Adaptive Computation with PML for Time-harmonic Scattering Problems

Zhiming Chen and Xuezhe Liu

ABSTRACT. We introduce an adaptive perfectly matched layer (PML) technique for solving the time harmonic scattering problems. The PML parameters such as the thickness of the layer and the fictitious medium property are determined through sharp a posteriori error estimates. The derived finite element a posteriori estimate for adapting meshes has the nice feature that it decays exponentially away from the boundary of the fixed domain where the PML layer is placed. This property makes the total computational costs insensitive to the thickness of the PML absorbing layers. A numerical example is included to illustrate the competitive behavior of the proposed adaptive method.

1. Introduction

We consider in this paper an adaptive perfectly matched layer (PML) technique for solving Helmholtz-type scattering problems with perfectly conducting boundary:

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \backslash \bar{D}, \tag{1.1a}$$

$$\frac{\partial u}{\partial \mathbf{n}} = -g \quad \text{on } \Gamma_D, \tag{1.1b}$$

$$\sqrt{r}\left(\frac{\partial u}{\partial r} - \mathbf{i}ku\right) \to 0 \quad \text{as } r = |x| \to \infty. \tag{1.1c}$$

Here $D \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary Γ_D, $g \in H^{-1/2}(\Gamma_D)$ is determined by the incoming wave, and $\mathbf{n}$ is the unit outer normal to Γ_D. We assume the wave number $k \in \mathbb{R}$ is a constant.

Since the work of Berenger [**1**] which proposed a PML technique for solving with the time dependent Maxwell equations, various constructions of PML absorbing layers have been proposed and studied in the literature (cf. e.g. Turkel and Yefet [**17**], Teixeira and Chew [**16**] for the reviews). Under the assumption that the exterior solution is composed of outgoing waves only, the basic idea of the PML

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technique is to surround the computational domain by a layer of finite thickness with specially designed model medium that would either slow down or attenuate all the waves that propagate from inside the computational domain. The PML equation for the time-harmonic scattering problem (1.1a) is derived in Collino and Monk [**7**] by a complex extension of the solution u in the exterior domain. It is proved in Lassas and Somersalo [**10**], Hohage, Schmidt and Zschiedrich [**9**] that the resultant PML solution converges exponentially to the solution of the original scattering problem as the thickness of the PML layer tends to infinite.

A posteriori error estimates are computable quantities in terms of the discrete solution and data that measure the actual discrete errors without the knowledge of exact solutions. They are essential in designing algorithms for mesh modification which equi-distribute the computational effort and optimize the computation. Ever since the pioneering work of Babuška and Rheinboldt [**15**], the adaptive finite element methods based on a posteriori error estimates have become a central theme in scientific and engineering computations. The ability of error control and the asymptotically optimal approximation property (see e.g. Morin, Nochetto and Siebert [**14**], Chen and Dai [**3**]) make the adaptive finite element method attractive for complicated physical and industrial processes (cf. e.g. Chen and Dai [**2**], Chen, Nochetto and Schmidt [**5**]). For the efforts to solve scattering problems using adaptive methods based on a posterior error estimate, we refer to the recent work Monk [**12**], Monk and Süli [**13**].

It is proposed in Chen and Wu [**6**] for scattering problem by periodic structures (the grating problem) that one can use the a posteriori error estimate to determine the PML parameters. Moreover, the derived a posteriori error estimate in [**6**] has the nice feature of exponential decay in terms of the distance to the boundary of the fixed domain where the PML layer is placed. This property leads to coarse mesh size away from the fixed domain and thus makes the total computational costs insensitive to the thickness of the PML absorbing layer.

In this paper we extend the idea of using a posteriori error estimates to determine the PML parameters and propose an adaptive PML technique for solving Helmholtz-type scattering problems (1.1a)-(1.1c). We remark that the results in this paper can be easily extended to solve the scattering problems with other boundary conditions such as Dirichlet or the impedance boundary condition on Γ_D, or to solve the acoustic wave propagation through inhomogeneous media with a variable wave number $k^2(x)$ inside some bounded domain. The main difficulty of the analysis is that in contrast to the grating problems in which there are only finite number of outgoing modes [**6**], now there are infinite number of outgoing modes expressed in terms of Hankel functions. We prove the exponentially decaying property of the PML solution without resorting to the integral equation technique in [**10**] or the representation formula in [**9**]. We remark that in [**10**], [**9**], it is required the fictitious absorbing coefficient must be linear after certain distance away from the boundary where the PML layer is placed.

The layout of the paper is as follows. In section 2 we describe the basic idea of PML technique by using the example of solving an one-dimensional scattering problem. In section 3 we recall the PML formulation for (1.1a)-(1.1c) and consider the existence, uniqueness and convergence of the PML formulation. In section 4 we introduce the finite element discretization and present sharp a posteriori error estimate which lays down the basis of the combined adaptive PML and finite element methods. In section 5 we discuss the implementation of the adaptive method and present a numerical example to illustrate the competitive behavior of the method.

2. The PML technique for 1D scattering problems

In this section we describe the basic idea of the PML technique using the example of solving the following 1D scattering problem

$$\frac{d^2H}{dx^2} + k^2 H = 0 \quad \text{in } (-L,\infty), \quad L > 0. \tag{2.1}$$

The equation (2.1) has two fundamental solutions $e^{\mathbf{i}kx}$ and $e^{-\mathbf{i}kx}$. Imposing the Sommerfeld radiation condition: $dH/dx - \mathbf{i}kx \to 0 \quad$ as $x \to \infty$, which allows only the outgoing wave for sufficiently large x, leads to $H = Ae^{\mathbf{i}kx}$ in $(-L,\infty)$ for some constant $A \in \mathbb{C}$.

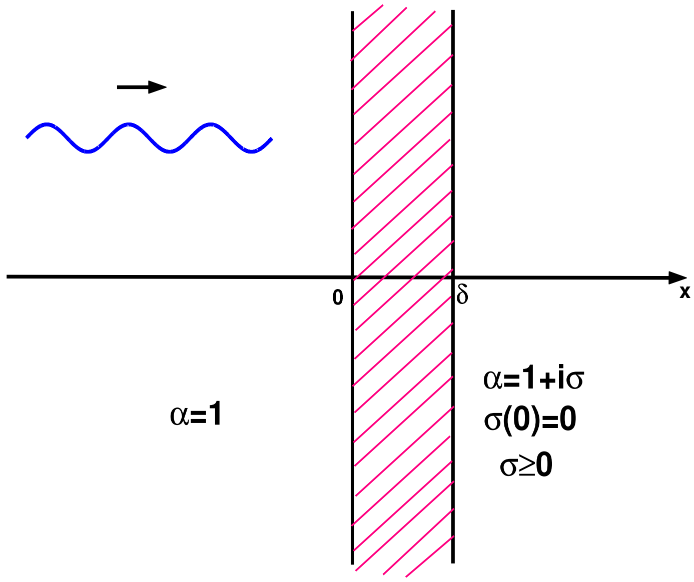

FIGURE 1. Setting of the 1D scattering problem with the PML layer.

Denote $\tilde{x} = \int_0^x \alpha(\tau)d\tau$, where $\alpha(x) = 1 + \mathbf{i}\sigma(x)$, and define

$$\hat{H} = Ae^{\mathbf{i}k\tilde{x}} = Ae^{\mathbf{i}kx} \cdot e^{-k\int_0^x \sigma(\tau)d\tau} \quad \text{in } \ (-L,\infty). \tag{2.2}$$

It is easy to check that $\hat{H} = H$ for $-L \le x \le 0$ and $\hat{H}$ decays exponentially for $x > 0$. Moreover, $\hat{H}$ satisfies:

$$\frac{d}{dx}\left(\frac{1}{\alpha(x)}\frac{d\hat{H}}{dx}\right) + k^2\alpha(x)\hat{H} = 0 \quad \text{in } (-L,\infty). \tag{2.3}$$

The PML technique for solving (2.1) in the unbounded domain is to solve the equation (2.3) in the bounded domain $(-L,\delta)$ with Dirichlet boundary condition $\hat{H} = 0$ at $x = \delta$.

In the practical applications, the medium property α is usually taken as power functions

$$\alpha(x) = 1 + \mathbf{i}\ \sigma_0(x/\delta)^m, \quad \sigma_0 > 0, \quad m \geq 1 \ \text{ integer.}$$

Notice that the exponentially decaying factor in (2.2)

$$e^{-k\int_0^x \sigma(\tau)d\tau} = e^{-k\frac{\sigma_0 x}{m+1}},$$

it is apparent that the error of the PML technique is controlled by the thickness of the layer δ and the parameters σ_0, m. m is usually taken as 2 or 3. In practical applications involving PML techniques, one cannot afford to use a very thick PML layer if uniform finite element meshes are used because it requires excessive grid points and hence more computer time and more storage. On the other hand, a thin PML layer requires a rapid variation of the artificial material property which deteriorates the accuracy if too coarse mesh is used in the PML layer.

3. The PML technique for 2D problems

Now we consider the scattering problem (1.1a)-(1.1c). Let D be contained in the interior of the circle $B_R = \{x \in \mathbb{R}^2 : |x| < R\}$. We start by introducing an equivalent variational formulation of (1.1a)-(1.1c) in the bounded domain $\Omega_R = B_R \backslash \bar{D}$. In the domain $\mathbb{R}^2 \backslash \bar{B}_R$, the solution u of (1.1a)-(1.1c) can be written under the polar coordinates as follows:

$$u(r,\theta) = \sum_{n\in\mathbb{Z}} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(kR)} \hat{u}_n e^{\mathbf{i}n\theta}, \quad \hat{u}_n = \frac{1}{2\pi}\int_0^{2\pi} u(R,\theta)e^{-\mathbf{i}n\theta}d\theta. \tag{3.1}$$

where $H_n^{(1)}$ is the Hankel function of the first kind and order n. The series in (3.1) converges uniformly for $r > R$ (cf. e.g. Colten and Kress [**8**]). Let $T : H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R)$, where $\Gamma_R = \partial B_R$, be the Dirichlet-to-Neumann operator defined as follows: for any $f \in H^{1/2}(\Gamma_R)$,

$$Tf = \sum_{n\in\mathbb{Z}} k\frac{H_n^{(1)'}(kR)}{H_n^{(1)}(kR)} \hat{f}_n e^{\mathbf{i}n\theta}, \quad \hat{f}_n = \frac{1}{2\pi}\int_0^{2\pi} f e^{-\mathbf{i}n\theta}d\theta. \tag{3.2}$$

It is known that T is well-defined and the solution u written as in (3.1) satisfies

$$\left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\Gamma_R} = Tu.$$

The scattering problem can be formulated in the fixed domain as:

$$\Delta u + k^2 u = 0 \quad \text{in } B_R\backslash\bar{D}, \tag{3.3a}$$

$$\frac{\partial u}{\partial \mathbf{n}} = -g \ \text{ on } \Gamma_D, \qquad \frac{\partial u}{\partial \mathbf{n}} = Tu \ \text{ on } \Gamma_R. \tag{3.3b}$$

Let $a : H^1(\Omega_R) \times H^1(\Omega_R) \to \mathbb{C}$ be the sesquilinear form:

$$a(\varphi,\psi) = \int_{\Omega_R} \left(\nabla\varphi\cdot\nabla\bar{\psi} - k^2\varphi\bar{\psi}\right)dx - \langle T\varphi,\psi\rangle_{\Gamma_R}, \tag{3.4}$$

where $\langle\cdot,\cdot\rangle_{\Gamma_R}$ stands for the inner product on $L^2(\Gamma_R)$ or the duality pairing between $H^{-1/2}(\Gamma_R)$ and $H^{1/2}(\Gamma_R)$. Similar notation applies for $\langle\cdot,\cdot\rangle_{\Gamma_D}$, $\langle\cdot,\cdot\rangle_{\Gamma_\rho}$. The scattering problem (1.1a)-(1.1c) is equivalent to the following weak formulation (cf. e.g. [**8**]): Given $g \in H^{-1/2}(\Gamma_R)$, find $u \in H^1(\Omega_R)$ such that

$$a(u,\psi) = \langle g,\psi\rangle_{\Gamma_D} \quad \forall\psi \in H^1(\Omega_R). \tag{3.5}$$

The existence of a unique solution of the variational problem (3.5) is known (cf. e.g. [**8**], McLean [**11**]).

Now we turn to the introduction of the absorbing PML layer. We surround the domain Ω_R with a PML layer $\Omega^{\rm PML} = \{x \in \mathbb{R}^2 : R < |x| < \rho\}$. The specially designed model medium in the PML layer should basically be so chosen that either the wave never reaches its external boundary or the amplitude of the reflected wave is so small that it does not essentially contaminate the solution in Ω_R.

Let $\alpha(r) = \alpha_1(r) + \mathbf{i}\alpha_2(r)$ be the model medium property which satisfies

$$\alpha_1, \alpha_2 \in C(\mathbb{R}), \alpha_1 \geq 1, \alpha_2 \geq 0, \text{and } \alpha(r) = 1 \quad \text{for } r \leq R.$$

Denote by $\tilde{r}$ the complex radius defined by

$$\tilde{r} = \tilde{r}(r) = \begin{cases} r & \text{if } r \leq R, \\ \int_0^r \alpha(t)dt = r\beta(r) & \text{if } r \geq R. \end{cases} \tag{3.6}$$

Here we remark that in contrast to the original PML condition which takes $\alpha_1 \equiv 1$ in the PML region, we allow a variable α_1 in order to attenuate both the outgoing and evanescent waves there. The advantage of this extension makes our method insensitive to the distance of the PML region to the structure.

Since $H_n^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{\mathbf{i}\left(z - \frac{\pi}{2}n - \frac{\pi}{4}\right)}$ as $|z| \to \infty$, Collino and Monk [**7**] obtained the PML equation by considering the following extension of u in the exterior domain $\mathbb{R}^2 \backslash \bar{B}_R$

$$w(r,\theta) = \sum_{n \in \mathbb{Z}} \frac{H_n^{(1)}(k\tilde{r})}{H_n^{(1)}(kR)} \hat{u}_n e^{\mathbf{i}n\theta}, \qquad \hat{u}_n = \frac{1}{2\pi} \int_0^{2\pi} u(R,\theta) e^{-\mathbf{i}n\theta} d\theta. \tag{3.7}$$

It is easy to see by using the property of Hankel function $H_n^{(1)}$ that w satisfies the PML equation

$$\nabla \cdot (A \nabla w) + \alpha\beta k^2 w = 0 \quad \text{in } \Omega^{\rm PML}, \tag{3.8}$$

where $A = A(x)$ is a matrix which satisfies, in polar coordinates,

$$\nabla \cdot (A \nabla) = \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\beta r}{\alpha} \frac{\partial}{\partial r} \right) + \frac{\alpha}{\beta r^2} \frac{\partial^2}{\partial \theta^2}. \tag{3.9}$$

The PML solution $\hat{u}$ in $\Omega_\rho = B_\rho \backslash \bar{D}$ is defined as the solution of the following system

$$\nabla \cdot (A \nabla \hat{u}) + \alpha\beta k^2 \hat{u} = 0 \quad \text{in } \Omega_\rho, \tag{3.10a}$$

$$\frac{\partial \hat{u}}{\partial \mathbf{n}} = -g \ \text{ on } \Gamma_D, \qquad \hat{u} = 0 \ \text{ on } \Gamma_\rho. \tag{3.10b}$$

The PML problem is equivalent to

$$\Delta \hat{u} + k^2 \hat{u} = 0 \quad \text{in } B_R \backslash \bar{D}, \tag{3.11a}$$

$$\frac{\partial \hat{u}}{\partial \mathbf{n}} = -g \ \text{ on } \Gamma_D, \qquad \frac{\partial \hat{u}}{\partial \mathbf{n}} = \hat{T}\hat{u} \ \text{ on } \Gamma_R, \tag{3.11b}$$

where $\hat{T} : H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R)$ is the approximate Dirichlet-to-Neumann operator defined as follows: Given $f \in H^{1/2}(\Gamma_R)$,

$$\hat{T}f = \left.\frac{\partial \zeta}{\partial \mathbf{n}}\right|_{\Gamma_R},$$

where $\zeta \in H^1(\Omega^{\rm PML})$ satisfies

$$\nabla \cdot (A\nabla\zeta) + \alpha\beta k^2\zeta = 0 \quad \text{in } \Omega^{\rm PML}, \tag{3.12a}$$

$$\zeta = f \quad \text{on } \Gamma_R, \qquad \zeta = 0 \quad \text{on } \Gamma_\rho. \tag{3.12b}$$

The error analysis for the PML technique depends on the following key estimate whose proof which depends on an application of the Macdonald formula for the modifed Bessel functions can be found in [**4**].

Lemma 3.1. *For any* $\nu \in \mathbb{R}, z \in \mathbb{C}_{++} = \{z \in \mathbb{C} : {\rm Im}\,(z) \geq 0, {\rm Re}\,(z) \geq 0\}$*, and* $\Theta \in \mathbb{R}$ *such that* $0 < \Theta \leq |z|$*, we have*

$$|H_\nu^{(1)}(z)| \leq e^{-{\rm Im}\,(z)\left(1-\frac{\Theta^2}{|z|^2}\right)^{1/2}}|H_\nu^{(1)}(\Theta)|. \tag{3.13}$$

Based on Lemma 3.1, we obtain that the extension function w in (3.7) satisfies the following estimate which confirms the crucial exponentially decaying property of the PML technique.

$$\| w \|_{H^{1/2}(\Gamma_\rho)} \leq e^{-k{\rm Im}\,(\tilde\rho)\left(1-\frac{R^2}{|\tilde\rho|^2}\right)^{1/2}} \| u \|_{H^{1/2}(\Gamma_R)}.$$

The existence and uniqueness of the solutions of the PML problem (3.12a)-(3.12b) and the well-posedness of the PML problem (3.10a)-(3.10b) can be found in [**4**]. In particular, the following estimate plays the key role.

Lemma 3.2. *There exist constants* $\hat C, C > 0$ *independent of* k, R, ρ*, and* σ *such that the following estimates are satisfied*

$$\| Tf - \hat T f \|_{H^{-1/2}(\Gamma_R)} \leq C\hat C^{-1}(1+kR)^2|\alpha_0|^2 e^{-k{\rm Im}\,(\tilde\rho)\left(1-\frac{R^2}{|\tilde\rho|^2}\right)^{1/2}} \| f \|_{H^{1/2}(\Gamma_R)}.$$

where $\alpha_0 = \alpha(0)$.

4. Finite element approximation

From now on we assume $g \in L^2(\Gamma_D)$. Let $b : H^1(\Omega_\rho) \times H^1(\Omega_\rho) \to \mathbb{C}$ be the sesquilinear form given by

$$b(\varphi, \psi) = \int_{\Omega_\rho} \left(A\nabla\varphi \cdot \nabla\bar\psi - \alpha\beta k^2\varphi\bar\psi\right) dx. \tag{4.1}$$

Denote by $H^1_{(0)}(\Omega_\rho) = \{v \in H^1(\Omega_\rho) : v = 0 \text{ on } \Gamma_\rho\}$. Then the weak formulation of (3.10a)-(3.10b) is: Given $g \in L^2(\Gamma_D)$, find $\hat u \in H^1_{(0)}(\Omega_\rho)$ such that

$$b(\hat u, \psi) = \int_{\Gamma_D} g\bar\psi ds \qquad \forall \psi \in H^1_{(0)}(\Omega_\rho). \tag{4.2}$$

Let Γ_ρ^h, which consists of piecewise segments whose vertices lie on Γ_ρ, be an approximation of Γ_ρ. Let Ω_ρ^h be the subdomain of Ω_ρ bounded by Γ_D and Γ_ρ^h. Let $\mathcal{M}_h$ be a regular triangulation of the domain Ω_ρ^h. We assume the elements $K \in \mathcal{M}_h$ may have one curved edge align with Γ_D so that $\Omega_\rho^h = \cup_{K\in\mathcal{M}_h} K$.

Let $V_h \subset H^1(\Omega_\rho^h)$ be the conforming linear finite element space over Ω_ρ^h, and $\overset{\circ}{V}_h = \{v_h \in V_h : v_h = 0 \text{ on } \Gamma_\rho^h\}$. In the following we will always assume that the functions in $\overset{\circ}{V}_h$ are extended to the domain Ω_ρ by zero so that any function

$v_h \in \overset{\circ}{V}_h$ is also a function in $H^1_{(0)}(\Omega_\rho)$. The finite element approximation to the PML problem (3.10a)-(3.10b) reads as follows: Find $u_h \in \overset{\circ}{V}_h$ such that

$$b(u_h, \psi_h) = \int_{\Gamma_D} g\bar{\psi}_h ds \qquad \forall\, \psi_h \in \overset{\circ}{V}_h. \tag{4.3}$$

For any $K \in \mathcal{M}_h$, we denote by h_K its diameter. Let $\mathcal{B}_h$ denote the set of all sides that do not lie on Γ_D and Γ^h_ρ. For any $e \in \mathcal{B}_h$, h_e stands for its length. For any $K \in \mathcal{M}_h$, we introduce the residual:

$$R_h := \nabla \cdot (A\nabla u_h|_K) + \alpha\beta k^2 u_h|_K. \tag{4.4}$$

For any interior side $e \in \mathcal{B}_h$ which is the common side of K_1 and $K_2 \in \mathcal{M}_h$, we define the jump residual across e:

$$J_e := (A\nabla u_h|_{K_1} - A\nabla u_h|_{K_2}) \cdot \nu_e, \tag{4.5}$$

using the convention that the unit normal vector ν_e to e points from K_2 to K_1. If $e = \Gamma_D \cap \partial K$ for some element $K \in \mathcal{M}_h$, then we define the jump residual

$$J_e := 2(\nabla u_h|_K \cdot \mathbf{n} + g) \tag{4.6}$$

For any $K \in \mathcal{M}_h$, denote by η_K the local error estimator which is defined by

$$\eta_K = \max_{x \in \tilde{K}} \omega(x) \cdot \left(\|h_K R_h\|^2_{L^2(K)} + \frac{1}{2} \sum_{e \subset \partial K} h_e \| J_e \|^2_{L^2(e)} \right)^{1/2}, \tag{4.7}$$

where $\tilde{K}$ is the union of all elements having nonempty intersection with K, and

$$\omega(x) = \begin{cases} 1 & \text{if } x \in \overline{\Omega_R}, \\ |\alpha_0\alpha| e^{-k\mathrm{Im}\,(\tilde{r})\left(1 - \frac{r^2}{|\tilde{r}|^2}\right)^{1/2}} & \text{if } x \in \Omega^{\mathrm{PML}}, \end{cases}$$

THEOREM 4.1. *There exists a constant C depending only on the minimum angle of the mesh $\mathcal{M}_h$ such that the following a posterior error estimate is valid*

$$\begin{aligned} \| u - u_h \|_{H^1(\Omega_R)} \quad \leq \quad & C\hat{C}^{-1}\Lambda(kR)^{1/2}(1 + kR) \left(\sum_{K \in \mathcal{M}_h} \eta_K^2 \right)^{1/2} \\ & + C\hat{C}^{-1}(1 + kR)^2 |\alpha_0|^2 e^{-k\mathrm{Im}\,(\tilde{\rho})\left(1 - \frac{R^2}{|\tilde{\rho}|^2}\right)^{1/2}} \| u_h \|_{H^{1/2}(\Gamma_R)}. \end{aligned} \tag{4.8}$$

Here $\Lambda(kR) = max \left(1, \frac{|H_0^{(1)'}(kR)|}{|H_0^{(1)}(kR)|} \right)$.

We remark that the important exponentially decaying factor $e^{-k\mathrm{Im}\,(\tilde{r})\left(1 - \frac{r^2}{|\tilde{r}|^2}\right)^{1/2}}$ in the PML region Ω^{PML} allows us to take thicker PML layers without introducing unnecessary fine meshes away from the fixed domain Ω_R. Recall that thicker PML layers allow smaller PML medium property, which enhances numerical stability.

In the following we provide a sketch of the proof of the theorem. For the full proof we refer to the paper [**4**]. The key issue is the derivation of the error representation formula. For any $\varphi \in H^1(\Omega_R)$, let $\tilde{\varphi}$ be its extension to Ω^{PML} such that

$$\begin{aligned} & \nabla \cdot (\bar{A}\nabla\tilde{\varphi}) + \overline{\alpha\beta} k^2 \tilde{\varphi} = 0 \quad \text{ in } \Omega^{\mathrm{PML}}, \\ & \tilde{\varphi} = \varphi \ \text{ on } \Gamma_R, \quad \tilde{\varphi} = 0 \ \text{ on } \Gamma_\rho. \end{aligned}$$

It can be proved that for any $\varphi, \psi \in H^1(\Omega^{\rm PML})$, we have $\langle \hat{T}\varphi, \psi\rangle_{\Gamma_R} = \langle \hat{T}\bar{\psi}, \bar{\varphi}\rangle_{\Gamma_R}$.

Thus, for any $\varphi \in H^1(\Omega_R)$ and $\varphi_h \in \overset{\circ}{V}_h$,

$$
\begin{aligned}
a(u-u_h,\varphi) &= \int_{\Gamma_D} g\bar{\varphi} - \int_{\Omega_R} (A\nabla u_h\cdot\nabla\bar{\varphi} - \alpha\beta k^2 u_h\bar{\varphi}) + \langle Tu_h,\varphi\rangle_{\Gamma_R} \\
&= \int_{\Gamma_D} g\bar{\varphi} - b(u_h,\varphi) + \int_{\Omega^{\rm PML}} (A\nabla u_h\cdot\nabla\bar{\tilde{\varphi}} - \alpha\beta k^2 u_h\bar{\tilde{\varphi}}) + \langle Tu_h,\varphi\rangle_{\Gamma_R} \\
&= \int_{\Gamma_D} g\bar{\varphi} - b(u_h,\varphi) - \left\langle \frac{\partial\bar{\tilde{\varphi}}}{\partial\mathbf{n}}, \bar{u}_h \right\rangle_{\Gamma_R} + \langle Tu_h,\varphi\rangle_{\Gamma_R} \\
&= \int_{\Gamma_D} g\bar{\varphi} - b(u_h,\varphi) - \langle \hat{T}\bar{\varphi}, \bar{u}_h\rangle_{\Gamma_R} + \langle Tu_h,\varphi\rangle_{\Gamma_R} \\
&= \int_{\Gamma_D} g\bar{\varphi} - b(u_h,\varphi) + \langle Tu_h - \hat{T}u_h,\varphi\rangle_{\Gamma_R} \\
&= \int_{\Gamma_D} g(\overline{\varphi-\varphi_h}) - b(u_h,\varphi-\varphi_h) + \langle Tu_h - \hat{T}u_h,\varphi\rangle_{\Gamma_R}.
\end{aligned}
$$

Starting from this error representation formula, by using some sharp estimates for the extension $\tilde{\phi}$ and standard techniques in finite element a posteriori error analysis, one can derive the a posteriori error estimate in Theorem 4.1.

5. Numerical Results

The implementation of the adaptive algorithm in this section is based on the PDE toolbox of MATLAB. We use the a posteriori error estimate in Theorem 3.1 to determine the PML parameters. According to the discussion in §2, we choose the PML medium property as the power function and thus we need only to specify the thickness $\rho - R$ of the layer and the medium parameter σ_0. Recall from Theorem 3.1 that the a posteriori error estimate consists of two parts: the PML error and the finite element discretization error. In our implementation we first choose ρ and σ_0 such that the exponentially decaying factor:

$$
\hat{\omega} = e^{-k{\rm Im}\,(\tilde{\rho})(1-\frac{R^2}{|\tilde{\rho}|^2})^{1/2}} \le 10^{-8}, \tag{5.1}
$$

which makes the PML error negligible compared with the finite element discretization errors. Once the PML region and the medium property are fixed, we use the standard finite element adaptive strategy to modify the mesh according to the a posteriori error estimate. Now we describe the adaptive algorithm we used in the paper.

Algorithm 5.1. Given tolerance TOL > 0. Let $m = 2$.

• Choose ρ and σ such that the exponentially decaying factor $\hat{\omega} \le 10^{-8}$;
• Set the computational domain $\Omega_\rho = B_\rho \backslash \bar{\Gamma}_D$ and generate an initial mesh $\mathcal{M}_h$ over Ω_ρ;
• While $\mathcal{E}_{FEM} = \left(\sum_{K\in\mathcal{M}_h}\eta_K^2\right)^{1/2} > $ TOL do
  - refine the mesh $\mathcal{M}_h$ according to the strategy:

if $\eta_K > \frac{1}{2}\max_{K\in\mathcal{M}_h}\eta_K$, refine the element $K \in \mathcal{M}_h$

- solve the discrete problem (3.3) on $\mathcal{M}_h$
- compute error estimators on $\mathcal{M}_h$

end while

Now we report a numerical example to demonstrate the competitive behavior of the proposed algorithm. In the computations we first prescribe ρ and then determine σ according to (5.1). We scale the error estimator for determining finite element meshes by a factor 0.15 as in the PDE toolbox of MATLAB.

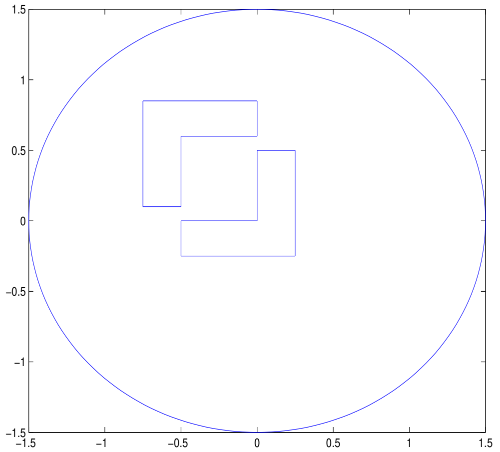

FIGURE 2. The geometry of the scatter for Example 1.

We consider the scattering from two perfectly conducting metals as plotted in Figure 5.1 with the incoming wave $u_I = e^{\mathbf{i}kx_1}$, $k = 2\pi$. We take $R = 1.5$.

Figure 3 shows the $\log N_k$-$\log \mathcal{E}_k$ curves, where N_k is the number of nodes of the mesh $\mathcal{M}_k$ and the $\mathcal{E}_k = (\sum_{K\in\mathcal{M}_k} \eta_K^2)^{1/2}$ is the associated a posteriori error estimate. It indicates that the meshes and the associated numerical complexity are quasi-optimal: $\mathcal{E}_k = CN_k^{-1/2}$ is valid asymptotically.

One of the important quantities in the scattering problems is the far field pattern:

$$u_\infty(\hat{x}) = \frac{e^{i\frac{\pi}{4}}}{\sqrt{8\pi k}} \int_{\partial D} \left(u(y)\frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \upsilon(y)} - \frac{\partial u(y)}{\partial \upsilon(y)} e^{-ik\hat{x}\cdot y} \right) ds(y), \qquad \hat{x} = \frac{x}{|x|}.$$

We compute the far field $u_\infty(\hat{x})$, $\hat{x} = (\cos(\theta), \sin(\theta))^T$ in the observation direction $\theta = \pi/4$. Figures 4 shows the far fields for different choices of PML parameters ρ and σ. We observe that our adaptive algorithm is robust with respect to the choice of the thickness of PML layer: the far fields of the scattering solutions are insensitive to the choices of the PML parameters.

In Figure 5 we show the mesh of the problem when $\rho = 4R$. The mesh has 5460 nodes.We observe that the mesh near the boundary Γ_ρ is rather coarse, as a consequence of the exponentially decaying factor in our finite element a posteriori error estimator. Figure 6 shows the surface plot of the real part of the solution over the mesh displayed in Figure 5.

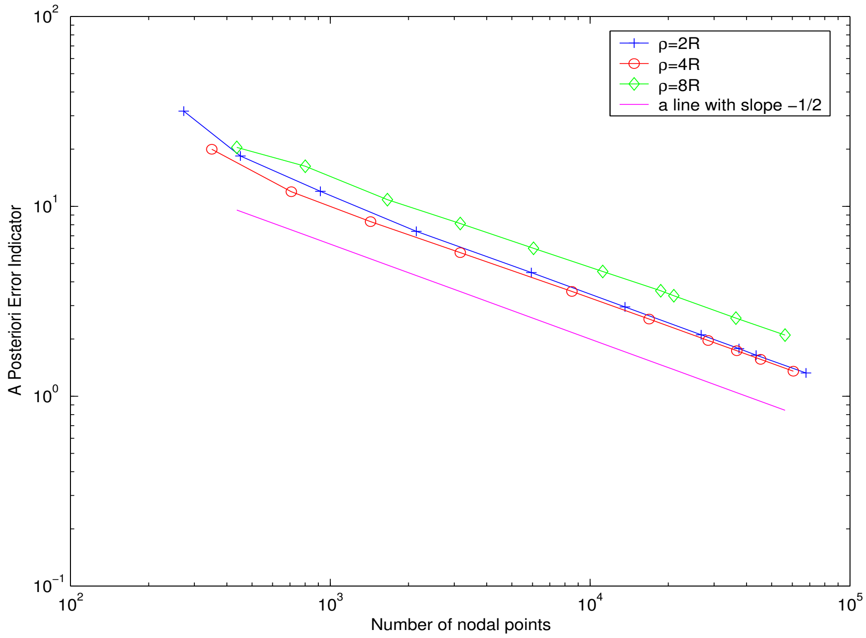

FIGURE 3. Quasi-optimality of the adaptive mesh refinements of the a posteriori error estimator.

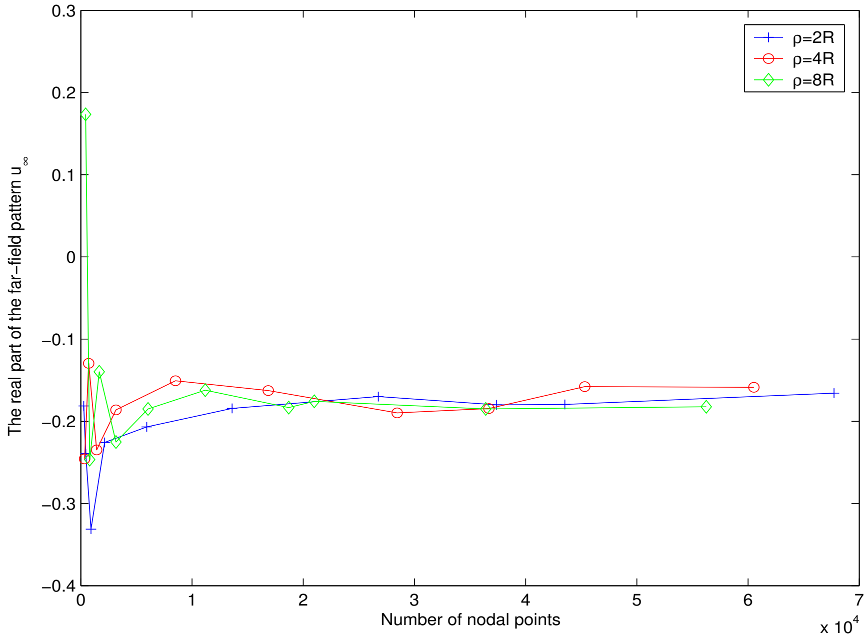

FIGURE 4. The far field of the observation angle $\theta = \pi/4$.

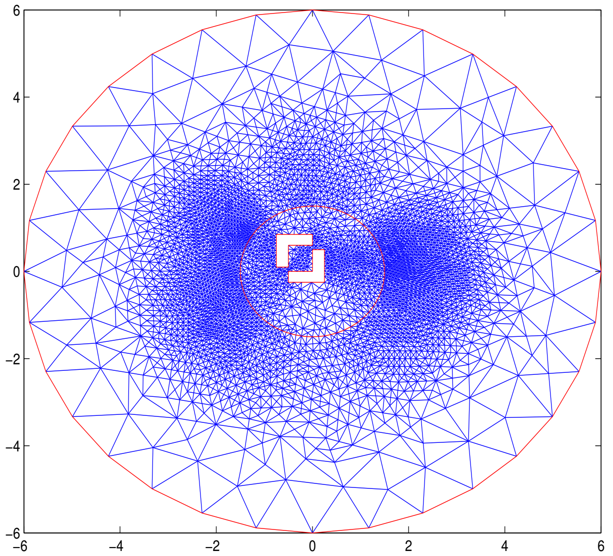

FIGURE 5. The mesh of 5460 nodes when $\rho = 4R$.

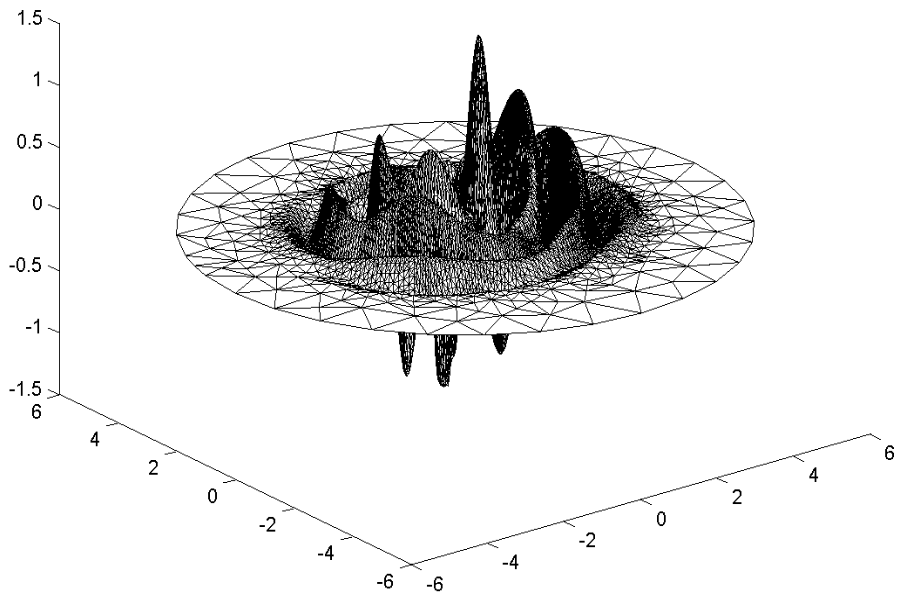

FIGURE 6. The surface plots of the real part of the solution when $\rho = 4R$.

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