Maximum Norm Error Estimates for the Finite Element Method Allowing Highly Refined Grids

Alfred H. Schatz

ABSTRACT. In this article we shall survey four papers [1,3–5], two of which ([3, 4]) are as yet unpublished, that are concerned with global L_∞ error estimates for the finite element method for elliptic problems on plane polygonal domains.

1. Introduction

The papers [1, 3–5] represent two different points of view concerning grid refinements. Before discussing this in more detail, let us formulate a model problem that we shall consider here. A generalization will be mentioned later on.

For simplicity, let Ω be a polygonal domain, $\partial\Omega$ its boundary, with vertices at $v_j = (x_{1j}, x_{2j})$, $j = 1, \ldots, m$, and interior angles

$$0 < \alpha_1 \le \alpha_2 \cdots \le \alpha_m \le 2\pi. \tag{1.1}$$

We will allow Ω to be nonconvex except for one of the papers in which Ω is required to be convex, i.e., $\alpha_m \le \pi$. Consider the model Dirichlet problem

$$-\Delta u = f \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega, \tag{1.2}$$

or in weak form, for $u \in \overset{\circ}{W}{}_2^1$

$$\int_\Omega \nabla u \nabla v dx = \int_\Omega f v dx, \text{ for all } v \in \overset{\circ}{W}{}_2^1(\Omega). \tag{1.3}$$

It is well known [2] that, even if f is smooth on $\overline{\Omega}$, then u may not be smooth near the vertices. For simplicity let $f \in C^\infty(\overline{\Omega})$, then for $x \in \Omega$ and near the vertex v_j the solution is known to satisfy [2]

$$|D^\gamma u| \le C r_j^{\beta_j - |\gamma|},\ |\gamma| = 1, 2 \text{ when } \frac{1}{2} \le \beta_j = \frac{\pi}{\alpha_j} < 2, \tag{1.4}$$

and where $r_j = |x - v_j|$, and $v_j = (x_{1j}, x_{2j})$.

We wish to discuss the finite element method error in $L_\infty(\Omega)$ when applied to the problem (1.2) under various conditions on the finite element spaces. Let $M_{\overline{h}}$, $0 < \overline{h} < \frac{1}{2}$, be a one parameter family of meshes that triangulate Ω. We assume

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that the triangulations are shape regular and edge to edge. For any triangle τ in any of the triangulations we set $h_\tau = \sqrt{\text{Area}(\tau)}$, $\overline{h} = \max_{\tau \in M_{\overline{h}}} h_\tau$, and $\underline{h} = \min_{\tau \in M_{\overline{h}}} h_\tau$. For any x that is an interior part of τ we set $h(x) = h_\tau$. If x is on the boundary of some triangle we set $h(x) = h_\tau$ where h_τ is the mesh size of any triangle containing x on its boundary.

For each $\overline{h}$ we take the finite element space $\overset{\circ}{S}_{\overline{h}}(\Omega)$ to be, for simplicity, the set of piecewise linear functions relative to the mesh $M_{\overline{h}}$. Higher order elements will be discussed in Section 5. Additional requirements will be placed on the mesh in each of the three papers to be considered. Under the assumption made so far, the finite element space $\overset{\circ}{S}_{\overline{h}}$ has many well known properties including the following approximation property.

P.1. Let τ be an arbitrary triangle in a triangulation $M_{\overline{h}}$, then for each $u \in W^s_\infty(\tau)$, $0 < s \le 2$, u_I the interpolant satisfies

$$\|u - u_I\|_{L_\infty(\tau)} \le C h^s_\tau \|u\|_{W^s_\infty(\tau)}. \tag{1.5}$$

The finite element approximation $u_h \in \overset{\circ}{S}_{\overline{h}}$ of u satisfies

$$\int_\Omega \nabla u_h \nabla \varphi dx = \int_\Omega f \varphi dx, \text{ for all } \varphi \in \overset{\circ}{S}_{\overline{h}} \tag{1.6}$$

or

$$\int_\Omega \nabla(u - u_h) \nabla \varphi dx = 0, \text{ for all } \varphi \in \overset{\circ}{S}_{\overline{h}}. \tag{1.7}$$

An outline of this article is as follows: In Section 2 we review the main result in [5]. In a standard way, we use the approximation property (P.1) and the a priori known bounds [2] on the derivatives, to construct a mesh for which the piecewise linear interpolant approximates the solution to optimal order $\overline{h}^2$ in L_∞. Here, roughly speaking, the mesh is quasi–uniform of size $\overline{h} = \max_\tau h_\tau$ a unit distance away from corners where u is smooth. This a priori given mesh was analyzed in [5] for the finite element L_∞ error for the solution of (1.7).

In Section 3 we discuss a part of the work in [1]. This paper takes a different point of view from that in [5]. Roughly speaking, instead of prescribing the mesh size a priori, it gives conditions that essentially limit how quickly the mesh size can change. The conditions given there allow highly refined grids for which the error in $L_\infty(\Omega)$ is essentially proportional to the interpolation error. This includes as a special case the mesh distribution discussed in Section 2 when Ω is convex. The requirement of convexity is the main limitation of this work. We mention that the result given there is used to analyze the error for meshes that are given a posteriori via a certain self adaptive algorithm. Section 4 contains a discussion of part of some new work [3]. Here we treat the case of nonconvex polygons. Under slightly different conditions on the mesh than in [1], which also allows highly refined grids, it is shown that the finite element error in $L_\infty(\Omega)$ is almost proportional to the interpolation error. In Section 5 we discuss the results for a generalization of problem (1.2). This includes problems with discontinuous coefficients. The proofs for the results in Sections 4 and 5 are obtained by gluing together local estimates. They are lengthy and technical and will be published elsewhere.

2. An optimal order grid given a priori

In this section we shall discuss part of the results in [5]. Our first aim here is to describe the choice of a mesh which is designed, using the properties P.1 of the solution and the approximation properties (1.5) of the piecewise linear finite elements to produce an interpolant with an optimal second order accuracy h^2.

Consider N_j, a unit sized sector of Ω centered at the vertex v_j for which

$$\frac{\pi}{\alpha_j} = \beta_j < 2. \tag{2.1}$$

Outside of these N_j, the solution is smooth and we choose a quasi–uniform mesh of size $\overline{h}$. Then

$$\|u - u_I\|_{L_\infty(\Omega/UN_j)} \leq C\overline{h}^2 \|u\|_{W^2_\infty(\Omega/UN_j)}. \tag{2.2}$$

Now consider any N_j (satisfying (2.1)) and consider any triangle τ_d, a distance d away from v_j. We wish to choose h_{τ_d}, the mesh size of τ_d, so that the interpolation error is order $\overline{h}^2$ in the maximum norm.

In view of approximation property (1.5), and the behavior (1.4) of the solution, this is satisfied if

$$\|u - u_I\|_{L_\infty(\tau_d)} \leq Ch^2_{\tau_d}\|u\|_{W^2_\infty(\tau_d)} \leq Ch^2_{\tau_d} d^{\beta_j - 2} \leq C\overline{h}^2. \tag{2.3}$$

This in turn is satisfied (modulo multiplicative constants) if h_{τ_d} is chosen so that

$$h_{\tau_d} = \overline{h} d^{1-\frac{\beta_j}{2}}, \ d > 0. \tag{2.4}$$

If $d = 0$, then since $u \in C^{\beta_j}$ in N_j, it follows from (1.5) that we wish to have

$$\|u - u_I\|_{L_\infty(\tau_d)} \leq Ch^{\beta_j}_{\tau_d} \leq C\overline{h}^2.$$

This is satisfied with the choice

$$h_{\tau_d} = (\overline{h})^{2/\beta_j}, \ d = 0. \tag{2.5}$$

The choices (2.4) and (2.5) motivate the choice of mesh with order $\overline{h}^2$ convergence. Let us summarize this choice. To begin with choose a quasi–uniform mesh of size $\overline{h}$ outside of the (maximal mesh size) unit sized sectors N_j satisfying (2.1). To constuct the mesh on the N_j we start with $d = 1$ and construct a mesh of size h_{τ_d} satisfying (2.4) for decreasing d until $d = \overline{h}^{2/\beta_j}$ where $h_{\tau_d} = \overline{h}^{2/\beta_j}$. The mesh is then completed with a quasi–uniform mesh of size $\overline{h}^{2/\beta_j}$ in a $\overline{h}^{2/\beta_j}$ neighborhood of v_j. The resulting mesh has the property that

$$\|u - u_I\|_{L_\infty(\Omega)} \leq C\overline{h}^2. \tag{2.6}$$

We now turn to the finite element error $u - u_h$ satisfying (1.7) and ask whether the estimate (2.6) holds for that? An answer given in [5] is yes, almost.

THEOREM 1. *[5] Let $\overset{\circ}{S}_{\overline{h}}$ be given as above and let $u - u_h$ satisfy (1.7) where u is a solution of (1.3). Then for each given $\varepsilon > 0$ there exists a constant $C(\varepsilon)$, independent of h, u and u_h such that*

$$\|u - u_h\|_{L_\infty(\Omega)} \leq C(\varepsilon)\overline{h}^{2-\varepsilon}.$$

For a more general treatment of this problem see [5].

3. $\mathbf{L}_\infty$ estimates for a class of meshes on convex polygons allowing highly refined grids

In this section we shall discuss part of the work [1]. Instead of prescribing a particular mesh, as in Section 2, conditions are given that define a rich family of meshes, on which the finite element method error in L_∞ behaves (almost) like the interpolation error. There is, however, a limitation to this work, namely the domain Ω is required to be convex, i.e., $\alpha_m < \pi$. The reason for this is the need, in the proof of L_∞ estimates, for $W_2^2(\Omega)$ regularity for the solutions of (1.3). We only mention here that the estimates are used to analyze a self adaptive scheme that controls the error $\|u - u_h\|_{L_\infty(\Omega)}$. For details see [1].

In order to state the result in [1] we shall need some restrictions on the mesh.

A.1. The mesh is shape regular and edge to edge.

A.2. There exist a function $h(x) \in W_\infty^1(\Omega)$ such that for any triangle τ and any $x \in \tau$ there exists a constant $0 < C \leq 1$ such that

$$Ch_\tau \leq h(x) \leq h_\tau. \tag{3.1}$$

A.3. Furthermore, for some ε sufficiently small

$$|\nabla h(x)| \leq \frac{\varepsilon}{\ln \frac{1}{\underline{h}}}, \quad \underline{h} = \min_{x \in \Omega} h(x). \tag{3.2}$$

Theorem 2. *(Eriksson [1]) Suppose the mesh satisfies A.1–A.3, Ω is convex, and $u - u_h$ satisfies (1.7). Then*

$$\|u - u_h\|_{L_\infty(\Omega)} \leq C\Big(\ln \frac{1}{h}\Big)^{3/2} \min_{\chi \in \overset{\circ}{S}_{\overline{h}}} \|h(x)\nabla(u - x)\|_{L_\infty(\Omega)}. \tag{3.3}$$

Remark. Eriksson mentions that the exponent 3/2 may be replaced by 1. He also shows that the optimal L_∞ mesh discussed in Section 2 satisfies the conditions of Theorem 2.

Remark. Obviously the estimate (3.3) is almost optimal.

Remark. One of the advantages of specifying a mesh as in Theorem 2 is its applicability to analyzing meshes that are created adaptively.

4. $\mathbf{L}_\infty$ error estimates on nonconvex polygons

In this section we shall first describe a special case of the results in [3], namely problem (1.3) on a nonconvex polygonal domain Ω. A generalization will then be briefly discussed in Section 5.

It is well known that near a vertex v_j with a non–convex corner having interior angle $\pi < \alpha_j$, the first derivatives of the solution may blow up like $r_j^{\beta_j - 1}$, where $\frac{1}{2} \leq \beta_j = \frac{\pi}{\alpha_j} < 1$. Thus we will treat neighborhoods of corners differently from the rest of the domain. Since our assumptions are a little different than those in Theorem 2, we shall need some preliminaries.

For $\rho > 0$, let

$$B(x, \rho) = \{y \in \Omega : |x - y| < \rho(x)\}.$$

We make the following assumptions regarding the mesh.

A.1. The mesh is shape regular and edge to edge.

A.4. There exists a constant $\gamma \geq 1$ such that if $\underline{h} = \min_{x\in\overline{\Omega}} h(x)$ and $\overline{h} = \max_{x\in\overline{\Omega}} h(x)$ then

$$\underline{h} \geq \overline{h}^{\gamma}. \tag{4.1}$$

A.5. (local quasi–uniformity) There exist a positive constant $\lambda \geq 1$ and a decreasing function $k(z) : (0, 1/2] \to [1, \infty]$ with $\lim_{z\to 0^+} k(z) = \infty$, such that for each $x \in \overline{\Omega}$ and all $y \in B(x, k(\overline{h})h(x))$

$$\frac{1}{\lambda} h(x) \leq h(y) \leq \lambda h(x). \tag{4.2}$$

Remark: A.5 can be replaced by a more complicated condition which leads to a slightly stronger result. This will be presented in [3].

Before stating our first result we shall need some notation.

For each vertex v_j consider the neighborhood $B(v_j, k(\overline{h})h(v_j))$, $j = 1, \dots, M$ and set

$$G_h = \Omega / \bigcup_{j=1}^{M} B(v_j, k(\overline{h})h(v_j)). \tag{4.3}$$

We remark that for f sufficiently smooth

$$u \in \begin{cases} W^1_{q_j} B(v_j, k(\overline{h})h(v_j)), \text{ for any } 2 \leq q_j < \frac{2}{1-\beta_j} \text{ when } \frac{1}{2} \leq \beta_j < 1 \\ \qquad\qquad\qquad\qquad \text{(nonconvex corner)} \\ W^{\beta_j}_{\infty} B(v_j, k(\overline{h})h(v_j)), \text{ for } 1 < \beta_j \qquad \text{(convex corner).} \end{cases} \tag{4.4}$$

Assuming that Ω is nonconvex, i.e., $\frac{1}{2} \leq \beta_M < 1$, it is well known that globally, $u \in W^2_q(\Omega)$ for any $1 < q < \frac{2}{2-\beta_M}$ and

$$\|u\|_{W^2_q(\Omega)} \leq C_q \|f\|_{L_q}, \ 1 < q < \frac{2}{2-\beta_M}, \tag{4.5}$$

where in general $C_q \to \infty$ as $q \to 1$.

We are now in a position to state a result.

THEOREM 3. *[3] Suppose that A.1, A.4, and A.5 are satisfied and that $u - u_h$ satisfies (1.7). Let $\varepsilon > 0$ be given, arbitrary but fixed. Then there exist positive constants C_ε, and σ (independent of h, u_h and u, and in addition σ is independent of ε) such that for all $\overline{h}$ sufficiently small*

(4.6)

$$\|u - u_h\|_{L_\infty(\Omega)} \leq C_\varepsilon h^{-\sigma\varepsilon} \Big(\big\|h\nabla(u-\chi)\big\|_{L_\infty(G_h)} + \sum_{j=1}^{n} \big\|\nabla(u-\chi)\big\|_{L_2(B(v_j, k(\overline{h})h(v_j)))} \Big)$$

for any $\chi \in S$.

Remark. We claim that this almost yields a quasi–optimal estimate (by taking ε small) with a loss of order of accuracy proportion to ε. Obviously the term $\|h\nabla(u-\chi)\|_{L_\infty(\tau_h)}$ is optimal and we need only consider the second term on the right in (4.6). Now using Hölder's inequality we arrive at

$$\sum_{j=1}^{m} \|\nabla(u-\chi)\|_{L_2(B(v_j, k(\overline{h})h(v_j)))} \leq C(k(\overline{h}))^{s_j} \|h^{s_j}(x)\nabla(u-\chi)\|_{L_{q_j}(v_j, k(\overline{h})h(v_j))}, \tag{4.7}$$

where we may take

$$S_j = \begin{cases} \beta_j - \varepsilon \ \text{ with } \ q_j = \frac{2}{1-\beta_j+\varepsilon}, & \text{for nonconvex corners } \frac{1}{2} \le \beta_j < 1 \\ \quad 1 \quad \text{ with } q_j = \infty, & \text{for convex corners } 1 \le \beta_j. \end{cases}$$

This is almost optimal provided $k(\overline{h}) \le Ch^{-\varepsilon}$. It is important to remark that a critical factor in the derivation of (4.7) is the fact that the Area$(B(v_j, k(\overline{h})h(v_j)) \le \pi(k(\overline{h})h(v_j))^2$.

5. A generalization, Dirichlet's problem with discontinuous coefficients

Let Ω be a simply connected polygonal domain in the plane that is the union of a finite number of polygonal domains Ω_k, $k = 1, \ldots, \ell$, i.e., $\overline{\Omega} = \bigcup_{k=1}^{\ell} \overline{\Omega}_k$. Let $a = a_k(x)$ on Ω_k, $k = 1, \ldots, \ell$ be a piecewise smooth function $0 < a_o \le a(x)$, and consider the boundary value problem for $u \in \overset{\circ}{W}{}_2^1(\Omega)$,

$$A(u, v) = \int_\Omega a \nabla u \nabla v dx = \int_\Omega f v dx, \ \forall \ v \in \overset{\circ}{W}{}_2^1(\Omega), \tag{5.1}$$

where for simplicity we shall assume that $f \in C^\infty(\Omega_k)$, $k = 1, \ldots, \ell$, (this regularity assumption can be greatly weakened). Call v_j a vertex if it is the vertex of some Ω_k, $k = 1, \ldots, \ell$, v_j, $j = 1, \ldots, m$.

Remark. (about regularity) $u \in C(\overline{\Omega})$ with possible singularities in derivatives at vertices. Furthermore, away from the vertices u is piecewise smooth, i.e., $u \in C^\infty(\overline{\Omega}_k/\text{vertices})$.

We make the following assumptions on the meshes and finite elements.

A.6. Ω is triangulated by a shape regular and edge to edge triangulation that also triangulates each Ω_k, $k = 1, \ldots, \ell$. Furthermore, each vertex v_j, $j = 1, \ldots, m$ coincides with a vertex of the triangulation. Relative to these triangulations we take our finite element spaces $\overset{\circ}{S}{}^r_{\overline{h}}(\Omega)$ to be the continuous functions on $\overline{\Omega}$, which vanish on $\partial\Omega$, and whose restrictions to each triangle are polynomials of degree $r - 1$ for $r \ge 2$.

A.4. Let $\max h(x) = \overline{h}$, then $\underline{h} = \min h(x) \ge h^\gamma$ for some $\gamma \ge 1$.

The next assumption is a special case of A5.

A.7. ("local" quasi–uniformity) There exists an $\varepsilon_0 > 0$ and constants $\lambda \ge 1$, $k \ge 1$ such that the following holds. For any given mesh M and any $x \in \overline{\Omega}$, then if $y \in B(x, k(h(x))^{1-\varepsilon_0})$

$$\frac{h(x)}{\lambda} \le h(y) \le \lambda h(x). \tag{5.2}$$

Remark. A.7 also holds for each $\varepsilon \le \varepsilon_0$.

Let $u_h \in \overset{\circ}{S}{}^r_{\overline{h}}(\Omega)$ be the unique solution of

$$A(u - u_h, \varphi) = 0 \quad \forall \ \varphi \in \overset{\circ}{S}{}^r_{\overline{h}}(\Omega). \tag{5.3}$$

For each vertex v_j and $0 < \varepsilon \leq \varepsilon_0$ set

$$N_j^\varepsilon = \{x \in \Omega : |x - v_j| \leq k(h(v_j))^{1-\varepsilon}\}$$
$$N^\varepsilon = \bigcup_{j=1}^{m} N_j^\varepsilon,\ G^\varepsilon = \Omega/N^\varepsilon$$

We are now in a position to state the main result for the problem (5.1). The proof is lengthy and will appear in [4].

THEOREM 4. *Suppose that A.7–A.9 hold. Then for each fixed ε, $0 < \varepsilon \leq \varepsilon_0$, there exist positive constants σ and C_ε such that, if $u \in W_\infty^1(\overline{\Omega/N^\xi}) \cap \overset{\circ}{W}{}_2^1(\Omega)$ and $u_h \in \overset{\circ}{S}{}_{\overline{h}}^r(\Omega)$ satisfy (3) the following holds for all $\overline{h}$ sufficiently small:*

$$\|e\|_{L_\infty(\Omega)} \leq C_\varepsilon h^{-\sigma\varepsilon}\Big(\big\|h(x)\nabla(u-\chi)\big\|_{L_\infty(G^\varepsilon)} + \sum_{j=1}^{m} \|\nabla(u-\chi)\|_{L_2(N_j^\varepsilon)}\Big), \tag{5.4}$$

for all $\chi \in \overset{\circ}{S}{}_{\overline{h}}^r$. Here C_ε and σ are independent of $h(x)$, u, u_h and σ is also independent of ε.

Remark. Each of the terms $\|\nabla(u - u_I)\|_{L_2(N_j^\varepsilon)}$ can be replaced in many ways. For example, using Hölder's inequality it follows that if $\nabla u \in L_{p_j}(v_j)$ for some $2 < p_j < \infty$ then

$$\|\nabla(u-\chi)\|_{L_2(N_j^\varepsilon)} \leq \big\|h(x)^{(1-\varepsilon)(1-\frac{2}{p_j})}(\nabla(u-\chi))\big\|_{L_{p_j}(N_j^\varepsilon)} \tag{5.5}$$

where we have used (A.9). Or for example if $\nabla u \in L_\infty(N_j^\varepsilon)$ then for any $\delta > 0$

$$\|\nabla(u-\chi)\|_{L_2(N_j^\varepsilon)} \leq C\Big\|r_j\Big(\ln\frac{1}{r_j}\Big)^{\frac{1+\delta}{2}}\nabla(u-\chi)\Big\|_{L_\infty(N_j^\varepsilon)} \tag{5.6}$$

It is easy to check that all of these terms are almost optimal, with a loss of $h^{\sigma\varepsilon}$, for some σ.

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DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NY 14853
E-mail address: schatz@math.cornell.edu