_j^n\cdot\Delta_+ u_j^n] + \frac{1}{2}\lambda_n\cdot\sum_{j=1}^N[\Delta_-(h_j^n\cdot\Delta_+ u_j^n)]^2 \\ &\equiv [\mathrm{I}] + [\mathrm{II}] + [\mathrm{III}] \end{aligned}$$

where $k_n = \Delta t_n$, the time step-length. From [1], we have

$$\begin{aligned} [\mathrm{I}] \;\le\; & \frac{1}{2k_n}\|u^n\|_h^2 + \left[C^2\left(4R + R\cdot\frac{(1+R)^2}{b_{\mathrm{III}}}\right) + C\cdot(-1+2R^2)\right. \\ & \left. + \frac{1}{8} + \frac{1}{4}R^2\left(1 + 2R + \frac{2}{b_{\mathrm{II}}}\cdot R(1+R)^2\right)\right]\sum_{j=1}^n |\Delta_+ a(u_j^n)|\cdot(\Delta_+ u_j^n)^2 \\ & + \frac{\lambda_n}{2}\left[b_{\mathrm{III}} - \frac{1}{4}(1 - 3R^3 - 2b_{\mathrm{II}})\right]\sum_{j=1}^N(\Delta_+ f(u_j^n) - \Delta_+ f(u_{j-1}^n))^2 \end{aligned}$$

where b_{II}, b_{III} are arbitrary fixed positive constants. Now we estimate the terms [II] and [III].

$$\begin{aligned} & \sum_{j=1}^N\left[-\frac{\lambda}{2}\Delta_0 f(u_j) + \frac{\lambda^2}{2}\Delta_-\left[\frac{\Delta_+ f(u_j)}{\Delta_+ u_j}\cdot\Delta_+ f(u_j)\right]\right]\cdot\Delta_-(h_j\cdot\Delta_+ u_j) \\ &= \sum_{j=1}^N\frac{\lambda}{2}h_j\cdot\Delta_+ u_j\left[(\Delta_+ f(u_{j+1}) - \Delta_+ f(u_j))\left(1 - \lambda\frac{\Delta_+ f(u_{j+1})}{\Delta_+ u_{j+1}}\right)\right. \\ &\quad \left. + (\Delta_+ f(u_j) - \Delta_+ f(u_{j-1}))\cdot\left(1 + \lambda\frac{\Delta_+ f(u_{j-1})}{\Delta_+ u_{j-1}}\right)\right] \\ &\quad - \sum_{j=1}^N\frac{1}{2}\lambda^2 h_j\cdot\Delta_+ u_j\cdot\Delta_+ f(u_j)\cdot\left[\frac{\Delta_+ f(u_{j+1})}{\Delta_+ u_{j+1}} - 2\frac{\Delta_+ f(u_j)}{\Delta_+ u_j} + \frac{\Delta_+ f(u_{j-1})}{\Delta_+ u_{j-1}}\right] \\ &\le \frac{\lambda}{2}\sum_{j=1}^N b_{\mathrm{III}}(\Delta_+ f(u_j) - \Delta_+ f(u_{j-1}))^2 \\ &\quad + R\left[(1+R)^2\frac{\beta^2}{b_{\mathrm{III}}} + 2\beta R\right]\sum_{j=1}^N|\Delta_+ a(u_j)|\cdot(\Delta_+ u_j)^2, \end{aligned}$$

where β is a constant as in (3.1d).

$$
\begin{aligned}
&\sum_{j=1}^{N} C\cdot\lambda\Delta_-[\,|\Delta_+a(u_j)|\cdot\Delta_+u_j]\cdot\Delta_-(h_j\cdot\Delta_+u_j)\\
&=\sum_{j=1}^{N} C\cdot\lambda[2|\Delta_+a(u_j)|\cdot h_j\cdot(\Delta_+u_j)^2-|\Delta_+a(u_j)|\cdot h_{j-1}\cdot\Delta_+u_j\cdot\Delta_+u_{j-1}\\
&\quad -|\Delta_+a(u_{j-1})|\cdot h_j\cdot(\Delta_+u_j)\cdot(\Delta_+u_{j-1})]\\
&\le\sum_{j=1}^{N} C\lambda[2\beta|\Delta_+a(u_j)|^2\cdot(\Delta_+u_j)^2+|\Delta_+a(u_j)|^2\cdot(\Delta_+u_j)^2+h_j^2\cdot(\Delta_+u_j)^2]\\
&\le\sum_{j=1}^{N} 2\cdot C\cdot R(1+\beta)^2\cdot|\Delta_+a(u_j)|\cdot(\Delta_+u_j)^2.
\end{aligned}
$$

Then it is easy to verify that

$$
\begin{aligned}
[\mathrm{II}] \;&\le\; \beta+R(1+R)^2\frac{\beta^2}{b_{\mathrm{III}}}+2\beta R^2+2C\cdot R(1+\beta)^2\cdot\sum_{j=1}^{N}|\Delta_+a(u_j)|(\Delta_+u_j)^2\\
&\quad+\frac{\lambda}{2}\sum_{j=1}^{N} b_{\mathrm{III}}\cdot(\Delta_+f(u_j)-\Delta_+f(u_{j-1}))^2,\\
[\mathrm{III}] \;&\le\; \sum_{j=1}^{N}\lambda[h_j^2\cdot(\Delta_+u_j)^2-h_j\cdot h_{j-1}\Delta_+u_j\cdot\Delta_+u_{j-1}]\le\sum_{j=1}^{N}2\lambda\cdot h_j^2(\Delta_+u_j)^2\\
&\le\; 4\beta^2R\sum_{j=1}^{N}|\Delta_+a(u_j)|\cdot(\Delta_-u_j)^2.
\end{aligned}
$$

Then, we obtain the following estimate

$$
\begin{aligned}
&\frac{1}{2k_n}\|u^{n+1}\|_h^2-\frac{1}{2k_n}\|u^n\|_h^2\\
&\le\Bigg[C^2\left(4R+R\frac{(1+R)^2}{b_{\mathrm{III}}}\right)+C\cdot(-1+2R^2+2R(1+\beta)^2)+\frac{1}{8}\\
&\quad+\frac{1}{4}R^2\left(1+2R+\frac{2}{b_{\mathrm{II}}}R(1+R)^2\right)+\beta+R(1+R)^2\frac{\beta^2}{b_{\mathrm{III}}}\\
&\quad+2R^2\beta+4\beta^2R\Bigg]\sum_{j=1}^{N}|\Delta_+a(u_j)|\cdot(\Delta_+u_j)^2\\
&\quad+\frac{\lambda}{2}\left[2b_{\mathrm{III}}-\frac{1}{4}(1-3R^2-2b_{\mathrm{II}})\right]\sum_{j=1}^{N}(\Delta_+f(u_j)-\Delta_+f(u_{j-1}))^2\\
&\equiv[A(R)\cdot C^2+B(R)\cdot C+D(R)]\sum_{j=1}^{N}|\Delta_+a(u_j)|\cdot(\Delta_+u_j)^2\\
&\quad+E(R)\cdot\sum_{j=1}^{N}(\Delta_+f(u_j)-\Delta_+f(u_{j-1}))^2.
\end{aligned}
$$

Thus we only need the following inequalities for the validity of Theorem 3.3:

$$\text{(i)} \qquad E(R) \le 0, \tag{3.12a}$$

$$\text{(ii)} \qquad A(R)C^2 + B(R)C + D(R) \le 0. \tag{3.12b}$$

We require that $R < \dfrac{1}{\sqrt{3}}$. Then we can choose proper constants $b_{\mathrm{II}} > 0$, $b_{\mathrm{III}} > 0$ such that $E(R) \equiv 0$. In addition, we have the following facts:

$$A(R) > 0 \quad \text{for all} \quad 0 < R < \frac{1}{\sqrt{3}}. \tag{3.13a}$$

$$\lim_{R \to 0+} A(R) = 0, \tag{3.13b}$$

$$\lim_{R \to 0+} B(R) = -1, \tag{3.13c}$$

$$\lim_{R \to 0+} D(R) = \frac{1}{8} + \beta. \tag{3.13d}$$

From (3.13), there exists a constant $0 < R_0 < \dfrac{1}{\sqrt{3}}$, such that if $R \le R_0$, then

$$D(R) - \frac{B^2(R)}{4 \cdot A(R)} \le 0, \qquad -\frac{B(R)}{2 \cdot A(R)} > 0. \tag{3.14}$$

so we choose $C = C(R) = -\dfrac{B(R)}{2 \cdot A(R)} > 0$. Then the requirement (3.12b) is satisfied. The theorem is proved.

Since (3.12) is too complicated to be analysed more clearly, it is very difficult to give a definite $R_0 = R_0(\beta)$. But for a given scheme, such as schemes (3.7), (3.9) and (3.10), we can give a definite R_0. We shall give a simple example in the next section.

§4. Example

We shall study the modified Richtmyer scheme (3.7) for the inviscid Burgers equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial (u^2/2)}{\partial x} = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{4.1}$$

We prove the following theorem in a similar manner to Section 3. The details of the proof are omitted.

Theorem 4.1. *Consider the scheme (3.7) and $f(u) = \frac{1}{2}u^2$. The scheme is stable in the sense of Definition 3.2 provided that R_0 and C satisfy the restrictions*

$$(i) \qquad b^2 - 4ad \ge 0, \tag{4.2a}$$

$$(ii) \qquad \frac{-b - \sqrt{b^2 - 4ad}}{2a} \le C \le \frac{-b + \sqrt{b^2 - 4ad}}{2a}, \tag{4.2b}$$

where

$$a = 8R_0, \tag{4.2c}$$

$$b = -2 + 3R_0 + 4R_0^2 + 2R_0^3, \tag{4.2d}$$

$$d = \frac{1}{12} + \frac{R_0}{4(1-R_0^2)} + 2R_0^2 + \frac{1}{2}R_0^3 + \frac{1}{2}R_0^4 + \frac{1}{8}R_0^5. \tag{4.2e}$$

(4.2f)

In particular, such a number C in (ii) above exists provided that $R_0 \le 0.20$.

For the inviscid Burgers equation (4.1), we choose the initial condition as

$$u_0(x) = \begin{cases} -1, & x < 0, \\ 1, & x \ge 0. \end{cases} \tag{4.3}$$

The Lax-Wendroff scheme and the MacCormack scheme produce the steady expansion jump function $u(x,t) = u_0(x)$ for all t, which is a nonphysical solution of problem (4.1) and (4.3). Fig. 4.1 shows the numerical solution obtained by applying the difference equation (3.9) to this initial value problem with $R = 0.4$, $C(R) = 0.2$ and $R = 0.2$, $C(R) = 0.4$ respectively. The schemes (3.7), (3.9) and (3.10) produce almost the same numerical solution for this problem, so we only present the result of the modified MacCormack scheme in Fig. 4.1. We find that the numerical solution is a stable and physical one. In addition, the Courant number R can be chosen larger that R_0 mentioned in Theorem 4.1.

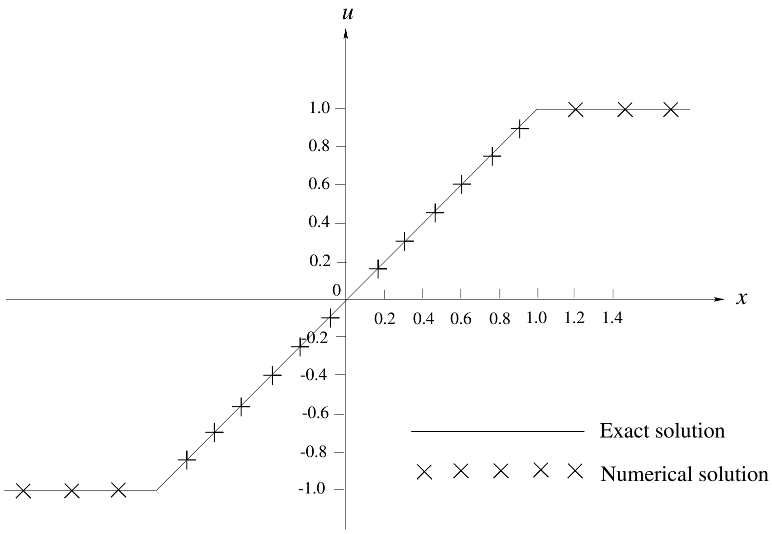

Fig. 4.1. Comparison of the numerical and exact physical solutions of the inviscid Burgers equation at $t = 50h/u_0$, $u_0 = 1$, with $R = 0.4$ and 0.2 respectively

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