# Superconvergence of numerical solutions to weakly singular Volterra integro-differential equations

**Tao Tang**

Department of Mathematics and Statistics, Simon Fraser University,
Burnaby, British Columbia, Canada V5A IS6

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**Summary.** We discuss the application of a class of spline collocation methods to first-order Volterra integro-differential equations (VIDEs) which contain a weakly singular kernel $(t - s)^{-\alpha}$ with $0 < \alpha < 1$. It will be shown that superconvergence properties may be obtained by using appropriate collocation parameters and graded meshes. The grading exponents of graded meshes used are not greater than $m$ (the polynomial degree) which is independent of $\alpha$. This is in contrast to the theories of spline collocation methods for Volterra (or Fredholm) integral equation of the second kind. Numerical examples are given to illustrate the theoretical results.

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## 1 Introduction

This paper is concerned with the superconvergence properties of collocation approximations for Volterra integro-differential equations (VIDEs)

$$y'(t) = f(t, y(t)) + \int_0^t (t-s)^{-\alpha} k(t, s, y(s))ds, \quad t \in I := [0, T], \tag{1.1}$$

with $0 < \alpha < 1$, and with given initial condition $y(0) = y_0$. For ease of exposition we shall employ the linear counterpart of (1.1),

$$y'(t) = a(t)y(t) + b(t) + \int_0^t (t-s)^{-\alpha} K(t, s)y(s)ds, \quad t \in I, \tag{1.2}$$

in the analysis of the principle properties of the collocation approximations; its extension to the non-linear equation (1.1) is straightforward following [4] (see also Remark

2 made in Sect. 2). The investigation of the superconvergence properties of collocation methods for VIDEs (1.1) is of interest since in many applications one is interested in generating approximations to $y(t)$ at the right endpoint $t = T$ of $I$ which exhibit a high order of convergence at that point. Related results for equations with singular solutions may be found in [3] (product integration for second kind Volterra integral equations); [5] and [6] (product integration for Fredholm integral equation of the second kind); [4] (collocation methods for Basset equation). For all of these work the so-called graded mesh (see (1.6) below for the definition) has been employed and in order to obtain the superconvergence at the grid points the grading exponents of order $O(1/(1-\alpha))$ have been used. However, it has been noted that the use of graded meshes, with the grading exponent of $O(1/(1-\alpha))$, has serve drawback which will restrict its applicability (see, e.g. [3]). Ideally we want to obtain the highest order of accuracy with the smallest (and practical) grading exponent: this is what we will do in this work for Volterra integral-differential equations.

Collocation methods for (1.1) were described in [1]. The methods generate, as approximations to the solution of (1.1), elements of the polynomial spline space

$$S_m^{(0)}(Z_N) := \{u \in C(I) : \quad u|_{\sigma_n} := u_n \in \pi_m, 0 \leqq n \leqq N-1\}, \tag{1.3}$$

associated with a given partition (or mesh sequence) $\Pi_N$: $0 = t_0 < t_1 < \ldots < t_N = T,\ N \geqq 1$, of the interval $I$. Here, $\pi_m$ is the set of (real) polynomials of degree not exceeding $m$ (with $m \geqq 1$), and we have set $\sigma_n := [t_n, t_{n+1}] (0 \leqq n \leqq N-1)$, and $Z_N := \{t_n : 1 \leqq n \leqq N-1\}$ (the set of interior grid points). The quantity $h$, $h := \max\{h_n := t_{n+1} - t_n,\ 0 \leqq n \leqq N-1\}$, is often called the diameter of the grid $\Pi_N$. The desired approximation to $y$ is the element $u \in S_m^{(0)}(Z_N)$ satisfying

$$u'(t) = f(t, u(t)) + \int_0^t (t-s)^{-\alpha} k(t, s, u(s)) ds, \quad t \in X(N), \tag{1.4}$$

with $X(N) := \bigcup_{n=0}^{N-1} \{t_{n,j} := t_n + c_j h_n : 0 \leqq c_1 < \ldots < c_m \leqq 1\}$. In other words, the collocation solution for (1.1) is the element in the polynomial spline space $S_m^{(0)}(Z_N)$ which solves (1.1) on the set of collocation points $X(N)$. This set is characterized, for a given partition $\Pi_N$, by the collocation parameters $\{c_j\}$. For the linearized VIDE (1.2), the collocation equation (1.4) can be written as

$$u'(t) = a(t)u(t) + b(t) + \int_0^t (t-s)^{-\alpha} K(t, s) u(s) ds, \quad t \in X(N). \tag{1.5}$$

If the mesh points $\{t_n\}_{n=0}^N$ are given by

$$t_n := \left(\frac{n}{N}\right)^r T, \quad (0 \leqq n \leqq N) \tag{1.6}$$

where $r > 1$, then $\Pi_N$ is called a graded mesh. If $r = 1$ then $\Pi_N$ is called a uniform mesh sequence.

The remainder of the paper consists of three sections. Section 2 is devoted to the main results of this paper and their proofs are given in Sect. 3. The final section presents one numerical example with the aim of illustrating the convergence behaviour of the method.

## 2 Main results

The main results of the present work are given by two theorems. The first one given the global superconvergence properties of the collocation solution $u \in S_m^{(0)}(Z_N)$.

**Theorem 1.** *Let the functions* $a$, $b$ *and* $K$ *in* (1.2) *be* $(m+1)$*-times continuously differentiable functions, and assume that* $b$ *and* $K$ *do not vanish identically. Assume that* $u \in S_m^{(0)}(Z_N)$ *is the collocation approximation defined by* (1.5)*, with collocation parameters* $\{c_j\}$ *such that*

$$J := \int_0^1 \prod_{i=1}^{m} (s - c_i) ds = 0. \tag{2.1}$$

*If one employs graded grids* (1.6) *with grading exponent* $r = (m+1-\alpha)/(2-\alpha)$*, then the resulting error* $e = y - u$ *satisfies*

$$\|e\|_\infty = O(N^{-(m+1-\alpha)}). \tag{2.2}$$

The second theorem gives the discrete (or local) convergence properties of the derivative of the collocation solution $u \in S_m^{(0)}(Z_N)$.

**Theorem 2.** *Let the function* $a$, $b$ *and* $K$ *in* (1.2) *be subject to the conditions stated in Theorem* 1*. Assume that* $u \in S_m^{(0)}(Z_N)$ *is the collocation approximation defined by* (1.5)*, with collocation parameters* $\{c_j\}$ *satisfying* (2.1)*, i.e.* $J = 0$*. If one employs graded grids* (1.6) *with grading exponent* $r = (m+1-\alpha)/(2-\alpha)$*, then the resulting error* $e = y - u$ *satisfies*

$$\max_{(n,j)} |e'(t_{nj})| = O(N^{-(m+1-\alpha)}). \tag{2.3}$$

*Remark 1.* The results of Theorems 1 and 2 also hold for $r > (m+1-\alpha)/(2-\alpha)$. However, numerical calculations suggest that in the range $[(m+1-\alpha)/(2-\alpha), \infty)$ smaller values of $r$ will give more accurate approximations. This will be illustrated in the final section. One of the reasons for this is that the numerical accuracy depends also on $h$, the diameter of $\Pi_N$. For two graded meshes, if the convergence orders are same then larger values of $h$ will in general give worse numerical solutions. It follows from (1.6) that $h \approx rT/N$ which suggests that the smallest $r$ in $[(m+1-\alpha)/(2-\alpha), \infty)$ should be chosen in the practical calculations. Further, the value of $r$ given in Theorems 1 and 2 is not greater than $m$ and is therefore capable of producing accurate results even when $1-\alpha$ is very small.

*Remark 2.* The corresponding results of Theorems 1 and 2 for nonlinear equations (1.1) and (1.4) can be obtained by using the technique introduced in [2] and [4]. The collocation error induced by the collocation solution $u$, the solution of (1.4), can be regarded as the solution of the initial-value problem

(2.4)
$$e'(t) = A(t)e(t) + \delta(t) + \int_0^t (t-s)^{-\alpha} H(t,s)e(s)ds + (Qe)(t), \ \ t \in I, \ e(0) = 0,$$

with $A(t) := f_y(t, y(t)), \ H(t,s) := k_z(t, s, y(s))$, and

$$(2.5) \qquad (Qe)(t) := -\frac{1}{2} f_{yy}(t, \cdot)e^2(t) - \frac{1}{2}\int_0^t (t-s)^{-\alpha} k_{zz}(t, s, \cdot)e^2(s)ds.$$

In the last expression the unspecified arguments represent intermediate values arising in the respective remainder terms of Taylor's formula. In order to estimate the perturbation term $(Qe)(t)$ we need the following results ([7]) which will also play an important role in the proofs of Theorems 1 and 2.

**Lemma 1.** *Let the functions $a$, $b$ and $K$ in* (1.2) *be $m$-times continuously differentiable functions, and assume that $b$ and $K$ do not vanish identically. If $u \in S_m^{(0)}(Z_N)$ is the collocation approximation defined by* (1.5)*, and if the underlying mesh sequence $\Pi_N$ consists of graded meshes of the form* (1.6)*, with grading exponent $r$, $r > m/(2-\alpha)$, then for all collocation parameters $\{c_j\}$ with $0 \leqq c_1 < \ldots < c_m \leqq 1$, the resulting error $e := y - u$ satisfies*

$$(2.6) \qquad (i) \quad \|e\|_\infty = O(N^{-m}),$$

$$(2.7) \qquad (ii) \quad e_n'(t) = O\left(N^{-m}\left(\frac{N}{n}\right)^{m-r(1-\alpha)}\right), \quad (1 \leqq n \leqq N-1),$$

$$(2.8) \qquad (iii) \quad e_0'(t) = O(N^{-r(1-\alpha)}),$$

*where $e_n$ denotes the restriction of $e$ to the subinterval $\sigma_n$, $0 \leqq n \leqq N-1$.*

Similar results of Lemma 1 hold for the nonlinear case if $f$ and $k$ are sufficiently smooth. Since $e(t) = O(N^{-m})$ we can see from (2.5) that $(Qe)(t) = O(N^{-2m})$ provided that the grading exponents $r$ in (1.6) satisfies $r > m/(2-\alpha)$. The small error term, $O(N^{-2m})$, will not change the convergence order $m + 1 - \alpha$ obtained in Theorem 1 and 2.

*Remark 3.* The integrated form of (1.1) or (1.2) is a second-kind Volterra integral equation whose kernel contains the convolution term $(t-s)^{1-\alpha}$. It is known that optimal order of convergence, $m$, is obtained (when using splines of degree $m-1$) if the grading exponent is $m/(2-\alpha)$ (cf.[3]). Therefore it is quite natural to have the results of Lemma 1. The result of Theorem 2 is also predictable from the above fact and from [4]. However, the result of Theorem 1 cannot be easily foreseen since the superconvergence property (2.3) holds in the whole interval $I$ rather than at the mesh points $\{t_n\}$ only.

## 3 Proofs

Throughout this paper $C$ denotes a positive constant which is independent of $N$ (or equivalently, the partition $\Pi_N$), but possibly with different values at different places. According to (1.5) the collocation approximation $u \in S_m^{(0)}(Z_N)$ of (1.2) satisfies the following "continuous" form

$$u'(t) = a(t)u(t) + b(t) + \int_0^t (t-s)^{-\alpha} K(t,s)u(s)ds - \delta(t), \quad t \in I, \tag{3.1}$$

with $\delta(0) = 0$; here, the residual function $\delta(t)$ vanishes on the set $X(N)$ of collocation points. If we now subtract (3.1) from (1.2) we are led to

$$e'(t) = a(t)e(t) + \int_0^t (t-s)^{-\alpha} K(t,s)e(s)ds + \delta(t), \quad t \in I, e(0) = 0. \tag{3.2}$$

The initial-value problem for (3.2) is equivalent to the following equation

$$\tilde{e}'(t) = \int_0^t (t-s)^{-\alpha} \widetilde{K}(t,s)\tilde{e}(s)ds + \tilde{\delta}(t), \quad t \in I, \tilde{e}(0) = 0, \tag{3.3}$$

where $\tilde{e}(t) := A(t)(t)$, $\widetilde{K}(t,s) := A(t)K(t,s)$, $\tilde{\delta}(t) := A(t)\delta(t)$ and $A(t) := \exp(-\int_0^t a(\theta)d\theta)$. This suggests that there is no loss of generality by taking $a(t) \equiv 0$ in (3.2), and we need only to consider the VIDE (3.3) in our error analysis. Further for easy of notations, we still use $e, K$ and $\delta$ instead of using $\tilde{e}, \widetilde{K}$ and $\tilde{\delta}$. Namely, we will restrict our discussion to the following equation,

$$e'(t) = \int_0^t (t-s)^{-\alpha} K(t,s)e(s)ds + \delta(t), \quad t \in I, e(0) = 0. \tag{3.4}$$

And the error function $e$ in (3.4) and its derivative $e'$ satisfy the estimates given by Lemma 1. It can be shown that (3.4) is equivalent to the following equation

$$e'(t) = \int_0^t (t-s)^{1-\alpha} K_1(t,s)e'(s)ds + \delta(t), \quad t \in I, e(0) = 0. \tag{3.5}$$

where

$$K_1(t,s) := \int_0^1 (1-\theta)^{-\alpha} K(t, \theta t + (1-\theta)s)d\theta. \tag{3.6}$$

The standard theory for the integral equation of the second-kind (3.5) gives that (see, e.g. [2])

$$e'(t) = \delta(t) + \int_0^t (t-s)^{-\alpha} Q(t,s)\delta(s)ds, \quad t \in I, e(0) = 0, \tag{3.7}$$

where the kernel $(t-s)^{-\alpha}Q(t,s)$ is the so-called resolvent kernel of $(t-s)^{1-\alpha}K_1(t,s)$ and it can be shown that if $K_1 \in C^1(S)$ then $Q \in C^1(S)$, where $S := \{(t,s); 0 \leqq s < t \leqq T\}$. Integrating (3.7) and using Dirichlet's formula we obtain

$$e(t) = \int_0^t \delta(s)ds + \int_0^t (t-s)^{1-\alpha} Q_1(t,s)\delta(s)ds, \quad t \in I, \tag{3.8}$$

where

$$Q_1(t,s) := \int_0^1 \theta^{-\alpha} Q(\theta t + (1-\theta)s, s)d\theta. \tag{3.9}$$

From now on we assume that the functions $a, b$ and $K$ be $m+1$ times continuously differentiable. Then the exact solution $y$ of (1.2) is in $C^{m+2}((0,T])$, $K_1$ in (3.6) is in $C^{m+1}(S)$, and $Q$ in (3.7) and $Q_1$ in (3.9) are both in $C^1(S)$. However, the regularity properties of the function $\delta$ in (3.4) may give difficulties in the error analysis, and we need to consider it in the following way. Namely, for $t \in \sigma_n = [t_n, t_{n+1}]$ and $1 \leqq n \leqq N-1$, we set

$$\delta(t) := \delta_1(t) - \delta_2(t), \tag{3.10}$$

where

$$\delta_2(t) := \int_{t_{n-1}}^t (t-s)^{-\alpha} K(t,s)e(s)ds. \tag{3.11}$$

From (3.4) and (3.11) we have

$$\delta_1(t) = e'(t) - \int_0^{t_{n-1}} (t-s)^{-\alpha} K(t,s)e(s)ds, \quad t \in \sigma_n, n \geqq 1. \tag{3.12}$$

From (3.4) and Lemma 1 we find that $\delta(t) = O(N^{-r(1-\alpha)})$ if $t \in [0, t_1]$. Using Lemma 1 and the arguments similar to those used in [7] we can obtain the following results.

**Lemma 2.** *For $t \in \sigma_n$, $1 \leqq n \leqq N-1$, the functions $\delta_1$ and $\delta_2$ satisfy that*

$$\text{(i)} \quad |\delta_1(t)| = O\left(N^{-m} + N^{-m}\left(\frac{N}{n}\right)^{m-r(1-\alpha)}\right), \tag{3.13}$$

$$\text{(ii)} \quad |\delta_1^{(m+1)}(t)| = O(h_n^{-\alpha-m}N^{-m} + |y^{(m+2)}(t)|), \tag{3.14}$$

$$\text{(iii)} \quad \max\{\delta_2(t) : t \in [t_1, T]\} = O(N^{-(m+1-\alpha)}). \tag{3.15}$$

**Lemma 3.** *For $1 \leqq i \leqq n-1$ and $t \geqq t_{i+1}$, we have*

$$\int_{t_i}^{t_{i+1}} \Big((t-s)^{1-\alpha} Q_1(t,s) - (t-t_i)^{1-\alpha} Q_1(t,t_i)\Big)ds = O(h_i^{2-\alpha}). \tag{3.16}$$

**Lemma 4.** *For* $t \in \sigma_n, 1 \leqq n \leqq N-1$,

$$h_n^{m+2}\|y^{(m+2)}\|_{\sigma_n} = O\left(N^{-(m+2)+\gamma}n^{-\gamma}\right), \quad \gamma := -r(2-\alpha)+m+2. \tag{3.17}$$

*Proof of Theorem 1.* Noting that $\delta(t) = O(N^{-r(1-\alpha)})$ if $t \in [0, t_1]$, we have

$$\int_0^{t_1} |\delta(s)|ds = O(N^{-r(2-\alpha)}) = O(N^{-(m+1-\alpha)}). \tag{3.18}$$

This, together with (3.8), (3.10) and Lemma 1, yields that

$$\begin{aligned} e(t) =& \sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}} \delta_1(s)ds + \sum_{i=1}^{n-1}\int_{t_i}^{t_{i-1}} (t-s)^{1-\alpha}Q_1(t,s)\delta_1(s)ds \\ &+ O\left(\int_{t_n}^{t} \delta_1(s)ds\right) + O(N^{-(m+1-\alpha)}), \quad t \in \sigma_n,\ 1 \leqq n \leqq N-1. \end{aligned} \tag{3.19}$$

For graded grids (1.6) we have $h_n = O(N^{-r}n^{r-1})$, which together with (3.13) and $r = (m+1-\alpha)/(2-\alpha)$, yields that

$$\int_{t_n}^{t} \delta_1(s)ds = O(h_n\|\delta_1\|_{\sigma_n}) = O(N^{-(m+1-\alpha)}), \quad t \in \sigma_n,\ 1 \leqq n \leqq N-1. \tag{3.20}$$

Further, since $\delta_1$ is a piecewise smooth function, the Lagrange interpolation formula gives that

$$\begin{aligned} \delta_1(t_i + vh_i) =& \sum_{j=1}^{m} L_j(v)\delta_1(t_i + c_jh_i) + \delta_1^{(m)}(t_i)h_i^m \prod_{j=1}^{m}(v-c_j)/m! \\ &+\delta_1^{(m+1)}(t_i+\theta_2h_i)\theta_1h_i^{m+1}\prod_{j=1}^{m}(v-c_j)/m! \\ &0 < v < 1, 1 \leqq i \leqq n-1, \end{aligned} \tag{3.21}$$

where $L_j(v)$ represents the $j$-th Lagrange fundamental polynomial associated with the $m$ collocation parameters; $0 < \theta_1 < \theta_2 < 1$ and both $\theta_1$ and $\theta_2$ depend on $v$. Since $\delta(t_i + c_jh_i) = 0$, (3.10) and (3.15) yield that $\delta_1(t_i + c_jh_i) = O(N^{-(m+1-\alpha)})$. Then we obtain from (3.21) that

(3.22)

$$\delta_1(t_i + vh_i) = O(N^{-(m+1-\alpha)}) + D_{m,i}\prod_{j=1}^{m}(v-c_j) + O(h_i^{m+1}\|\delta_1^{(m+1)}\|_{\sigma_i}),$$

where $D_{m,i}$ are constants independent of $v$. Since the collocation parameters $\{c_j\}$ such that $J = 0$, we have from (3.22) and Lemmas 2 and 4 that

$$\begin{aligned}(3.23)\qquad \int_{t_i}^{t_{i+1}} \delta_1(s)ds &= h_i \int_0^1 \delta_1(t_i + vh_i)dv \\ &= O(N^{-(m+2-\alpha)}) + O(h_i^{m+2}\|\delta_1^{(m+1)}\|_{\sigma_i}) \\ &= O(N^{-(m+2-\alpha)} + h_i^{2-\alpha}N^{-m} + N^{-(m+2)+\gamma}i^{-\gamma}), \\ \gamma &= -r(2-\alpha) + m + 2.\end{aligned}$$

Under the hypothesis that $r = (m+1-\alpha)/(2-\alpha)$, we have $\gamma = 1 + \alpha$. Now if we go back to look at (3.19), then for $t \in \sigma_n, n \geqq 1$, it can be rewritten as

$$\begin{aligned}e(t) = &\sum_{i=1}^{n-1}(1 + (t-t_i)^{1-\alpha}Q_1(t,t_i)) \int_{t_i}^{t_{i+1}} \delta_1(s)ds \\ &+\sum_{i=1}^{n-1}\int_{t_i}^{t_{i+1}} ((t-s)^{1-\alpha}Q_1(t,s) - (t-t_i)^{1-\alpha}Q_1(t,t_i))\delta_1(s)ds \\ &+O(N^{-(m+1-\alpha)}),\end{aligned}$$

here we have made use of (3.20) and the fact that $r(2-\alpha) = m+1-\alpha$. Then Lemmas 1 and 3 and (3.23) yield that

$$\begin{aligned}|e(t)| \leqq\ &C\sum_{i=1}^{n-1}(N^{-(m+2-\alpha)} + h_i^{2-\alpha}N^{-m} + N^{-(m+1-\alpha)}i^{-(1+\alpha)}) \\ (3.24)\qquad &+C\sum_{i=1}^{n-1}\left(h_i^{2-\alpha}N^{-m} + N^{-m}h_i^{2-\alpha}\left(\frac{i}{N}\right)^{r(1-\alpha)-m}\right) + CN^{-(m+1-\alpha)}.\end{aligned}$$

Since $h_i = O(N^{-r}i^{r-1})$ and $r = (m+1-\alpha)/(2-\alpha)$ we can show that

$$(3.25)\qquad \sum_{i=1}^{n-1} N^{-m}h_i^{2-\alpha}\left(\frac{i}{N}\right)^{r(1-\alpha)-m} = O(N^{-(m+1-\alpha)}).$$

Using (3.24) and (3.25), together with $\|e\|_{\sigma_0} = O(h_0^{2-\alpha}) = O(N^{-r(2-\alpha)}) = O(N^{-(m+1-\alpha)})$, we readily deduce (2.2) $\square$

*Proof of Theorem 2.* It follows from the standard theory of the second-kind integral equation (see, e.g. [2]) that $e'(t)$ in (3.5) must also satisfy the equation

$$(3.26)\qquad e'(t) = \delta(t) + \int_0^t (t-s)^{1-\alpha}Q_2(t,s)\delta(s)ds,$$

where $Q_2 \in C^1(S)$. If $t = t_{nj}$ is a collocation point, then $\delta(t) = 0$, and (3.26) gives that

$$e'(t_{nj}) = \int_0^{t_{nj}} (t_{nj} - s)^{1-\alpha} Q_3(t_{nj}, s)\delta(s)ds. \tag{3.27}$$

Then (2.3) follows by employing the same proof technique used in the proof of Theorem 1. □

## 4 Numerical exmaple

For numerical verification of Theorems 1 and 2 we consider the following example

$$y'(t) = a(t)y(t) + b(t) + \int_0^t \lambda(t-s)^{-\alpha} y(s)ds, \quad y(0) = 0, t \in [0,1], \tag{4.1}$$

with $a(t) \equiv -1, \lambda = -1$ and $\alpha = 1/2$, and with $b(t)$ chosen so that $y(t) = t^{2-\alpha}$. This equation was tested in [1] in which a detailed description of the discretization of the collocation equations (1.4) and (1.5) was provided.

The grading exponents for the graded grids (1.6) are $r = r_1 = 1$ (uniform mesh), $r = r_2 = (m+1-\alpha)/(2-\alpha)$ (given by Theorems 1 and 2) and $r = r_3 = m/(1-\alpha)$ (given by [1]). The numerical results are obtained with $m = 3$. Since in many practical applications one is more interested in generating an accurate approximation at the right endpoint $t = T$ of the interval of integration, we first investigate the convergence rate at $t = T$. In Tables 1 and 2 we list the collocation error and its derivative obtained by using Gauss points and Labatto points respectively, at the right endpoint $t = t_N = 1$. As we expected, the collocation approximation with uniform meshes is less accurate at $t = t_N$ than the one where the underlying meshes are the properly graded ones; and the collocation approximation obtained with $r = r_3$ is less accurate than the one with $r = r_3$ is less accurate than the one with $r = r_1$. In both tables, the numerical results for $e(t_N)$ in the columns corresponding to $r = r_2$ and $r = r_3$ reflect the convergence order $m+1-\alpha(= 3.5)$ which is consistent with the theoretical prediction of Theorem 1 . For both $r = r_2$ and $r_3$, Table 1 suggests that the convergence order for $e'(t_N)$ is $m = 3$ only (cf. [7]); while Table 2 gives the higher order $m + 1 - \alpha = 3.5$. This is consistent with the prediction of Theorem 2 since in the later case $t_N$ is a collocation point.

If $m > 2$, it is difficult to calculate the maximum values of the error function (i.e. $\|e\|_\infty$) numerically. In order to investigate the attainable order of (global) convergence, we will use an applicable method to estimate the maximum errors. The idea of the technique is illustrated in the analysis for Table 3 in which we list the maximum errors at certain points of $I$. The notations $\|e(t_n + d_j h_n)\|$ and $\|e'(t_n + c_j h_n)\|$ in Table 3 are defined by

$$\|e(t_n + d_j h_n)\| := \max\{|e(t_n + d_j h_n)| : 0 \leqq n \leqq N-1, 0 \leqq j \leqq m\} \tag{4.2}$$
$$\|e'(t_n + c_j h_n)\| := \max\{|e'(t_n + c_j h_n)| : 0 \leqq n \leqq N-1, 1 \leqq j \leqq m\} \tag{4.3}$$

**Table 1.** $m = 3$ and collocation at Gauss points.

| N | $r = r_1 = 1$ | | $r = r_2 \approx 2.33$ | | $r = r_3 = 6$ | |
|---|---|---|---|---|---|---|
| | $\|e(t_N)\|$ | $\|e'(t_N)\|$ | $\|e(t_N)\|$ | $\|e'(t_N)\|$ | $\|e(t_N)\|$ | $\|e'(t_N)\|$ |
| 10 | 5.19D-6 | 5.35D-5 | 5.41D-7 | 6.69D-5 | 9.42D-6 | 9.56D-4 |
| 20 | 1.75D-6 | 1.80D-5 | 4.69D-8 | 8.03D-6 | 8.55D-7 | 1.28D-4 |
| 40 | 6.09D-7 | 6.27D-6 | 4.11D-9 | 9.78D-7 | 7.58D-8 | 1.62D-5 |
| 80 | 2.13D-7 | 2.21D-6 | 3.61D-10 | 1.20D-7 | 6.70D-9 | 2.03D-6 |

**Table 2.** $m = 3$ and collocation at Labatto points.

| N | $r = r_1 = 1$ | | $r = r_2 \approx 2.33$ | | $r = r_3 = 6$ | |
|---|---|---|---|---|---|---|
| | $\|e(t_N)\|$ | $\|e'(t_N)\|$ | $\|e(t_N)\|$ | $\|e'(t_N)\|$ | $\|e(t_N)\|$ | $\|e'(t_N)\|$ |
| 10 | 6.22D-5 | 5.45D-4 | 4.30D-6 | 1.48D-5 | 6.58D-5 | 1.10D-4 |
| 20 | 2.04D-5 | 1.98D-4 | 3.58D-7 | 1.44D-6 | 5.12D-6 | 1.28D-5 |
| 40 | 6.99D-6 | 7.06D-5 | 3.04D-8 | 1.34D-7 | 4.14D-7 | 1.27D-6 |
| 80 | 2.44D-6 | 2.51D-5 | 2.61D-9 | 1.23D-8 | 3.47D-8 | 1.20D-7 |

**Table 3.** $m = 3$ and collocation at Gauss points.

| N | $r = (m + 1 - \alpha)/(2 - \alpha) \approx 2.33$ | | $r = m/(1 - \alpha) = 6$ | |
|---|---|---|---|---|
| | $\|e(t_n) + d_j h_n\|$ | $\|e'(t_n + c_j h_n)\|$ | $\|e(t_n) + d_j h_n\|$ | $\|e'(t_n + c_j h_n)\|$ |
| 10 | 3.66D-6 | 4.62D-6 | 3.01D-5 | 7.87D-5 |
| 20 | 3.25D-7 | 3.93D-7 | 2.47D-6 | 7.61D-6 |
| 40 | 2.87D-8 | 3.35D-8 | 1.92D-7 | 6.82D-7 |
| 80 | 2.54D-9 | 2.88D-9 | 1.46D-8 | 5.95D-8 |

where $\{c_j\}_{j=1}^m$ are the collocation parameters which are the Gauss points (the error analysis for the numerical results obtained by using Labatto points can be carried out in a similar way); and $\{d_j\}_{j=0}^m$ are given $m + 1$ parameters which are assumed to be $d_0 = 0$ and $d_j = c_j$ for $1 \leqq j \leqq m$. It can be seen from Table 3 that the convergence orders of $\|e'(t_n + c_j h_n)\|$, for the numerical approximations obtained by using graded grids with $r = r_2$ and $r = r_3$, is $m + 1 - \alpha (= 3.5)$. This confirms the prediction of Theorem 2. Moreover, the numerical results of Table 3 also imply a global convergence order for $\|e\|_\infty$, $m + 1 - \alpha$, as predicted in Theorem 1. This can be seen from the following illustration. Since the error function $e$ is a piecewise smooth function, the Lagrange interpolation formula gives, for $0 \leqq n \leqq N - 1$ and $0 < v < 1$, that

$$e(t_n + vh_n) = \sum_{j=0}^{m} L_j(v) e(t_n + d_j h_n) + R_n(v),$$

with

$$R_n(v) := y^{(m+1)}(t_n + \theta_n h_n) h_n^{m+1} \prod_{j=0}^{m} (v - d_j)/(m+1)!, \quad 0 < \theta_n < 1,$$

where $L_j(v)$ represents the $j$-th Lagrange fundamental polynomial associated with the $m+1$ parameters $\{d_j\}$. Using the similar techniques of [7], we can show that $R_n(v) = O(h_n^{m+1}\|y^{m+1}\|_{\sigma_n}) = O(N^{-(m+1-\alpha)})$. Hence

$$|e(t_n + vh_n)| \leqq C\|e(t_n + d_j h_n)\| + O\Big(N^{-(m+1-\alpha)}\Big). \qquad (4.4)$$
$$0 \leqq n \leqq N-1, \quad 0 < v < 1.$$

Table 3 shows that, when $d_0 = 0$ and $\{d_j\}_{j=1}^m$ are the collocation parameters (i.e., the Gauss points), $\|e(t_n + d_j h_n)\| = O(N^{-(m+1-\alpha)})$ which holds for all $0 \leqq n \leqq N-1$. This and (4.4) imply that $\|e\|_\infty = O(N^{-(m+1-\alpha)})$, which confirms the theoretical estimate (2.2).

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