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# A Hermite-Type Collocation Method for the Solution of an Integral Equation with a Certain Weakly Singular Kernel

TERESA DIOGO
*Institute of Mathematics, University of Kent, Canterbury CT2 7NF, England*
SEAN MCKEE AND TAO TANG
*Department of Mathematics, University of Strathclyde, Glasgow G1 1XH, Scotland*

*Dedicated to Professor A. R. Mitchell on the occasion of his 70th Birthday*

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A Hermite-type collocation method is considered for the solution of a second-kind Volterra integral equation with a certain weakly singular kernel. The constructed approximation is a cubic spline in the continuity class $C^1$. The method is shown to be convergent of order four. Some numerical examples are included.

## 1. Introduction

CONSIDER the Volterra-type integral equation

$$y(t) + \int_0^t p(t,s)y(s)\,\mathrm{d}s = g(t), \tag{1.1a}$$

$$p(t,s) := \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{[\ln(t/s)]}} \left(\frac{s}{t}\right)^{\mu} \frac{1}{s}, \tag{1.1b}$$

where $\mu > 0$ is a constant, $y(t)$ is the unknown function, and $g(t)$ is a given function. Equations of this type can arise from certain diffusion problems after the application of Watson transforms and the use of the convolution theorem (Bartoshevich, 1975; Sub-Sizonenko, 1979). They have been studied theoretically by the above authors, Rooney (1983), and Lamb (1980, 1985) who provided somewhat complicated $L_2$ solutions but none is useful if a tabulated solution is required. In this work, we shall be concerned with the use of polynomial splines for the numerical solution of (1.1). Specifically, cubic splines whose second derivative may be discontinuous will be considered. Such functions have been employed by several authors for the solution of Volterra integral equations (VIEs) of the second kind with a smooth kernel. Hung (1970) proved that splines of degree $m$ and in the continuity class $C^{m-1}$ yield divergent collocation methods when $m > 2$. He also showed, however, that, by relaxing the continuity requirements, some convergent methods can be obtained. In particular, he developed a fourth-order method to obtain a cubic spline solution in $C^1$ which is a Hermite interpolation polynomial. In his approach, both the original VIE and its differentiated form were requires to be satisfied at the mesh points (Hermite-type collocation). Later, Netravali (1973) showed that, with this kind of technique, one can still obtain a convergent cubic spline approximation in $C^2$. Related methods may be found in El Tom (1974, 1976), where approximations in various polynomial spline spaces have been considered. In addition, Bowgen (1980) developed a block-by-block method, of which Hung's method is a particular case, to produce a piecewise cubic spline solution. This approximation, expressed in a B-spline basis, is of continuity $C^2$ within each block, but its second derivative may be discontinuous at the block boundaries. For the application of cubic Hermite polynomials to the solution of Fredholm integral equations, one can refer to Arthur (1973).

Although there has been some work on Hermite-type collocation methods for second-kind Volterra integral equations with smooth kernels, very few studies have dealt with weakly singular kernels (see, for example, Papatheodorou & Jesanis (1980), who considered Volterra integro-differential equations with weakly singular kernels). In this paper, we shall investigate a Hermite-type collocation methods for (1.1a) with a singular kernel of the form (1.1b). Two low-order product integration methods for the solution of (1.1) have been considered in Tang *et al.* (1990). Here approximations to $y(t)$ of orders 1 and 2 were obtained with the so-called Euler method and trapezoidal method respectively.

In order to obtain higher orders of convergence, a certain degree of smoothness on the exact solutions of (1.1) is required. In general, a unique solution of (1.1) exists for all $\mu > 0$ (see, for example, Lamb, 1980, 1985) but it is not clear if the solution is sufficiently smooth. In the following section we shall show that a (unique) smooth solution exists when $\mu > 1$. The smoothness property in the case $0 < \mu \leqslant 1$ has not been available, and therefore in this paper we shall restrict ourselves to the case $\mu > 1$. The basic idea of this work is to derive two (linear) Volterra equations for $y(t)$ and $y'(t)$ by transforming the original integral equation. Having the coupled equations for both $y$ and $y'$, we can then employ piecewise cubic Hermite polynomials to obtain numerical solutions of (1.1). In general this approach can be used in calculations if two coupled equations for $y$ and $y'$ exist and it is expected that higher orders of convergence might be achieved if the solution is sufficiently smooth. In Section 2 of this paper it will be shown that the derivative of the solution of (1.1) satisfies a Volterra-type equation whose kernel is of a relatively simple form. Section 3 introduces the numerical approach and a convergence analysis is given in Section 4. Unlike most weakly singular kernels, then kernel $p$ in (1.1) does not satisfy $\int_0^t p(t,s)\mathrm{d}s \to 0$ as $t \to 0$ and this fact gives rise to serious difficulties in the error analysis. Numerical results are presented in the final section which support the theoretical estimates obtained in this paper.

## 2. Preliminaries

Let $V_m$ be the normed space defined by

$$V_m := \{\phi : \phi(t) \in C^m[0,T]\},$$

where $m$ is a non-negative integer, $T$ a positive number, and

$$\|\phi\|_m := \max\left\{\left|\frac{\mathrm{d}^j\phi}{\mathrm{d}t^j}\right|, \, t \in [0,T], \, 0 \leqslant j \leqslant m\right\}.$$

LEMMA 1 *If the function $g$ in* (1.1) *belongs to $V_m$ and $\mu > 1$, then* (1.1) *possesses a unique solution $y \in V_m$.*

*Proof.* Choosing an arbitrary function $v \in V_m$, we define $u = S(v)$ such that

$$u(t) + \int_0^t p(t,s)v(s)\,\mathrm{d}s = g(t), \quad t \in [0,T]. \tag{2.1}$$

Setting $s = \lambda t$, we have

$$\int_0^t p(t,s)v(s)\,\mathrm{d}s = \frac{1}{\sqrt{\pi}} \int_0^1 \frac{1}{\sqrt{[\ln(1/\lambda)]}} \lambda^{\mu-1} v(\lambda t)\,\mathrm{d}\lambda. \tag{2.2}$$

Since $v \in \mathrm{C}^m[0,T]$ and $g \in \mathrm{C}^m[0,T]$, we obtain from (2.1) and (2.2) that

$$u^{(j)}(t) = -\frac{1}{\sqrt{\pi}} \int_0^1 \frac{1}{\sqrt{[\ln(1/\lambda)]}} \lambda^{\mu-1+j} v^{(j)}(\lambda t)\,\mathrm{d}\lambda + g^{(j)}(t), \tag{2.3}$$

where $0 \leqslant j \leqslant m$, and $u^{(j)} := \mathrm{d}^j u/\mathrm{d}t^j$, etc. If $u_1 = S(v_1)$ and $u_2 = S(v_2)$, we obtain from (2.3) that

$$\begin{aligned} |u_1^{(j)} - u_2^{(j)}| &\leqslant \frac{1}{\sqrt{\pi}} \int_0^1 \frac{1}{\sqrt{\ln(1/\lambda)}} \lambda^{\mu-1+j} |v_1^{(j)}(\lambda t) - v_2^{(j)}(\lambda t)|\,\mathrm{d}\lambda \\ &\leqslant \frac{1}{\sqrt{\pi}} \int_0^1 \frac{1}{\sqrt{\ln(1/\lambda)}} \lambda^{\mu-1}\,\mathrm{d}\lambda\,\|v_1 - v_2\|_m. \end{aligned} \tag{2.4}$$

Noting that the coefficient of the last term of (2.4) equals $1/\sqrt{\mu}$ it follows that

$$\|u_1 - u_2\|_m \leqslant \frac{1}{\sqrt{\mu}} \|v_1 - v_2\|_m. \tag{2.5}$$

If $\mu > 1$, then the inequality (2.5) implies that the operator $S$ is a contraction mapping. Since $V_m$ is a complete normed space, $S$ has a unique fixed point $y \in V_m$ such that $y = S(y)$. This completes the proof of Lemma 1. □

LEMMA 2 *Equation* (1.1) *can be transformed into the equivalent equation*

$$y(t) - \int_0^t p_1(t,s)y(s)\,\mathrm{d}s = g_1(t), \tag{2.6a}$$

*where*

$$p_1(t,s) := \left(\frac{s}{t}\right)^{\mu} \frac{1}{s}, \tag{2.6b}$$

$$g_1(t) := -\int_0^t p(t,s)g(s)\,\mathrm{d}s + g(t). \tag{2.6c}$$

*Proof.* Consider

$$y(s) + \int_0^s p(s,\lambda) y(\lambda)\,\mathrm{d}\lambda = g(s). \tag{2.7}$$

Multiplying both sides of (2.7) by $p(t,s)$ and integrating the resulting equation from 0 to $t$, we obtain

$$\int_0^t p(t,s)y(s)\,\mathrm{d}s + \int_0^t y(\lambda)\int_\lambda^t p(t,s)p(s,\lambda)\,\mathrm{d}s\,\mathrm{d}\lambda = \int_0^t p(t,s)g(s)\,\mathrm{d}s. \tag{2.8}$$

Here we have made use of Dirichlet's formula which states that

$$\int_0^t \int_0^s \phi(s,\lambda)\,\mathrm{d}\lambda\,\mathrm{d}s = \int_0^t \int_\lambda^t \phi(s,\lambda)\,\mathrm{d}s\,\mathrm{d}\lambda,$$

provided both integrals exist. Combining (2.8) with (1.1) gives (2.6). □

LEMMA 3 *The derivative of $y$, the solution of* (1.1), *satisfies*

$$y'(t) - \int_0^t p_2(t,s)y'(s)\,\mathrm{d}s = g_2(t), \tag{2.9a}$$

*where*

$$p_2(t,s) := \left(\frac{s}{t}\right)^{\mu+1}\frac{1}{s}, \tag{2.9b}$$

$$g_2(t) := -\int_0^t p(t,s)\frac{s}{t}g'(s)\,\mathrm{d}s + g'(t). \tag{2.9c}$$

*Proof.* By a change of variables, (2.6a) can be written in the form

$$y(t) - \int_0^1 \lambda^{\mu-1}y(t\lambda)\,\mathrm{d}\lambda = -\frac{1}{\sqrt{\pi}}\int_0^1 \frac{1}{\sqrt{[\ln(1/\lambda)]}}\lambda^{\mu-1}g(t\lambda)\,\mathrm{d}\lambda + g(t). \tag{2.10}$$

Differentiating both sides of (2.10) with respect to $t$ yields (2.9). □

## 3. A Hermite-type collocation method

In order to approximate the solution of (1.1), we now introduce the space $S_3$ of the piecewise cubic Hermite polynomials (see, for example, Prenter, 1975). If $\Pi_N$ denotes a uniform mesh

$$0 = t_0 < t_1 < \cdots < t_N = T, \quad N \geqslant 1,$$
$$t_n = nh, \quad 0 \leqslant n \leqslant N, \quad h = T/N,$$

then

$$S_3 := \{u : u(t) = \phi_{1n}(t)u_n + \phi_{2n}(t)u_{n+1} + \psi_{1n}(t)u'_n + \psi_{2n}(t)u'_{n+1},$$
$$t \in [t_n, t_{n+1}],\ 0 \leqslant n \leqslant N-1\},$$

where $u_j = u(t_j),\ u'_j = u'(t_j),\ j = n,\ n+1$, and

$$\begin{aligned}
\phi_{1n}(t) &= (t - t_{n+1})^2[h + 2(t - t_n)]/h^3, &(3.1a)\\
\phi_{2n}(t) &= (t - t_n)^2[h + 2(t_{n+1} - t)]/h^3, &(3.1b)\\
\psi_{1n}(t) &= (t - t_n)(t - t_{n+1})^2/h^2, &(3.1c)\\
\psi_{2n}(t) &= (t - t_n)^2(t - t_{n+1})/h^2. &(3.1d)
\end{aligned}$$

DEFINITION Let $f \in C^m[0,T],\ m > 1$, be a given function. Then $p(t) \in S_3$ is the Hermite cubic interpolant to $f$ if

$$p(t_n) = f(t_n), \quad p'(t_n) = f'(t_n), \quad 0 \leqslant n \leqslant N.$$

LEMMA 4 (Prenter, 1975) *Assume $f \in C^4[0,T]$. Then*

$$\|f^{(\lambda)} - p^{(\lambda)}\|_\infty = O(h^{4-\lambda}), \quad \lambda = 0, 1. \tag{3.2}$$

We seek an approximate solution $u$ of (1.1) in the space $S_3$, which satisfies

$$\begin{aligned}
u(t_n) - \int_0^{t_n} p_1(t_n, s)u(s)\,\mathrm{d}s &= \tilde{g}_1(t_n), &(3.3a)\\
u'(t_n) - \int_0^{t_n} p_2(t_n, s)u'(s)\,\mathrm{d}s &= \tilde{g}_2(t_n), \quad 0 \leqslant n \leqslant N, &(3.3b)
\end{aligned}$$

with

$$\begin{aligned}
u(0) &:= y(0) = g(0) \Big/ \left(1 + \frac{1}{\sqrt{\mu}}\right), &(3.3c)\\
u'(0) &:= y'(0) = g'(0) \Big/ \left(1 + \frac{1}{\sqrt{(\mu + 1)}}\right), &(3.3d)
\end{aligned}$$

and where

$$\begin{aligned}
\tilde{g}_1(t_n) &= -\int_0^{t_n} p(t_n, s)\tilde{g}(s)\,\mathrm{d}s + g(t_n), &(3.3e)\\
\tilde{g}_2(t_n) &= -\int_0^{t_n} p(t_n, s)\frac{s}{t_n}\tilde{g}'(s)\,\mathrm{d}s + g'(t_n). &(3.3f)
\end{aligned}$$

Noting that

$$\begin{aligned}
\lim_{t\to 0^+} \int_0^t p(t, s)y(s)\,\mathrm{d}s &= \lim_{t\to 0^-} \int_0^1 \frac{1}{\sqrt{[\pi \ln(1/\lambda)]}}\lambda^{\mu-1}y(t\lambda)\,\mathrm{d}\lambda\\
&= y(0)\int_0^1 \frac{1}{\sqrt{[\pi \ln(1/\lambda)]}}\lambda^{\mu-1}\,\mathrm{d}\lambda = \frac{y(0)}{\sqrt{\mu}}, &(3.4)
\end{aligned}$$

we obtain (3.3c) by letting $t \to 0^+$ in (1.1a). Similar arguments applied to (2.9) then lead to (3.3d). Here the function $\tilde{g} \in S_3$ is the Hermite cubic interpolant to $g$. The approximations $\tilde{g}_1(t_n)$ and $\tilde{g}_2(t_n)$ will be replaced by the exact values $g_1(t_n)$ and $g_2(t_n)$ if the integrals involved can be calculated analytically.

## 4. Convergence

LEMMA 5 (Bowgen, 1980) *For* $t \in [t_n, t_{n+1}],\ 0 \leqslant n \leqslant N$, *we have*

$$|\phi_{1n}(t)| + |\phi_{2n}(t)| = 1, \quad |\psi_{1n}(t)| + |\psi_{2n}(t)| \leqslant \tfrac{1}{4}h, \tag{4.1a}$$

$$\phi'_{in}(t) = O(h^{-1}), \quad \psi'_{in}(t) = O(1), \quad i = 1, 2. \tag{4.1b}$$

Define the error function $e(t) := y(t) - u(t),\ t \in [0, T]$. Set $t = t_n$ in (2.6a) and subtract (3.3a) from the resulting equation. Combining (2.9a) and (3.3b) in a similar way, it follows that $e(t)$ satisfies the error equations

$$e(t_n) - \int_0^{t_n} p_1(t_n, s)e(s)\,\mathrm{d}s = g_1(t_n) - \tilde{g}_1(t_n), \tag{4.2a}$$

$$e'(t_n) - \int_0^{t_n} p_2(t_n, s)e'(s)\,\mathrm{d}s = g_2(t_n) - \tilde{g}_2(t_n), \tag{4.2b}$$

for $0 \leqslant n \leqslant N$.

LEMMA 6 *If* $g \in C^4[0, T]$, *then*

$$\text{(i)} \qquad g_1(t_n) - \tilde{g}_1(t_n) = O(h^4), \tag{4.3a}$$

$$\text{(ii)} \qquad g_2(t_n) - \tilde{g}_2(t_n) = O(h^3), \tag{4.3b}$$

$$\text{(iii)} \qquad t_n e'(t_n) = (1 - \mu)e(t_n) + t_n O(h^3) + O(h^4), \tag{4.3c}$$

*for all* $0 \leqslant n \leqslant N$.

*Proof.* (i) Setting $t = t_n$ in (2.6c) and subtracting (3.3e) from the resulting equation gives

$$|g_1(t_n) - \tilde{g}_1(t_n)| \leqslant \frac{1}{\sqrt{\mu}} \max_{0 \leqslant s \leqslant T} |g(s) - \tilde{g}(s)|.$$

Applying (3.2), we obtain (4.3a).
(ii) Similarly, (4.3b) follows from (2.9c), (3.3f), and the use of (3.2).
(iii) From (4.2a) and (4.3a) we have

$$e(t_n) = \int_0^{t_n} p_1(t_n, s)e(s)\,\mathrm{d}s + O(h^4). \tag{4.4}$$

Integration by part gives

$$e(t_n) = \frac{1}{\mu}e(t_n) - \frac{t_n}{\mu}\int_0^{t_n} p_2(t_n, s)e'(s)\,\mathrm{d}s + O(h^4). \tag{4.5}$$

Using (4.2b) and (4.3b), then

$$e(t_n) = \frac{1}{\mu}e(t_n) - \frac{t_n}{\mu}[e'(t_n) + O(h^3)] + O(h^4), \tag{4.6}$$

which implies (4.3c). $\square$

THEOREM 1 *If $\mu > 1$ and $g \in C^4[0,T]$, then the error function $e(t) := y(t) - u(t)$ satisfies*

$$\|e^{(\lambda)}\|_\infty := \max\{|e^{(\lambda)}(t)|,\ t \in [0,T)\} = O(h^{4-\lambda}), \quad \lambda = 0, 1. \tag{4.7}$$

*Proof.* Let $E_1 := \max\{|e(t_n)|, \quad 1 \leqslant n \leqslant N\}$, and $E_2 := \max\{|he'(t_n)|,\ 1 \leqslant n \leqslant N\}$. For $t \in [t_n, t_{n+1}]$, we have

$$u(t) = \phi_{1n}(t)u_n + \phi_{2n}(t)u_{n+1} + \psi_{1n}(t)u'_n + \psi_{2n}(t)u'_{n+1}. \tag{4.8}$$

As $g \in C^4[0,T]$ then, by Lemma 1, $y(t) \in C^4[0,T]$. It follows from Lemma 4 that

$$y(t) = \phi_{1n}(t)y(t_n) + \phi_{2n}(t)y(t_{n+1}) + \psi_{1n}(t)y'(t_n) + \psi_{2n}(t)y'(t_{n+1}) + O(h^4). \tag{4.9}$$

Then, subtracting (4.8) from (4.9) and using (4.1a), we obtain

$$|e(t)| \leqslant E_1 + \tfrac{1}{4}E_2 + O(h^4). \tag{4.10}$$

Equation (4.2a) implies

$$E_1 \leqslant \frac{1}{\mu} \max_{0 \leqslant t \leqslant T} |e(t)| + O(h^4), \tag{4.11}$$

which, together with (4.10), results in

$$E_1 \leqslant \frac{1}{\mu}(E_1 + \tfrac{1}{4}E_2) + O(h^4). \tag{4.12}$$

On the other hand, from (4.3c)

$$n|he'(t_n)| \leqslant (\mu - 1)|e(t_n)| + nhO(h^3) + O(h^4). \tag{4.13}$$

Then

$$E_2 \leqslant (\mu - 1)E_1 + O(h^4). \tag{4.14}$$

Substituting (4.14) into (4.12) gives

$$E_1 \leqslant \frac{1}{\mu}[E_1 + \tfrac{1}{4}(\mu - 1)E_1] + O(h^4). \tag{4.15}$$

Hence

$$E_1 \leqslant \frac{4\mu}{3(\mu - 1)} O(h^4). \tag{4.16}$$

Using (4.14) and (4.16), we obtain from (4.10) that

$$|e(t)| \leqslant O(h^4). \tag{4.17}$$

The estimate $\|e'\|_\infty = O(h^3)$ follows easily from

$$e'(t) = \phi'_{1n}(t)e(t_n) + \phi'_{2n}(t)e(t_{n+1}) + \psi'_{1n}(t)e'(t_n) + \psi'_{2n}(t)e'(t_{n+1}) + O(h^3), \tag{4.18}$$

by using (4.1b) and the fact that $E_i = O(h^4),\ i = 1, 2.$ □

THEOREM 2 *Let $g \in C^5[0,T]$ and assume that $g'(t)$ can be obtained analytically. Then $e'(t)$ satisfies*

$$e'(t_n) = O(h^4), \tag{4.19}$$

*for $t_n \geqslant \varepsilon > 0$, $\varepsilon$ being a fixed constant.*

*Proof.* In this case, $g' \in C^4[0,T]$, which implies

$$\|g' - \tilde{g}'\|_\infty = O(h^4), \tag{4.20}$$

where $\tilde{g}'$ is the cubic Hermite interpolant to $g'$. Following the proof of Lemma 6, we can replace (4.3c) by

$$t_n e'(t_n) = (1-\mu)e(t_n) + t_n O(h^4) + O(h^4), \tag{4.21}$$

which gives, using (4.16),

$$|e'(t_n)| = \frac{1}{t_n} O(h^4) + O(h^4). \tag{4.22}$$

Then, for $t_n \geqslant \varepsilon > 0$,

$$|e'(t_n)| \leqslant \frac{1}{\varepsilon} O(h^4) + O(h^4) = O(h^4). \quad \square \tag{4.23}$$

## 5. Numerical examples

Equations (3.3a,b) can be written in the form

$$u(t_n) - \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} p_1(t_n,s)u(s)\,\mathrm{d}s = \tilde{g}_1(t_n), \tag{5.1a}$$

$$u'(t_n) - \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} p_2(t_n,s)u'(s)\,\mathrm{d}s = \tilde{g}_2(t_n), \quad 1 \leqslant n \leqslant N. \tag{5.1b}$$

By a change of variables in the integrals over the subintervals $[t_i, t_{i+1}]$, $0 \leqslant i \leqslant n-1$, (5.1a,b) becomes

$$u(t_n) - \sum_{i=0}^{n-1} h \int_0^1 p_1(t_n, t_i + vh)u(t_i + vh)\,\mathrm{d}v = \tilde{g}_1(t_n), \tag{5.2a}$$

$$u'(t_n) - \sum_{i=0}^{n-1} h \int_0^1 p_2(t_n, t_i + vh)u'(t_i + vh)\,\mathrm{d}v = \tilde{g}_2(t_n). \tag{5.2b}$$

In the above

$$u(t_i + vh) = l_1(v)u_i + l_2(v)u_{i+1} + d_1(v)hu_i' + d_2(v)hu_{i+1}', \quad 0 \leqslant v \leqslant 1, \tag{5.3a}$$

$$
\begin{aligned}
l_1(v) &= 2v^3 - 3v^2 + 1, && (5.3b)\\
l_2(v) &= -2v^3 + 3v^2, && (5.3c)\\
d_1(v) &= v^3 - 2v^2 + v, && (5.3d)\\
d_2(v) &= v^3 - v^2. && (5.3e)
\end{aligned}
$$

Similarly, for the derivative of the approximate solution in (5.2b), we have

$$u'(t_i + vh) = \frac{1}{h}[l_1'(v)u_i + l_2'(v)u_{i+1} + d_1'(v)hu_i' + d_2'(v)hu_{i+1}']. \qquad (5.4)$$

We note that

$$
\begin{aligned}
p_1(t_n, t_i + vh) &= \left(\frac{v+i}{n}\right)^{\mu} \frac{1}{t_i + vh},\\
p_2(t_n, t_i + vh) &= \left(\frac{v+i}{n}\right)^{\mu+1} \frac{1}{t_i + vh}.
\end{aligned} \qquad (5.5)
$$

In order to write (5.2a,b) in the form of an algorithm which more clearly exhibits the recursive nature of the method, it will be convenient to introduce the following notation. Define

$$
\begin{aligned}
L_k^j(n,i) &:= \frac{1}{n^{\mu+1}} \int_0^1 (v+i)^{\mu+j-1} l_k^{(j)}(v)\,\mathrm{d}v, && (5.6a)\\
D_k^j(n,i) &:= \frac{1}{n^{\mu+1}} \int_0^1 (v+i)^{\mu+j-1} d_k^{(j)}(v)\,\mathrm{d}v, && (5.6b)
\end{aligned}
$$

where the superscript $(j)$, $j = 0, 1$, denotes the derivative of order $j$. We also set

$$
\begin{aligned}
G_{jn} &:= \tilde{g}_{j+1}(t_n),\\
U_n^j &:= u^{(j)}(t_n), \quad j = 0, 1.
\end{aligned} \qquad (5.7)
$$

TABLE 1
*Example 1: $|e(t_n)|$ and $|e'(t_n)|$ (in brackets)*

| $t_n$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ |
|---|---|---|---|
| 0.4 | 5.0D-6 (6.2D-6) | 3.2D-7 (4.0D-7) | 2.0D-8 (2.5D-8) |
| 0.8 | 7.2D-6 (4.5D-6) | 4.5D-7 (2.8D-7) | 2.9D-8 (1.8D-8) |
| 1.0 | 8.1D-6 (4.0D-6) | 5.1D-7 (2.5D-7) | 3.2D-8 (1.6D-8) |
| 1.5 | 9.9D-6 (3.3D-6) | 6.2D-7 (2.1D-7) | 3.9D-8 (1.3D-8) |
| 2.0 | 1.2D-5 (2.9D-6) | 7.2D-7 (1.8D-7) | 4.5D-8 (1.1D-8) |
| 2.5 | 1.3D-5 (2.3D-6) | 8.1D-7 (1.6D-7) | 5.1D-8 (1.0D-8) |
| 3.0 | 1.4D-5 (2.4D-6) | 8.9D-7 (1.5D-7) | 5.5D-8 (9.2D-9) |
| 4.0 | 1.6D-5 (2.0D-6) | 1.0D-6 (1.3D-7) | 6.4D-8 (8.0D-9) |

TABLE 2
*Example 2:* $|e(t_n)|$ *and* $|e'(t_n)|$ *(in brackets)*

| $t_n$ | $h = 0.1$ | $h = 0.05$ | $h = 0.025$ |
|---|---|---|---|
| 0.4 | 1.1D-6 (1.3D-6) | 8.4D-8 (1.1D-7) | 6.0D-9 (7.4D-9) |
| 0.8 | 3.1D-6 (1.9D-6) | 1.8D-7 (1.1D-7) | 1.1D-8 (6.7D-9) |
| 1.0 | 5.2D-6 (2.6D-6) | 3.1D-7 (1.6D-7) | 1.9D-8 (9.5D-9) |
| 1.5 | 1.0D-5 (3.4D-6) | 6.3D-7 (2.1D-7) | 3.9D-8 (1.3D-8) |
| 2.0 | 1.5D-5 (3.8D-6) | 9.3D-7 (2.3D-7) | 5.8D-8 (1.4D-8) |
| 2.5 | 2.0D-5 (3.9D-6) | 1.2D-6 (2.4D-7) | 7.6D-8 (1.5D-8) |
| 3.0 | 2.4D-5 (4.0D-6) | 1.5D-6 (2.5D-7) | 9.3D-8 (1.6D-8) |
| 4.0 | 3.2D-5 (4.0D-6) | 2.0D-6 (2.5D-7) | 1.3D-7 (1.5D-8) |

Then (5.2a,b) yield

$$\begin{aligned} &h^j U_n^j - L_2^j(n,n-1)u_n - D_2^j(n,n-1)hu_n' \\ &\quad = L_1^j(n,n-1)u_{n-1} + D_1^j(n,n-1)hu_{n-1}' \\ &\qquad + \sum_{i=0}^{n-2}[L_1^j(n,i)u_i + L_2^j(n,i)u_{i+1} + D_1^j(n,i)hu_i' + D_2^j(n,i)hu_{i+1}'] + h^j G_{jn}. \end{aligned} \tag{5.8}$$

For $j = 0, 1$ $(1 \leqslant n \leqslant N)$, (5.8) constitutes a system of two equations in the unknowns $u_n = u(t_n)$ and $hu_n' = hu'(t_n)$. The starting values $u_0$ and $u_0'$ are given by (3.3c) and (3.3d) respectively. If the function $g$ in (1.1) is such that the values $g_1(t_n)$ and $g_2(t_n)$ in (2.6c) and (2.9c) can be evaluated analytically, then these values will be used in place of the approximations $\tilde{g}_1(t_n)$ and $\tilde{g}_2(t_n)$ respectively. This was done in the examples considered. Algorithm (5.8) was implemented as a Fortran program in double precision. The program was applied to equation (1.1) in the following two examples.

EXAMPLE 1 We set $\mu = \frac{3}{2}$ and chose $g$ such that the exact solution is $y(t) = t^{4.5}$.

EXAMPLE 2 We set $\mu = \frac{3}{2}$ and chose $g$ such that the exact solution is $y(t) = t^{4.5}\ln t$ .

Tables 1 and 2 contain a selection of the numerical results obtained. The modulus of the errors $e(t_n) := y(t_n) - u(t_n)$ for various values of the stepsize $h$ are shown in these tables indicating convergence of order 4 (cf. (4.7)). We have listed (in brackets) the errors $|e'(t_n)|$ for the derivative $u'$. The values produced indicate convergence of $u'(t)$ to $y'(t)$ with $h^4$, at the mesh points $t = t_n$, away from the origin cf. (4.19)).

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