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An Euler-type method for two-dimensional Volterra integral equations of the first kind

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Two-dimensional first-kind Volterra integral equations (VIEs) are studied. The first-kind equations are reduced to second kind, and by obtaining an appropriate integral inequality, existence and uniqueness are demonstrated. The equivalent discrete integral inequality then permits convergence of discretization methods; and this is illustrated for the Euler method. Finally, a class of nonlinear telegraph equations is shown to be equivalent to (two-dimensional) Volterra integral equations, thereby providing existence and uniqueness results for this class of equations. Furthermore, the telegraph equation may be solved by the numerical method for two-dimensional VIEs, and a simple numerical example is given.

1. Introduction

Consider the first-kind Volterra integral equation

$$\int_0^t \int_0^s k(t,s,u,v)x(u,v)\,\mathrm{d}u\,\mathrm{d}v = f(t,s), \qquad (t,s) \in \Omega := [0,T] \times [0,S]. \tag{1.1}$$

To be consistent, we require

$$f(t,0) \equiv 0, \qquad f(0,s) \equiv 0, \qquad \text{for } (t,s) \in \Omega. \tag{1.2}$$

Assume that k and f are smooth and that

$$k(t,s,t,s) \neq 0 \qquad \text{for all } (t,s) \in \Omega. \tag{1.3}$$

By obtaining an appropriate Gronwall inequality we shall study the existence and uniqueness of (1.1).

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Differentiating (1.1) with respect to t and s yields the following two-dimensional second-kind Volterra integral equation

$$x(t,s) = \int_0^t k_1(t,s,u)x(u,s)\,\mathrm{d}s + \int_0^s k_2(t,s,v)x(t,v)\,\mathrm{d}v + \int_0^t \int_0^s k_3(t,s,u,v)x(u,v)\,\mathrm{d}u\,\mathrm{d}v + F(t,s) \tag{1.4}$$

where

$$k_1(t,s,u) := -\frac{\partial k}{\partial t}(t,s,u,s)/k(t,s,t,s),$$
$$k_2(t,s,v) := -\frac{\partial k}{\partial s}(t,s,t,v)/k(t,s,t,s),$$
$$k_3(t,s,u,v) := -\frac{\partial^2 k}{\partial t \partial s}(t,s,u,v)/k(t,s,t,s),$$
$$F(t,s) := \frac{\partial^2 f}{\partial t \partial s}(t,s,u,v)/k(t,s,t,s).$$

Since k and f are smooth and k satisfies (1.3), there exist positive constants K_1, K_2, K_3 and F_0 such that

$$|k_1(t,s,u)| \leqslant K_1, \qquad |k_2(t,s,v)| \leqslant K_2, \qquad |k_3(t,s,u,v)| \leqslant K_3, \tag{1.5}$$
$$|F(t,s)| \leqslant F_0, \qquad \text{for } (t,s) \in \Omega, (u,v) \in \Omega.$$

It is readily shown that if (1.2) and (1.3) hold, then (1.1) and (1.4) are equivalent.

The numerical solution of equations of the type of (1.1) seems to have first been considered by Bel'tyukov and Kuznechikhina (1973, 1976). In the first paper a class of cubature formulas was derived and applied to (1.1) leading to a nonlinear algebraic system; in the latter an explicit Runge–Kutta method was considered but a convergence proof was not provided. More recently, Ries (1988) has investigated the application of spline collocation methods and studied their convergence in the case $k \equiv 1$.

A special case of (1.4) occurs when $k_1 = k_2 \equiv 0$. In this case, (1.4) reduces to

$$x(t,s) = \int_0^t \int_0^s k_3(t,s,u,v)x(u,v)\,\mathrm{d}u\,\mathrm{d}v + F(t,s). \tag{1.6}$$

The above second-kind equation may arise from certain hyperbolic differential equations (see Dobner 1987 for an equivalent formulation of the Darboux problem). In Singh (1976) a bivariate cubic spline approximation was obtained for the solution of (1.6) and in Brunner & Kauthen (1989) an exhaustive analysis of polynomial spline collocation and iterated methods was given.

We note that there are very few studies dealing with the derivation and numerical analysis of two-dimensional VIEs; this is in contrast with two-dimensional Fredholm integral equations (FIEs). Examples of physical problems leading to multidimensional weakly singular FIEs include some radiation transfer problems as well as certain interior-exterior boundary value problems (see Vainikko, 1993). The book by Mikhlin (1965)

contains the author's results on the theory of multidimensional singular integrals and their application to potential problems. A detailed description of recent results concerning smoothness properties of the solution of these equations and their numerical treatment by piecewise polynomial collocation methods can be found in Vainikko (1993) (see also Vainikko (1991a, b, 1992a, b), Graham (1981), Graham & Schneider (1985), Kangro (1990a, b)). Second-kind integral equations of mixed Volterra–Fredholm type (VFIEs) arise in connection with parabolic boundary value problems and in certain epidemic models (Diekmann, 1978; Pachpatte, 1986). In Haçia (1979) a Nyström type method is given for the numerical treatment of VFIEs while in Kauthen (1989) continuous time collocation methods were considered.

In the present work we are interested in solving (1.1) and (1.4) by a Euler-type method. The paper is organized as follows. In Section 2 we prove existence of a unique continuous solution to (1.1) and (1.4) by applying the Picard fixed point iterative procedure to (1.4). Section 3 deals with uniqueness. In Section 4 a new Gronwall inequality is obtained and in Section 5 a Euler-type discretization method is considered for the solution of (1.1). By using a discrete Gronwall inequality, first-order convergence is proved. In principle the arguments employed here can be extended to analyse convergence of higher-order multistep methods and a numerical example illustrating the application of the trapezoidal method to (1.1) is given. Finally in Section 6 a generalized telegraph equation is shown to be equivalent to a two-dimensional second-kind VIE of the general form (1.4) thus leading to an integral method for its solution. A numerical example is also included.

2. Existence

Existence and uniqueness results for equation (1.4) were obtained by Gronwall (1915) and Goursat (1942), who used techniques based on Picard's method of successive approximations (compare also Volterra 1896). We point out that in the case of equation (1.6), the standard resolvent kernel can be found in terms of a Neumann series (see Beesack 1984, Kauthen 1986 and Brunner & Kauthen 1989). Goursat (1942) provided an explicit expression for the solution of (1.4): a three-term integral formula involving three functions which plays the role of the resolvent kernel. It can be considered as the analogue of the solution for the one-dimensional case (that is, (1.4) with $k_1 = k_2 \equiv 0$). A different approach based on the method of weighted norms introduced by Bielecki (1956) was used by Kwapisz (1992) in the framework of L^p spaces. For a survey on the results proved by Bielecki's method for integral and integro-differential equations, see Corduneanu (1984). Among the references containing results for general multidimensional integral equations, we mention Walter (1967, 1970), Suryanarayana (1972), Beesack (1984, 1985), Kwapisz (1984, 1991), Kwapisz & Turo (1974a, b, 1975).

In this work, we shall provide an alternative way to prove the existence of a solution to (1.4), based on the Picard iterative procedure. We construct the following sequence:

$$x_{n+1}(t,s) = \int_0^t k_1(t,s,u)x_n(u,s)\,\mathrm{d}u + \int_0^s k_2(t,s,v)x_n(t,v)\,\mathrm{d}v + \int_0^t \int_0^s k_3(t,s,u,v)x_n(u,v)\,\mathrm{d}u\,\mathrm{d}v + F(t,s), \qquad n = 0, 1, 2, \ldots, \tag{2.1}$$

$$x_0(t,s) = 0. \tag{2.2}$$

We choose a positive constant β:

$$\beta = K_1 + K_2 + \sqrt{(K_1 + K_2)^2 + 2K_3}. \tag{2.3}$$

It is easy to verify that

$$\frac{K_1}{\beta} + \frac{K_2}{\beta} + \frac{K_3}{\beta^2} = \frac{1}{2}. \tag{2.4}$$

Next, we shall employ mathematical induction to show that

$$|x_{n+1}(t,s) - x_n(t,s)| \leqslant \left(\frac{1}{2}\right)^n F_0 e^{\beta(t+s)}, \qquad (t,s) \in \Omega, \tag{2.5}$$

for $n = 0, 1, 2, \ldots$. It is easy to verify that $x_1(t,s) = F(t,s)$. Thus the above inequality is true for $n = 0$. Assume that (2.5) is true for $n = N$. It follows from this assumption, (1.5) and (2.1) that

$$\begin{aligned}
|x_{N+2}(t,s) - x_{N+1}(t,s)| &\leqslant K_1 \int_0^t |x_{N+1}(u,s) - x_N(u,s)|\,du \qquad (2.6)\\
&+ K_2 \int_0^s |x_{N+1}(t,v) - x_N(t,v)|\,dv + K_3 \int_0^t \int_0^s |x_{N+1}(u,v) - x_N(u,v)|\,du\,dv\\
&\leqslant K_1 \int_0^t \left(\frac{1}{2}\right)^N F_0 e^{\beta(u+s)}\,du + K_2 \int_0^s \left(\frac{1}{2}\right)^N F_0 e^{\beta(t+v)}\,dv\\
&+ K_3 \int_0^t \int_0^s \left(\frac{1}{2}\right)^N F_0 e^{\beta(u+v)}\,du\,dv\\
&= F_0 \left(\frac{1}{2}\right)^N \left\{ \frac{K_1}{\beta} e^{\beta s}(e^{\beta t} - 1) + \frac{K_2}{\beta} e^{\beta t}(e^{\beta s} - 1) + \frac{K_3}{\beta^2}(e^{\beta t} - 1)(e^{\beta s} - 1) \right\}\\
&\leqslant F_0 \left(\frac{1}{2}\right)^N \left\{ \frac{K_1}{\beta} + \frac{K_2}{\beta} + \frac{K_3}{\beta^2} \right\} e^{\beta(t+s)}\\
&= F_0 \left(\frac{1}{2}\right)^{N+1} e^{\beta(t+s)}
\end{aligned}$$

where in the last step we have used (2.4). This implies that (2.5) holds for $n = N + 1$. Therefore, (2.5) is true for all $n \geqslant 0$.

From (2.5) we know that (2.1) is a Cauchy sequence in $C(\Omega)$. The standard theory ensures the existence of a continuous solution $x(t,s)$ such that

$$x(t,s) = \lim_{n \to \infty} x_n(t,s), \qquad (t,s) \in \Omega. \tag{2.7}$$

Letting $n \to \infty$ in (2.1) we obtain the existence of a solution to (1.4), and hence (1.1).

3. Uniqueness

Let us assume that there is another solution $x^* \in C(\Omega)$ satisfying (1.4), i.e.

$$\begin{aligned}
x^*(t,s) = &\int_0^t k_1(t,s,u) x^*(u,s)\,du + \int_0^s k_2(t,s,v) x^*(t,v)\,dv \qquad (3.1)\\
&+ \int_0^t \int_0^s k_3(t,s,u,v) x^*(u,v)\,du\,dv + F(t,s).
\end{aligned}$$

Subtracting (3.1) from (2.1) gives

$$\begin{aligned} x_{n+1}(t,s) - x^*(t,s) = &\int_0^t k_1(t,s,u)(x_n(u,s) - x^*(u,s))\,\mathrm{d}u \\ &+ \int_0^s k_2(t,s,v)(x_n(t,v) - x^*(t,v))\,\mathrm{d}v \\ &+ \int_0^t \int_0^s k_3(t,s,u,v)(x_n(u,v) - x^*(u,v))\,\mathrm{d}u\,\mathrm{d}v. \end{aligned} \tag{3.2}$$

Using a similar argument to that employed in Section 2, we can obtain

$$|x_n(t,s) - x^*(t,s)| \leqslant \left(\frac{1}{2}\right)^n \max_{\Omega} |x^*(t,s)| \mathrm{e}^{\beta(t+s)}, \qquad (t,s) \in \Omega,\ n = 0, 1, 2, \ldots \tag{3.3}$$

Letting $n \to \infty$ yields $x(t,s) \equiv x^*(t,s)$. This demonstrates that (1.4), and thereby (1.1), has a unique solution.

4. Gronwall inequality

Gronwall-type inequalities often play an important rôle in obtaining bounds for numerical methods of Volterra-type equations. An inequality related to equation (1.4), which may be used in error analysis, is the following:

$$\begin{aligned} y(t,s) \leqslant K_1 &\int_0^t y(u,s)\,\mathrm{d}u + K_2 \int_0^s y(t,v)\,\mathrm{d}v \\ &+ K_3 \int_0^t \int_0^s y(u,v)\,\mathrm{d}u\,\mathrm{d}v + F_0, \qquad (t,s) \in \Omega, \end{aligned} \tag{4.1}$$

where $y(t,s)$ is bounded, integrable, and non-negative on Ω.

Several authors have generalized Gronwall-type inequalities to the case of functions of two or more variables (see e.g. Haçia, 1997; Mitrinovic *et al.*, 1991 and the references therein). In some cases these have been used to analyse numerical methods for particular Volterra integral equations in higher dimensions (see e.g. Brunner & Kauthen 1989).

The function y in (4.1) satisfies the following equation:

$$\begin{aligned} y(t,s) = K_1 &\int_0^t y(u,s)\,\mathrm{d}u + K_2 \int_0^s y(t,v)\,\mathrm{d}v \\ &+ K_3 \int_0^t \int_0^s y(u,v)\,\mathrm{d}u\,\mathrm{d}v + F_0 + R(t,s), \end{aligned} \tag{4.2}$$

for $(t,s) \in \Omega$, where the function $R(t,s)$ is defined by

$$\begin{aligned} R(t,s) := y(t,s) - K_1 &\int_0^t y(u,s)\,\mathrm{d}u - K_2 \int_0^s y(t,v)\,\mathrm{d}v \\ &- K_3 \int_0^t \int_0^s y(u,v)\,\mathrm{d}u\,\mathrm{d}v - F_0. \end{aligned} \tag{4.3}$$

It follows from (4.1) and (4.3) that

$$R(t,s) \leqslant 0, \qquad |R(t,s)| \leqslant \text{constant}, \qquad \text{for } (t,s) \in \Omega, \tag{4.4}$$

where in order to obtain the second inequality we have used the assumption that y is bounded.

Again, we consider the iterated sequence:

$$y_{n+1}(t,s) = K_1 \int_0^t y_n(u,s)\,\mathrm{d}u + K_2 \int_0^s y_n(t,v)\,\mathrm{d}v \tag{4.5}$$
$$+K_3 \int_0^t \int_0^s y_n(u,v)\,\mathrm{d}u\,\mathrm{d}v + F_0 + R(t,s), \qquad n = 0,1,2,\ldots,$$
$$y_0(t,s) = 0. \tag{4.6}$$

Since R is a bounded function, using a similar argument to that employed in Section 2, we can show that $\lim_{n\to\infty} y_n(t,s)$ is the solution of (4.2). Next, we shall prove the following inequality:

$$y_n(t,s) \leqslant F_0 \mathrm{e}^{\gamma(t+s)}, \qquad \text{for } (t,s) \in \Omega,\ n = 0,1,2,\ldots, \tag{4.7}$$

where γ is defined by

$$\gamma = \tfrac{1}{2}\big(K_1 + K_2 + \sqrt{(K_1+K_2)^2 + 4K_3}\big). \tag{4.8}$$

It is easy to verify that

$$\frac{K_1}{\gamma} + \frac{K_2}{\gamma} + \frac{K_3}{\gamma^2} = 1. \tag{4.9}$$

Again we use mathematical induction. It is clear that (4.7) is true for $n = 0$. Assume that (4.7) holds for $n = N$. It follows from this assumption, (4.4) and (4.5) that

$$\begin{aligned} y_{N+1}(t,s) &\leqslant K_1 \int_0^t y_N(u,s)\,\mathrm{d}u + K_2 \int_0^s y_N(t,v)\,\mathrm{d}v \\ &\quad + K_3 \int_0^t \int_0^s y_N(u,v)\,\mathrm{d}u\,\mathrm{d}v + F_0 \\ &\leqslant K_1 F_0 \int_0^t \mathrm{e}^{\gamma(u+s)}\,\mathrm{d}u + K_2 F_0 \int_0^s \mathrm{e}^{\gamma(t+v)}\,\mathrm{d}v \\ &\quad + K_3 F_0 \int_0^t \int_0^s \mathrm{e}^{\gamma(u+v)}\,\mathrm{d}u\,\mathrm{d}v + F_0 \\ &= F_0\left(\frac{K_1}{\gamma} + \frac{K_2}{\gamma} + \frac{K_3}{\gamma^2}\right)\mathrm{e}^{\gamma(t+s)} \\ &\quad - F_0\left(\frac{K_1}{\gamma}\mathrm{e}^{\gamma s} + \frac{K_2}{\gamma}\mathrm{e}^{\gamma t} + \frac{K_3}{\gamma^2}\mathrm{e}^{\gamma t} + \frac{K_3}{\gamma^2}\mathrm{e}^{\gamma s}\right) + F_0\frac{K_3}{\gamma^2} + F_0 \\ &\leqslant F_0\mathrm{e}^{\gamma(t+s)} - F_0\left(\frac{K_1}{\gamma} + \frac{K_2}{\gamma} + \frac{2K_3}{\gamma^2}\right) + F_0\frac{K_3}{\gamma^2} + F_0 \\ &= F_0\mathrm{e}^{\gamma(t+s)}, \end{aligned} \tag{4.10}$$

where, in the last two steps, we have employed (4.9). This completes the proof of inequality (4.7). Letting $n \to \infty$ in (4.7) gives the following Gronwall-type inequality

$$y(t, s) \leqslant F_0 e^{\gamma(t+s)}, \tag{4.11}$$

where γ is given by (4.8).

4.1 *Special cases*

Case 1: $K_1 = 0, K_2 = 0$.
In this case, using a result from Beckenbach & Bellman (1965) (see also Bondge *et al.*, 1980; Fink, 1981; Beesack 1984, 1985; Mitrinovic *et al.*, 1991, p 401), gives

$$y(t, s) \leqslant F_0 \exp(K_3 ts). \tag{4.12}$$

However, inequality (4.11) suggests that

$$y(t, s) \leqslant F_0 \exp\left(\sqrt{K_3}(t + s)\right), \tag{4.13}$$

which is sharper than (4.12), at least when both t and s are large.

Case 2: $K_3 = 0$.
This case, again using a result of Beckenbach & Bellman (1965), gives

$$y(t, s) \leqslant F_0 \exp(K_1 t + K_2 s + K_1 K_2 ts). \tag{4.14}$$

Inequality (4.11) suggests that

$$y(t, s) \leqslant F_0 \exp((K_1 + K_2)(t + s)), \tag{4.15}$$

which is also sharper than (4.14) when t and s are large.

5. Numerical analysis

In this section we consider a Euler-type method and its convergence to the two-dimensional Volterra integral equation (1.1). First we develop a discrete version of the Gronwall inequality defined by (4.1) and (4.11), then we reduce the first-kind discrete equation (equation (5.1)) to a discrete second-kind equation, and then apply consistency. The argument can in principle be generalized, along the lines of Holyhead *et al.* (1975) (see also Holyhead & McKee, 1976), to demonstrate the convergence of linear multistep methods generally.

First of all consider the net of grid points

$$\Omega_{hk} := \{(t_i, s_j) : t_i = t_{i-1} + h, i = 1, 2, \ldots, N, t_0 = 0, t_N = T; \\ s_j = s_{j-1} + \kappa, j = 1, 2, \ldots, M, s_0 = 0, s_m = S\}$$

upon which we define the grid functions

$$y_{ij}, \qquad i = 0, 1, \ldots, N, j = 0, 1, \ldots, M.$$

We shall assume that the grid function y_{ij} satisfies the finite difference scheme

$$h\kappa \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} k(t_n, s_m, t_i, s_j) y_{ij} = f_{nm}, \qquad n = 1, 2, \ldots, N, m = 1, 2, \ldots, M, \quad (5.1)$$

where $f_{nm} := f(t_n, s_m)$.

5.1 *The discrete Gronwall inequality*

We shall prove the following lemma:

LEMMA 5.1 Consider the grid function $x_{nm} \geqslant 0$ $(n = 0, 1, \ldots, N; m = 0, 1, \ldots, M)$ satisfying

$$x_{nm} \leqslant hK_1 \sum_{i=0}^{n-1} x_{im} + \kappa K_2 \sum_{j=0}^{m-1} x_{nj} + h\kappa K_3 \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} x_{ij} + \Delta, \quad (5.2)$$

$$n = 0, 1, \ldots, N, m = 0, 1, \ldots, M.$$

Let

$$\gamma = \tfrac{1}{2}\left(K_1 + K_2 + \sqrt{(K_1 + K_2)^2 + 4K_3} \right)$$

and let K_i $(i = 1, 2, 3)$ and Δ be independent of h and κ and strictly positive. Then

$$x_{nm} \leqslant \Delta \exp(\gamma(Nh + M\kappa)). \quad (5.3)$$

Proof. From Section 4 we observe that if $x(t, s)$ satisfies the inequality

$$x(t, s) \leqslant K_1 \int_0^t x(u, s)\, \mathrm{d}u + K_2 \int_0^s x(t, v)\, \mathrm{d}v + K_3 \int_0^t \int_0^s x(u, v)\, \mathrm{d}u\, \mathrm{d}v + \Delta \quad (5.4)$$

for $(t, s) \in \Omega$, then

$$x(t, s) \leqslant \Delta \exp(\gamma(t + s)) \quad (5.5)$$

provided $K_i > 0$ $(i = 1, 2, 3)$ and $\Delta > 0$ and where

$$\gamma = \tfrac{1}{2}\left(K_1 + K_2 + \sqrt{(K_1 + K_2)^2 + 4K_3} \right).$$

Rewrite (5.2) as

$$x_{nm} \leqslant K_1 \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} x_{im}\, \mathrm{d}u + K_2 \sum_{j=0}^{m-1} \int_{s_j}^{s_{j+1}} x_{nj}\, \mathrm{d}v + K_3 \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \int_{t_i}^{t_{i+1}} \int_{s_j}^{s_{j+1}} x_{ij}\, \mathrm{d}u\, \mathrm{d}v + \Delta. \quad (5.6)$$

Define the piecewise constant function

$$x(t,s) := x_{ij} \qquad \text{if } (t,s) \in [t_i, t_{i+1}) \times [s_j, s_{j+1}),$$

and let $t := t_n$, $s := s_m$.

We can express (5.6) in the form

$$x(t,s) \leqslant K_1 \int_0^t x(u,s)\,\mathrm{d}u + K_2 \int_0^s x(t,v)\,\mathrm{d}v + K_3 \int_0^t \int_0^s x(u,v)\,\mathrm{d}u\,\mathrm{d}v + \Delta.$$

From (4.11) we can deduce that

$$x(t,s) \leqslant \Delta \exp(\gamma(t+s))$$

where

$$\gamma = \tfrac{1}{2}\left(K_1 + K_2 + \sqrt{(K_1+K_2)^2 + 4K_3}\right).$$

But we defined $t := t_n$ and $s := s_m$. Thus

$$x(t_n, s_m) \leqslant \Delta \exp(\gamma(t_n + s_m))$$

or

$$x_{nm} \leqslant \Delta \exp(\gamma(nh + m\kappa))$$

$$x_{nm} \leqslant \Delta \exp(\gamma(Nh + M\kappa))$$

□

5.2 *Convergence*

Note: by Euler's method we shall mean equation (5.1). Replacing y_{ij} by $y(t_i, s_j)$ (the solution of (1.1) at the grid points $t = t_i$ and $s = s_j$) it is not difficult to show that

$$h\kappa \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} k(t_n, s_m, t_i, s_j) y(t_i, s_j) - f(t_n, s_m) = \sigma_{nm} \tag{5.7}$$

where $\sigma_{nm} = O(h+\kappa)$, provided $\dfrac{\partial y}{\partial u}$, $\dfrac{\partial y}{\partial v}$ and $\dfrac{\partial k}{\partial u}$, $\dfrac{\partial k}{\partial v}$ are continuous. Subtract (5.1) from (5.7) to obtain

$$h\kappa \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} k(t_n, s_m, t_i, s_j)(y(t_i, s_j) - y_{ij}) = \sigma_{nm}.$$

Performing a discrete equivalent of differentiation with respect to t and s (that is performing the forward differences $\{(\sigma_{n+1,m+1} - \sigma_{n,m+1})/h - (\sigma_{n+1,m} - \sigma_{nm})/h\}/\kappa$) results in

$$
\begin{aligned}
&k(t_{n+1},s_{m+1}, t_n, s_m)(y(t_n, s_m) - y_{nm}) \\
&\quad + h\sum_{i=0}^{n-1}\left(\frac{k(t_{n+1}, s_{m+1}, t_i, s_m) - k(t_n, s_{m+1}, t_i, s_m)}{h}\right)(y(t_i, s_m) - y_{im}) \\
&\quad + \kappa\sum_{j=0}^{m-1}\left(\frac{k(t_{n+1}, s_{m+1}, t_n, s_j) - k(t_{n+1}, s_m, t_n, s_j)}{\kappa}\right)(y(t_n, s_j) - y_{nj}) \\
&\quad + h\kappa\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left\{\left[(k(t_{n+1}, s_{m+1}, t_i, s_j) - k(t_n, s_{m+1}, t_i, s_j))/h\right.\right. \\
&- \left.\left.(k(t_{n+1}, s_m, t_i, s_j) - k(t_n, s_m, t_i, s_j))/h\right]/\kappa\right\}(y(t_i, s_j) - y_{ij}) \\
&= \{(\sigma_{n+1,m+1} - \sigma_{n,m+1})/h - (\sigma_{n+1,m} - \sigma_{n,m})/h\}/\kappa.
\end{aligned}
$$

If h, κ are sufficiently small then

$$k(t_{n+1}, s_{m+1}, t_n, s_m) \neq 0.$$

Define $e_{nm} := |y(t_n, s_m) - y_{nm}|$, take the modulus and the triangular inequality repeatedly to obtain

$$
\begin{aligned}
e_{nm} &\leqslant h\sum_{i=0}^{n-1}\left|\frac{\partial k}{\partial t}(\xi_n, s_{m+1}, t_i, s_m)\right|\tau_{nm}e_{im} + \kappa\sum_{j=0}^{m-1}\left|\frac{\partial k}{\partial s}(t_{n+1}, \zeta_m, t_n, s_j)\right|\tau_{nm}e_{nj} \\
&+h\kappa\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\left(\rho\left|\frac{\partial^2 k}{\partial t^2}(\xi_n', s_{m+1}, t_i, s_j)\right| + \left|\frac{\partial^2 k}{\partial t\partial s}(\xi_n'', \zeta_m', t_i, s_j)\right|\right)\tau_{nm}e_{ij} + \Delta_{nm}, \qquad (5.8)
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_{nm} &:= \tau_{nm}|\{(\sigma_{n+1,m+1} - \sigma_{n,m+1})/h - (\sigma_{n+1,m} - \sigma_{n,m})/h\}/\kappa|, \\
\tau_{nm} &:= 1/|k(t_{n+1}, s_{m+1}, t_n, s_m)|, \\
\xi_n, \xi_n', \xi_n'', &\in (t_n, t_{n+1}), \zeta_m, \zeta_m' \in (s_m, s_{m+1}) \text{ and } \rho = h/\kappa.
\end{aligned}
$$

We note that $\Delta_{nm} = O(h + \kappa)$ provided the following are continuous:

$$\frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}, \frac{\partial^3 k}{\partial t^2\partial u}, \frac{\partial^3 k}{\partial t\partial s\partial u}, \frac{\partial^3 k}{\partial t^2\partial v}, \frac{\partial^3 k}{\partial t\partial s\partial v}.$$

We further note that ρ must be a constant independent of h and κ: this implies that h and κ must tend to zero at the same rate.

Recall that we chose $h < h_0, \kappa < \kappa_0$ sufficiently small to ensure that $k(t, s, t - h, s - \kappa) \neq 0$; or, equivalently, that there exists C, independent of h, κ, such that

$$\left|\frac{1}{k(t, s, t - h, s - \kappa)}\right| \leqslant C.$$

We can therefore define the following:

$$C_1 = \max_{t,s,u,v} \left| \frac{\frac{\partial k}{\partial t}(t,s,u,v)}{k(t,s,t-h,s-\kappa)} \right|,$$

$$C_2 = \max_{t,s,u,v} \left| \frac{\frac{\partial k}{\partial s}(t,s,u,v)}{k(t,s,t-h,s-\kappa)} \right|,$$

$$C_3 = \max_{t,s,u,v} \left\{ \frac{|\frac{\partial^2 k}{\partial t \partial s}(t,s,u,v)| + \rho|\frac{\partial^2 k}{\partial t^2}(t,s,u,v)|}{|k(t,s,t-h,s-\kappa)|} \right\}.$$

Inequality (5.8) now becomes

$$e_{nm} \leqslant hC_1 \sum_{i=0}^{n-1} e_{im} + \kappa C_2 \sum_{j=0}^{m-1} e_{nj} + h\kappa C_3 \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} e_{ij} + \Delta_{nm}. \tag{5.9}$$

From Lemma 5.1, provided $C_i > 0$ $(i = 1, 2, 3)$, we obtain

$$e_{nm} \leqslant \Delta_{nm} \exp(\gamma(nh + m\kappa))$$

with

$$\gamma = \tfrac{1}{2}\left(C_1 + C_2 + \sqrt{(C_1 + C_2)^2 + 4C_3} \right).$$

Since $\Delta_{nm} = O(h + \kappa)$, convergence of order $h + \kappa$ has been demonstrated.

If any of the $C_j = 0$ (implying the rather trivial case that one of the derivatives of the kernel is identically zero) then other forms of the discrete Gronwall lemma may be developed (analogous with the results of Section 4) and applied to yield sharper bounds.

5.3 *Numerical verification*

In order to illustrate the performance of the Euler and trapezoidal methods applied to (1.1), the following example has been considered, with $T = 2$, $S = 2$ and $h = \kappa$.

EXAMPLE 5.1 We set $k(t,s,u,v) = \sin(s+u) + \sin(t+v) + 3$ and choose $f(t,s)$ such that the exact solution of (1.1) is $x(u,v) = \cos(u+v)$.

By the trapezoidal method we mean the following discretization:

$$\frac{h\kappa}{4} \sum_{i=1}^{n} \sum_{j=1}^{m} (k_{i-1,j-1} y_{i-1,j-1} + k_{i-1,j} y_{i-1,j} + k_{i,j-1} y_{i,j-1} + k_{i,j} y_{ij}) = f_{nm}$$

with

$$k_{i,j} = k(t_n, s_m, t_i, s_j).$$

Further we define the L_∞ error and L_∞ rate of convergence to be, respectively,

$$\|e(h)\|_\infty := \max\{|y(ih, jh) - y_{ij}|, 0 \leqslant i, j \leqslant N\},$$

$$\text{rate:} = \log(\|e(h)\|_\infty / \|e(h/2)\|_\infty)/\log(2).$$

Tables 1 and 2 display the errors at selected grid points and the associated rates of convergence of Euler's method and the trapezoidal method.

TABLE 1
Euler's method

(u, v)	$h = 0{\cdot}1$	$h = 0{\cdot}05$	$h = 0{\cdot}025$	$h = 0{\cdot}0125$
(1,1)	9·45D−2	4·06D−2	2·27D−2	1·13D−2
(1,2)	1·97D−2	1·23D−2	6·76D−3	3·53D−3
(2,1)	1·97D−2	1·23D−2	6·76D−3	3·53D−3
(2,2)	8·76D−2	4·06D−2	1·94D−2	9·46D−3
$\|e(h)\|_\infty$	1·09D−1	5·49 D−2	2·75D−2	1·34D−2
Rate		0·99	1·0	1·0

TABLE 2
The trapezoidal method

(u, v)	$h = 0{\cdot}1$	$h = 0{\cdot}05$	$h = 0{\cdot}025$	$h = 0{\cdot}0125$
(1,1)	3·93D−3	9·80D−4	2·45D−4	6·12D−5
(1,2)	2·20D−3	5·47D−4	1·37D−4	3·41D−5
(2,1)	2·20D−3	5·47D−4	1·37D−4	3·42D−5
(2,2)	8·17D−3	2·03D−3	5·07D−4	1·27D−4
$\|e(h)\|_\infty$	8·17D−3	2·03 D−3	5·07D−4	1·27D−4
Rate		2·0	2·0	2·0

6. A generalized telegraph equation

Consider the nonlinear Cauchy problem (Garabedian, 1964)

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial}{\partial t}(f(x,t,w)) = \frac{\partial^2 w}{\partial x^2} + \frac{\partial}{\partial x}(g(x,t,w)) + h(x,t,w) \tag{6.1}$$

with $(x, t) \in W := \{(x, t) : x - t \geqslant 0 \text{ and } x + t \geqslant 0\}$, and subject to

$$u(x, x) = \alpha(x) \qquad \text{and} \qquad u(t, -t) = \beta(t).$$

In the special case when $f(x, t, w) \equiv w$ and $g(x, t, w) \equiv 0$ we obtain the 'forced' telegraph equation: this arises from electromagnetic waves in a conducting medium and can be derived directly from Maxwell's equations (see the Appendix). Performing the change of variables $x' = x + t, t' = x - t$, we obtain

$$4\frac{\partial^2 U}{\partial x' \partial t'} + \frac{\partial}{\partial x'}(-f + g) + \frac{\partial}{\partial t'}(f + g) = -h, \tag{6.2}$$

or equivalently,

$$\frac{\partial^2 U}{\partial x' \partial t'} - \frac{\partial F}{\partial x'} - \frac{\partial G}{\partial t'} = H, \tag{6.3}$$

where

$$\begin{aligned} U(x',t') &:= (u((x'+t')/2,(x'-t')/2))/4,\\ F(x',t',U) &:= (f(x',t',U)-g(x',t',U))/4,\\ G(x',t',U) &:= -(f(x',t',U)+g(x',t',U))/4,\\ H(x',t',U) &:= -h(x',t',U)/4. \end{aligned}$$

Integrating (6.3) with respect to x' and t' yields

$$\begin{aligned} \int_0^T\int_0^X \frac{\partial^2 U(x',t')}{\partial x'\partial t'}\,dx'\,dt' &= \int_0^T\int_0^X H(x',t',U(x',t'))\,dx'\,dt' \\ &+\int_0^T\int_0^X \frac{\partial G}{\partial t'}(x',t',U(x',t'))\,dx'\,dt' + \int_0^T\int_0^X \frac{\partial F(x',t',U(x',t'))}{\partial x'}\,dx'\,dt', \end{aligned} \tag{6.4}$$

which gives

$$\begin{aligned} U(X,T) = &\int_0^T\int_0^X H(x',t',U(x',t'))\,dx'\,dt' \\ &+\int_0^X (G(x',T,U(x',T))-G(x',0,U(x',0)))\,dx' \\ &+\int_0^T (F(X,t',U(X,t'))-F(0,t',U(0,t')))\,dt' \\ &+U(X,0)+U(0,T)-U(0,0). \end{aligned} \tag{6.5}$$

Defining

$$\begin{aligned} R(X,T) := &-\int_0^X G(x',0,U(x',0))\,dx' - \int_0^T F(0,t',U(0,t'))\,dt' \\ &+U(X,0)+U(0,T)-U(0,0), \end{aligned}$$

we are led to the following two-dimensional second-kind Volterra integral equation

$$\begin{aligned} U(X,T) = &\int_0^T\int_0^X H(x',t',U(x',t'))\,dx'\,dt' + \int_0^X G(x',T,U(x',T))\,dx' \\ &+\int_0^T F(X,t',U(x,t'))\,dt' + R(X,T). \end{aligned} \tag{6.6}$$

We note $R(X,T)$ is a known function from the initial conditions since $R(X,T)$ only involves $U(x',0)$ and $U(0,t')$, $(x',t')\in[0,X]\times[0,T]$, and $U(x',0)=u(x/2,x/2)=\alpha(x/2)$ and $U(0,t')=u(t/2,-t/2)=\beta(t/2)$.

Extending the results of Sections 2 and 3 to nonlinear equations (where the kernels in (6.6) are globally Lipschitz with respect to the dependent variable) we observe that the existence and uniqueness of (6.1) can be demonstrated. Indeed, rather surprisingly, (6.1) has been shown to be well-posed on the wedge

$$W := \{(x,t) : x-t\geqslant 0 \text{ and } x+t\geqslant 0\}$$

while only requiring knowledge of the solution on the lines $x=\pm t$; information about the derivatives would appear to be superfluous.

6.1 *A Computation*

In Section 5 a simple discretization of the two-dimensional first-kind Volterra integral equation (1.1) and its convergence analysis were given. The argument for second-kind equations follows similarly. We thus have an integral method for solving the class of nonlinear partial differential equations (6.1). Solving (6.6) rather than (6.1) may be interesting if a high-order continuous solution is sought and a spline collocation method can be used. To verify the efficacy of the method we shall solve a simple example.

Consider

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial \cos u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial \sin u}{\partial x} - \sin u \cos t \sin x - \cos u \sin t \cos x, \tag{6.7}$$

with the initial conditions given on $x = \pm t$, that is, $u(x, x) = \sin^2 x$, $u(t, -t) = -\sin^2 t$. The exact solution is $u(x, t) = \sin x \sin t$. With the transformation of variables $x' = x + t$, $t' = x - t$, (6.7) may be rewritten in the form

$$\begin{aligned} U(X, T) = &- \int_0^T \int_0^X \{\sin U(x', t') \cos(\tfrac{1}{2}(x' - t')) \sin(\tfrac{1}{2}(x' + t')) \\ &+ \cos U(x', t') \sin(\tfrac{1}{2}(x' - t')) \cos(\tfrac{1}{2}(x' + t'))\} \, dx' \, dt' \\ &+ \int_0^X \sin U(x', T) \, dx' + \int_0^T \cos U(X, t') \, dt' \\ &- \int_0^X \sin(\sin^2 x') \, dx' + \int_0^T \cos(\sin^2 t') \, dt' \\ &+ \sin^2 \frac{X}{2} - \sin^2 \frac{T}{2}. \end{aligned} \tag{6.8}$$

A Euler-type discretization of (6.8) (with $h = \kappa$) is

$$U_{IJ} = h^2 \sum_{i=1}^{I} \sum_{j=1}^{J} H(ih, jh, U_{ij}) + h\left(\sum_{i=1}^{I} G(ih, Jh, U_{iJ}) + \sum_{j=1}^{J} F(Ih, jh, U_{Ij}) \right) + R_{IJ},$$

$$I = 1, 2, \ldots, N; \; J = 1, 2, \ldots, M,$$

where H, G, F and R are defined appropriately. The errors shown in Table 3 at a sample of grid points were obtained displaying first-order convergence.

7. Concluding remarks

This paper has introduced two-dimensional Volterra integral equations of both the first and second kind. The existence and uniqueness of solutions to these equations have been analysed and the solution of a discretization method has been shown to converge to that of the underlying integral equation. In the course of this analysis it has been found necessary to develop a new Gronwall inequality. A special class of partial differential equations arising from electromagnetic waves has been shown to be equivalent to these two-dimensional Volterra equations, thus providing both uniqueness and existence of their solutions and a constructive means of obtaining their approximate solution. Finally, it is clear that the analysis of this paper extends to N-dimensional Volterra integral equations; we dealt with the case $N = 2$ largely for reasons of clarity.

TABLE 3
Euler's method

(x,t)	$h = 0{\cdot}1$	$h = 0{\cdot}05$	$h = 0{\cdot}025$	$h = 0{\cdot}0125$
(1,0)	7·03D−4	1·73D−4	4·28D−5	1·07D−5
(3/2,-1/2)	3·47D−3	1·27D−3	5·21D−4	2·31D−4
(3/2,1/2)	3·01D−3	1·16D−3	4·94D−4	2·25D−4
(2,0)	9·32D−4	2·38D−4	6·00D−5	1·51D−5
$\|e(h)\|_\infty$	3·72D−3	1·49 D−3	6·47D−4	2·99D−4
Rate		1·3	1·2	1·1

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Appendix

Consider Maxwell's equations on the usual notation (see e.g. Coulson 1961):

(i) $\operatorname{div}\mathbf{D} = 4\pi\rho$

(ii) $\operatorname{div}\mathbf{B} = 0$

(iii) $\operatorname{curl}\mathbf{H} = 4\pi\mathbf{j} + \dfrac{1}{c}\dfrac{\partial\mathbf{D}}{\partial t}$

(iv) $\operatorname{curl}\mathbf{E} = -\dfrac{1}{c}\dfrac{\partial\mathbf{B}}{\partial t}$ together with the constitutive relations

(v) $\mathbf{D} = \kappa\mathbf{E}$

(vi) $\mathbf{B} = \mu\mathbf{H}$

(vii) $\mathbf{j} = \sigma(\mathbf{E})$.

Note that (vii) is a nonlinear version of Ohm's law and models a number of conducting materials, for instance, semiconductors. In any material capable of some degree of conduction there will be energy dissipation and it will now be shown that $\mathbf{E}$ satisfies a nonlinear telegraph equation.

Operate with curl on both sides of (iv) to obtain

(viii) $$\operatorname{grad}\operatorname{div}\mathbf{E} - \nabla^2\mathbf{E} = -\frac{\mu}{c}\frac{\partial}{\partial t}(\nabla \wedge \mathbf{H})$$

where we have used the vector identity

$$\operatorname{curl}\operatorname{curl} \equiv \operatorname{grad}\operatorname{div} - \nabla^2$$

and the fact that $\mathbf{B} = \mu\mathbf{H}$.
We note that

$$\operatorname{div}\mathbf{D} = \kappa\operatorname{div}\mathbf{E} = 4\pi\rho$$

and that

$$\operatorname{curl}\mathbf{H} = 4\pi\mathbf{j} + \frac{\kappa}{c}\frac{\partial\mathbf{E}}{\partial t}$$

so that (viii) becomes

(ix) $$\frac{\mu\kappa}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}+\frac{4\pi\mu}{c}\frac{\partial}{\partial t}(\boldsymbol{\sigma}(\mathbf{E}))=\nabla^2\mathbf{E}-\frac{4\pi}{\kappa}\operatorname{grad}\rho.$$

Consider now the case of one spatial dimension.We note that in the linear case (i.e. $\mathbf{j}=\sigma_0\mathbf{E}$, Ohm's law) we simply obtain the telegraph equation. In the intermediate case when $j_k=\sigma_k(E_k)$, $k=1,2,3$, (ix) takes the form of three uncoupled equations, each of type (6.1). For the general nonlinear case $\mathbf{j}=\boldsymbol{\sigma}(\mathbf{E})$ we obtain three coupled nonlinear equations of type (6.1), each of which may be reduced to a second-kind Volterra integral equation, resulting in a coupled system of Volterra integral equations.