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Recent Progress on A-posteriori Error Analysis for the p and h-p Finite Element Methods

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Abstract. This paper investigates recent progress on a-posteriori error analysis for the high-order finite element method(FEM). The paper will discuss the differences between a-posteriori error estimations for lower-order FEM and those for high-order FEM, and analyzes the technical and methodological differences on a-posteriori error estimations for high-order FEM in one dimension and in high dimensions. Based on a newly established mathematical framework of the Jacobi-weighted spaces, a-posterior error estimators and indicators for the p- and h-p version of FEM are proposed, and its prospective is discussed.

1. Introduction

Since Babuška and Rheinboldt proposed [10, 11] in the late 1970s the revolutionary concepts of a-posteriori error estimation and the adaptivity, the adaptive finite element algorithms have become powerful and reliable computational tools for large-scale scientific and engineering computations. The adaptive method has rapidly developed in the past decades theoretically and practically, and has integrated in many commercial and research codes such as ANSYS, PHLEX, STRESS CHECK, and STRIPE.

There are two basic approaches for a-posteriori error estimation of FEM. The first one is based on the solutions of local auxiliary problems, and the second one is based on residuals, including the jump of normal derivative of finite element solutions on internal edges of elements. For the lower-order FEM which is refereed as the h-version of FEM, a-posteriori error estimation and the adaptive mesh generation have been well developed in the past two decades, and we refer to the books [4, 12, 31] for an overview. The first approach for the h-version of FEM has been developed in various papers such as [1, 2, 5]. The second approach is also widely used for the h-version, see. e.g. [2, 3, 27]

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In contrast to the lower-order FEM, a-posteriori error estimation for the high-order FEM such as the p-version of FEM, the h-p version of FEM and the spectral method is much less developed and lacks of substantial progress in the past two decades. There are only few papers on this topic in the current literature, see, e.g., [14, 25, 26]. Obviously, a-posteriori error estimation for the high-order FEM is a challenging issue except in one-dimensional case. Many useful and effective methods and the techniques developed for the h-version of FEM can not be or have not been applied to the high-order FEM, such as the super-convergence and patch recovery techniques [13, 32–35]. The comprehensive analysis for a-posteriori error estimators and indicators based on the residuals for the p- and h-p of FEM in one dimension was well known in the mid of the 1980s [19, 29], but these results can not be established in high dimensions, and techniques used for one-dimensional analysis can not be applied to two- and three-dimensional cases. We have tried for two decades in order to find an appropriate mathematical framework for a-posteriori error estimation of the high-order FEM in two- and three-dimensional cases. In the mean time, the engineers have proposed and implemented various types of error estimators and indicators in commercial and research codes based on their own experiences. The first residual based error estimator and error indicator for p-version of FEM was used for two-dimensional elasticity problems in [17], and an error estimator for the p-version of FEM based on the extrapolation techniques was implemented in the commercial code STRESS CHECK [18, 30]. The approach of solving local auxiliary problems was utilized in adaptive h-p codes in the 1980s and 1990s [27, 28]. Unfortunately, these error indicators and estimators used by engineers lack of theoretical support.

Since the very late 1990s, the approximation theory for the p- and h-p version of FEM has been well developed in the mathematical framework of the Jacobi-weighted Besov and Sobolev spaces [6–9]. In this mathematical framework, a-priori error analysis leads to a rigorous proof of the optimal convergence for the p- and h-p version (with quasi-uniform meshes) of FEM [6, 8, 9, 22]. The new mathematical framework and the recent breakthrough of the a-priori error estimation for the p- and h-p version (with quasi-uniform meshes) of FEM have had significant impacts on various aspects of the p- and h-p FEM, including a-posteriori error analysis. The investigation on the effectiveness of the new mathematical framework for a-posteriori error estimation and on further development of this framework for the need of a-posteriori error analysis was started a couple of years ago. The primary results of the investigation have been presented at international conferences [20] and will appear in forthcoming paper [21], which seem very promising.

This paper will summarize the theoretical results of a-posteriori error estimation for the high-order FEM and analyze the similarities and differences of methodology and techniques between one-dimensional case and two-dimensional case. We will also present major progress on residual based a-posterior error estimation in the mathematical framework of Jacobi-weighted Sobolev spaces. In concluding remarks, a comprehensive comparison between these quasi-optimal results in two-dimensional case and the optimal results in one-dimensional case will be given.

2. A-posteriori analysis in one dimension

We begin by considering again the model problem

$$\begin{cases} -u'' = f \text{ in } \Omega = (-1, 1), \\ \\ u(-1) = u(1) = 0. \end{cases} \tag{2.1}$$

Let $\mathcal{T} = \{-1 = x_0 < x_1 < \cdots < x_M = 1\}$ and $\Omega_j = (x_{j-1}, x_j), 1 \leq j \leq M$ be the mesh and elements on $\Omega = (-1, 1)$, and let $p = \{p_1, p_2, \cdots, p_M\}$ be the polynomial degree. By $S_0^{p,1}(\Omega, \mathcal{T}) = S^p(\Omega, \mathcal{T}) \cap H_0^1(\Omega)$ we denote finite element subspace, where $S^p(\Omega, \mathcal{T})$ contains all piecewise polynomials of the degree p_i on Ω_i. Then the finite element solution $u_S \in S_0^{p,1}(\Omega, \mathcal{T})$ satisfies variational equation,

$$B(u_S, v) = F(v), \quad \forall v \in S_0^{p,1}(\Omega, \mathcal{T}) \tag{2.2}$$

where

$$B(u, v) = \int_{-1}^{1} u'v'dx, \qquad F(v) = \int_{-1}^{1} fvdx. \tag{2.3}$$

Let $u_0 \in H^k(\Omega)$ denote the solution of the equation (2.1), then the error $e = u_0 - u_S$ satisfies

$$B(e, v) = R(v) = F(v) - B(u_S, v) \qquad \forall v \in S_0^{p,1}(\Omega, \mathcal{T}). \tag{2.4}$$

Let $W_j(x) = (x - x_{j-1})(x_j - x)$ be the weight function on Ω_j, and let $L^2_{W_j}(\Omega_j)$ and $L^2_W(\Omega)$ be the weighted L^2-space on Ω_j and Ω, respectively, with the norms

$$\|u\|^2_{L^2_W(\Omega, \mathcal{T})} = \sum_{j=1}^{M} \|u\|^2_{L^2_{W_j}(\Omega_j)}, \qquad \|u\|_{L^2_{W_j}(\Omega_j)} = \|uW_j^{1/2}\|_{L^2(\Omega_j)}. \tag{2.5}$$

The error indicators η_j are defined as

$$\eta_j^2 = \frac{1}{p_j(p_j+1)} \|r_j\|^2_{L^2_{W_j}(\Omega_j)} \tag{2.6}$$

where $r_j = \Pi^0_{p_j} f + u''_S$, $\Pi^0_{p_j}$ is the Legendre projection on $\mathcal{P}_{p_j}(\Omega_j) = \{$ polynomials of degree $\leq p_j$ on $\Omega_j\}$.

The error estimator η is defined by

$$\eta = \sum_{j=1}^{M} \eta_j^2 + \frac{1}{p_j(p_j+1)} \|f - \Pi^0_{p_j} f\|^2_{L^2_{W_j}(\Omega_j)}. \tag{2.7}$$

The relation between the error e and the indicators η_i and η is given in the following theorem.

THEOREM 2.1. *Let e and η be the error and the error estimator defined as above. Then there holds*

$$\|e\|^2_{E(\Omega)} \leq \eta^2 \leq C\|e\|^2_{E(\Omega)} + \sum_{j=1}^{M} \frac{1}{p_j(p_j+1)} \|f - \Pi^0_{p_j} f\|^2_{L^2_{W_j}(\Omega_j)} \tag{2.8}$$

where C is independent of the mesh and the element degree p and $\|e\|_{E(\Omega)}$ denotes the energy norm $\sqrt{B(e,e)}$.

For the proof of the above theorem, we refer to [29]. We shall make comments on the methodology and technique which lead to the above claims. Stating with the variational equation (2.2), we have for any $v \in H_0^1(\Omega)$,

$$\begin{aligned} B(e,v) &= B(e, v - \Pi_p v) \\ &= \sum_{j=1}^{M} \int_{\Omega_j} (u_S'' + f)(v - \Pi_p v) dx \\ &= \sum_{j=1}^{M} \int_{\Omega_j} (u_S'' + \Pi_{p_j}^0 f)(v - \Pi_p v) dx + \int_{\Omega_j} (f - \Pi_{p_j}^0 f)(v - \Pi_p v) dx \end{aligned} \tag{2.9}$$

where Π_p is an operator: $H_0^1(\Omega) \to S_0^{p,1}(\Omega, \mathcal{T})$ such that

$$(v - \Pi_p v)(x_j) = 0, \quad 0 \le j \le M.$$

We decompose $\Pi_p = \Pi_{p_1} \Pi_{p_2} \cdots \Pi_{p_M}$, where Π_{p_j} is an operator : $H_0^1(\Omega) \to P_{p_j}(\Omega_j)$ such that $v = \Pi_p v$ at the ending points of Ω_j. It is known that

$$\int_{\Omega_j} \frac{(v - \Pi_{p_j} v)^2}{W_j(x)} dx \le C \frac{1}{p_j(p_j+1)} \|v'\|^2_{L^2(\Omega_j)}. \tag{2.10}$$

For the construction of Π_{p_j} and the property (2.10) we refer to [19]. Actually Π_{p_j} is an integration of the Legendre projection of $v'(x)$ on $P_{p_j-1}(\Omega_j)$, and can be viewed as a quasi Jacobi projection of $v(x)$ with the weight function $W_j^{\beta}(x)$ with $\beta = -1$. Therefore we characterize the operator Π_{p_j} as a quasi Jacobi projection on $\mathcal{P}_{p_j}(\Omega_j)$ with the weight $\beta = -1$ and denoted by $\Pi_{p_j}^{-1}$. Due to the property (2.10) of the quasi Jacobi projection $\Pi_{p_j}^{-1}$ we have

$$|B(e,v)| \le \sum_{j=1}^{M} (\|r_j\|_{L^2_{W_j}(\Omega_j)} + \|f - \Pi_{p_j}^0 f\|_{L^2_{W_j(\Omega_j)}}) \frac{\|v'\|_{L^2(\Omega_j)}}{\sqrt{p_j(p_j+1)}} \tag{2.11}$$

which leads to the first inequality of (2.8).

The second inequality of (2.8) is a consequence of the following estimation for the indicator η_j on each Ω_j,

$$\eta_j \le C\|e\|_{E(\Omega_j)} + \frac{1}{\sqrt{p_j(p_j+1)}} \|f - \Pi_{p_j}^0 f\|_{L^2_{W_j}(\Omega_j)}. \tag{2.12}$$

The variational equation (2.2) gives for any $v \in S_0^{p,1}(\Omega, \mathcal{T})$

$$\begin{aligned} B(e,v) &= \int_{\Omega} (u_0' - u_S') v' dx \\ &= \sum_{j=1}^{M} \int_{\Omega_j} (u_S'' + f) v dx + \sum_{j=1}^{M-1} [u_S'](x_j) v(x_j). \\ &= \sum_{j=1}^{M} \int_{\Omega_j} (u'' + \Pi_{p_j}^0 f) v dx + \sum_{j=1}^{M-1} [u_S'](x_j) v(x_j) + \sum_{j=1}^{M} \int_{\Omega_j} (f - \Pi_{p_j}^0 f) v dx \end{aligned} \tag{2.13}$$

where $[u_S'](x_j)$ denotes the jump of u_S' at x_j. In particular, for $v = W_j r_j$ on Ω_j and vanishes on $\Omega \backslash \Omega_j$, we have

$$
\begin{aligned}
\int_{\Omega_j} |r_j|^2 W_j(x)dx &= B(e,v) - \int_{\Omega_j} (f - \Pi^0_{p_j} f)vdx \\
&\le \|e'\|_{L^2(\Omega_j)}\|v'\|_{L^2(\Omega_j)} + \|f - \Pi^0_{p_j} f\|_{L^2_{W_j}(\Omega_j)}\|r_j\|_{L^2_{W_j}(\Omega_j)}.
\end{aligned} \tag{2.14}
$$

Note that

$$\|v'\|^2_{L^2(\Omega_j)} \le 2(\|r_j' W_j\|^2_{L^2(\Omega_j)} + h_j^2\|r_j\|^2_{L^2(\Omega_j)}).$$

Due to the properties of the Jacobi-weighted norms of polynomial, including the inverse inequality, there hold

$$\|r_j' W_j\| \le Cp_j^2\|r_j\|_{L^2_{W_j}(\Omega_j)}, \quad \|r_j\|^2_{L^2(\Omega_j)} \le Cp_j^2 h_j^{-2}\|r_j\|_{L^2_{W_j}(\Omega_j)}. \tag{2.15}$$

which together with (2.14) yields (2.12).

Let us comment on the quality of the error estimator and indicators. To this end, we assume that $f \in S^p(\Omega, \mathcal{T})$. Then there holds due to (2.8)

$$C_1 \le \frac{\|e\|_{E(\Omega)}}{\eta} \le C_2 = 1, \tag{2.16}$$

with the upper bound C_2 and the lower bound C_1 independent of degree p and mesh size. Hence the error estimator η is a reliable assessment of the error e in the energy norm. The quasi Jacobi projection $\Pi^{-1}_{p_j}$ and its important property (2.10) are the key for the upper bound $C_2 = 1$. The inverse inequality (2.15) on polynomial in Jacobi-weights norms is essential to the lower bound C_1. We will come back to these two important issues in two-dimensional setting, and discuss the similarities and differences on the projection operator and the inverse inequality between one-dimensional analysis and two-dimensional analysis in next sections.

3. A-posteriori analysis in two dimensions

Consider a boundary value problem on a polygon domain Ω :

$$\begin{cases} -\Delta u + u = f \text{ in } \Omega \\ u|_\Gamma = 0, \text{ on } \Gamma = \partial\Omega. \end{cases} \tag{3.1}$$

The corresponding variational problem is to find $u \in H^1_0(\Omega)$ such that

$$B(u,v) = F(v) \quad \forall v \in H^1_0(\Omega) \tag{3.2}$$

where B is a bilinear form on $H^1_0(\Omega) \times H^1_0(\Omega)$:

$$B(u,v) = \int_\Omega (\nabla u \cdot \nabla v + uv)\, dx,$$

and F is a linear functional on $H^1(\Omega)$:

$$F(v) = \int_\Omega fvdx \quad \forall v \in H^1(\Omega).$$

Let $\mathcal{T} = \{K_i, 1 \le i \le M\}$ be a partition of Ω with shape-regular quadrilateral or triangular elements K_i, and let $\partial\mathcal{T} = \{\gamma_\ell, 1 \le \ell \le L\}$ with inter-element edges γ_ℓ. By F_i we denote a mapping of $Q = (-1,1)^2$ onto elements K_i. Then the subspace of piecewise polynomials over $\mathcal{T}$ for the p-version of FEM is defined as usual, i.e.

$$S^{p,1}_0(\Omega,\mathcal{T}) = S^p(\Omega,\mathcal{T}) \bigcap H^1_0(\Omega) = \{\varphi \,|_{K_i} = \psi_p \circ F_i, \psi_p \in \mathcal{P}_p(Q)\} \bigcap H^1_0(\Omega)$$

where $\mathcal{P}_p(Q)$ is a set of polynomials of (separate) degree $\leq p$ on Q, and $S^p(\Omega, \mathcal{T})$ denotes a set of piecewise polynomials of degree $p = (p_1, p_2 \cdots, p_M)$.

The h-p version FE solution $u_S \in S_0^{p,1}(\Omega, \mathcal{T})$ satisfies

$$B(u_S, v) = F(v) \quad \forall v \in S_0^{p,1}(\Omega, \mathcal{T}). \tag{3.3}$$

By e, e_i, r, r_i and R, R_{γ_ℓ} we denote the error, residue and jump of normal derivative along the internal edges γ_ℓ for the finite element solution u_S:

$$\begin{aligned} e &= u - u_S, \quad e_i = e\,|_{K_i}; \\ r &= f + \triangle u_S - u_S,\ r_i = r\,|_{K_i}; \\ R &= \left[\frac{\partial u_S}{\partial n}\right], R_{\gamma_\ell} = \left[\frac{\partial u_S}{\partial n}\right]|_{\gamma_\ell}\ . \end{aligned}$$

A weight function $W_i^\beta(x)$ is defined on each element K_j

$$W_i^\beta(x) = \prod_{l=1}^{L_i} |dist(x, \gamma_l^i)|^\beta,$$

where γ_l^i denote edges of $K_i, L_i = 3$ or 4. The Jacobi-weighted L^2-spaces over Ω and Ω_i are furnished with the norms

$$\|u\|^2_{L^2_\beta(\Omega)} = \sum_{i=1}^{M} \|u\|^2_{L^2_\beta(K_i)}, \quad \|u\|^2_{L^2_\beta(K_i)} = \|uW_i^{\beta/2}\|^2_{L^2(K_i)}.$$

Similarly, a weight function $W_{i,l}^\beta(x)$ is defined on each edge of K_i

$$W_{i,l}^\beta(x) = \prod_{m=1}^{2} |dist(x, V_{i,l}^m)|^\beta$$

where $V_{i,l}^m$ are two vertices of the edge γ_l^i. The corresponding Jacobi-weighted spaces is associated with the norm

$$\|g\|_{L^2_\beta(\gamma_\ell^i)} = \|gW_{i,\ell}^{\beta/2}\|_{L^2(\gamma_\ell^i)}.$$

A local error indicator η_i on the element K_i is introduced in [26]

$$\eta_i^2 = \eta_{i,B}^2 + \eta_{i,E}^2$$

with the indicators $\eta_{i,B}$ associated with $L^2(K_i)$-projection of the residual r_i.

$$\eta_{i,B}^2 = \frac{h_i^2}{p_i^2}\|\Pi^0_{p_i} f + \Delta u_S - u_S\|^2_{L^2_\beta(K_i)} = \frac{h_i^2}{p_i^2}\|r_{ip}\|^2_{L^2_\beta(K_i)},$$

and the indicators associated with the jump of $\dfrac{\partial u}{\partial n}$ on the internal edge of K_i

$$\eta_{i,E}^2 = \sum_{l=1}^{L_i} \frac{h_i}{2p_i}\|[\frac{\partial u_S}{\partial n}]\|^2_{L^2_\beta(\gamma_l^i)}.$$

Here, $\Pi^0_{p_i}$ denotes the $L^2(K_i)$-projection on $\mathcal{P}_{p_i}(K_i)$, $r_{ip} = \Pi^0_{p_i} r = \Pi^0_{p_i} f + \triangle u_S - u_S$. The error estimator η is defined as usual

$$\eta^2 = \sum_{i=1}^{M} \eta_i^2.$$

It was claimed in [26] that

$$\|e\|^2_{H^1(\Omega)} \leq C \left(\sum_{i=1}^{M} (p_i^{2\beta} \eta_i^2 + \frac{h_i^2}{p_i^2} \|f - \Pi^0_{p_i} f\|^2_{L^2(K_i)}) \right), \tag{3.4}$$

$$\eta_i^2 \leq C(\varepsilon) p_i^{\max\{1+2\varepsilon-2\beta,0\}} \left(p_i \|e\|^2_{H^1(Q_{K_i})} + p_i^{2\varepsilon} \frac{h_i^2}{p_i^2} \|f - \Pi^0_{p_i} f\|^2_{L^2(Q_{K_i})} \right) \tag{3.5}$$

where Q_{K_i} is a neighborhood of the element K_i.

We refer to [26] for the derivation of the above claims (3.4)-(3.5), and we are interested in the interpretation of the above claims, the methodology and technique used in the proof. To evaluate precisely the equality of the error estimator and indicators, we assume again that $f \in S^p(\Omega, \mathcal{T})$. Then for uniform degree $p_j = p$ and $\beta \in (0, 1/2]$, (3.4) and (3.5) lead to

$$C_1(p) \leq \frac{\|e\|_{E(\Omega)}}{\eta} \leq C_2(p) \tag{3.6}$$

with

$$C_2(p) = \tilde{C}_2 p^{\beta}, \qquad C_1(p) = \tilde{C}_1 p^{1-\beta+\varepsilon}.$$

The ratio τ of the upper bound $C_2(p)$ and the lower bound $C_1(p)$ is an important index for the quality of the residual based error estimator and indicators. The error estimator is optimal if $\tau = O(1)$, and is quasi optimal if $\tau = O(p^{\varepsilon}), \varepsilon > 0$ arbitrary. Obviously, the result (3.4) and (3.5) are not optimal, or even close to quasi-optimal because

$$\tau = O(p^{2\beta-\varepsilon}). \tag{3.7}$$

To derive the estimate (3.4), a $hp-$element interpolation operator Π^p_V of Clèment type is introduced in [25, 26], which eventually determines the quality of the estimator. We start with the observation

$$\begin{aligned}
\|e\|^2_{E(\Omega)} &= B(e,e) = B(e, e - \Pi^p_V e) \\
&= \sum_{i=1}^{M} \int_{K_i} (f + \triangle u_S - u_S)(e - \Pi^p_V e) dx + \int_{\partial K_i} \frac{\partial u_S}{\partial n} (e - \Pi^p_V e) ds \\
&\leq \sum_{i=1}^{M} \|r_i\|_{L^2(K_i)} \|e - \Pi^p_V e\|_{L^2(K_i)} + \sum_{\gamma_l \in \partial \mathcal{T}} \|[\frac{\partial u_S}{\partial n}]\|_{L^2(\gamma_l)} \|e - \Pi^p_V e\|_{L^2(\gamma_l)}. \qquad (3.8)
\end{aligned}$$

It is shown in [26] that

$$\begin{aligned}
\|e - \Pi^p_V e\|_{L^2(Q_{K_i})} &\leq C \|e\|_{E(Q_{K_i})}, \qquad (3.9)\\
\|e - \Pi^p_V e\|_{L^2(\gamma_l)} &\leq \sqrt{\frac{h^*}{p^*+1}} \|e\|_{E(Q^*_{K_i})}.
\end{aligned}$$

where Q_{K_i} is the neighborhood of K_i and $Q^*_{K_i}$ is the neighborhood of Q_{K_i}, $h^* = \max_{K_i \subset Q^*_{K_i}} h_i$ and $p^* = \min_{K_i \subset Q^*_{K_i}} p_i$. (3.9) leads to (3.4). It seems the $hp-$element interpolation Π^p_V is not a desired operator for the optimal or quasi optimal estimator, we need to find a better one for a-posteriori error estimator for high-order FEM in high dimensions.

Remark 3.1 Although [26] did not mention the dependence of C_1 and C_2 on ε and β, it is very important to point out that the upper bound $C_2(p)$ and the lower bound $C_1(p)$ of the ratio τ depends not only on p, but also on ε and β. The constants $\tilde{C}_1$ and $\tilde{C}_2$ may blow up as $\beta \to 1$ and $\varepsilon \to 0$.

Remark 3.2 It is implicitly assumed in [26] that $[\frac{\partial u_S}{\partial n}]$ is a polynomial or pull-back polynomial on internal edges. It is not true for meshes containing quadrilateral elements in the *p*- and *h-p* FEM codes.

4. A-posteriori error estimation in the framework of Jacobi-weighted Sobolev spaces

The Jacobi-weighted Sobolev and Besov spaces have been proved the most appropriate functional spaces for a-priori error analysis for the *p*- and *h-p* version with quasi-uniform meshes of FEM, it is expected to provide with a useful mathematical framework for the a-posteriori error estimation. Although the investigation was started a couple of years ago, various results presented at international conferences seem quite encouraging [20,21]. For convenience and effectiveness of explaining the main ideas, we concentrate ourself on a-posteriori error estimation for the *p*-version, the results here can be generalize to the *h-p* -version by a scaling argument.

Let $Q = I^2 = (-1,1)^2$. For $\beta = (\beta_1, \beta_2)$ and $\alpha = (\alpha_1, \alpha_2)$ with $\beta_i > -1$ and integers $\alpha_i \geq 0$, we define the weight functions as in the previous section. Let

$$\begin{aligned} W_{\beta\pm\alpha}(x) &= W_{\beta_1\pm\alpha_1,\beta_2\pm\alpha_2}(x) \\ &= (1-x_1^2)^{\beta_1\pm\alpha_1}(1-x_2^2)^{\beta_2\pm\alpha_2}. \end{aligned}$$

The Jacobi-weighted Sobolev spaces $H^{k,\beta}(Q), k \geq 0$ are introduced in [8,24] with the norms

$$\|u\|^2_{H^{k,\beta}(Q)} = \sum_{|\alpha|\leq k} \int_Q |\partial^\alpha u|^2 W_{\beta+\alpha}(x)dx.$$

For important approximation properties of Jacobi-projections for functions $\in H^{k,\beta}(Q)$, we refer to [6,8,9] and [24].

The $\tilde{H}^{k,\beta}(Q)$ is defined as a dual to $H^{k,\beta}(Q)$ with the norm

$$\|u\|^2_{\tilde{H}^{k,\beta}(Q)} = \sum_{|\alpha|\leq k} \int_Q |\partial^\alpha u|^2 W_{\beta-\alpha}(x)dx.$$

The structure of the space $\tilde{H}^{k,\beta}(Q)$ is quite different from one of the spaces$H^{k,\beta}(Q)$. We are interested particularly the spaces with $k = 0,1$ and $0 \leq \beta_1, \beta_2 < 1$. We write

$$\|u\|_{\tilde{H}^{0,\beta}(Q)} = \|u\|_{L^2_\beta(Q)} = \|uW_{\beta/2}\|_{L^2(Q)}$$

and

$$\|u\|_{H^{0,-\beta}(Q)} = \|u\|_{L^2_{-\beta}(Q)} = \|uW_{-\beta/2}\|_{L^2(Q)}.$$

We next to define Jacobi-weighted spaces $H^{k,\beta}(K_i)$ and $\tilde{H}^{k,\beta}(K_i)$ over elements. Let F_i be the mapping of Q onto element K_i, and $\tilde{u}_i = u \circ F_i$. We define

$$\|u\|_{\tilde{H}^{k,\beta}(K_i)} = \|\tilde{u}\|_{\tilde{H}^{k,\beta}(Q)}, \quad \|u\|_{H^{k,-\beta}(K)} = \|\tilde{u}\|_{H^{k,-\beta}(Q)}$$

and

$$\|u\|_{L^2_{\pm\beta}(K_i)} = \|\tilde{u}\|_{L^2_{\pm\beta}(Q)}.$$

Since there is no scaling for the p-version , we may assume that all elements are standard and axis-parallel squares. For simplicity we drop sub-index i and l from the notation K_i and γ_l^i without causing confusion. The Jacobi-weighted spaces $H^{k,-\beta}(\mathcal{T})$ and $\tilde{H}^{k,\beta}(\mathcal{T})$ are defined as

$$H^{k,-\beta}(\mathcal{T}) = \bigcap_K H^{k,-\beta}(K), \quad \tilde{H}^{k,\beta}(\mathcal{T}) = \bigcap_K \tilde{H}^{k,\beta}(K)$$

furnished with "broken" norms

$$\|u\|^2_{H^{k,-\beta}(\mathcal{T})} = \sum_K \|\tilde{u}\|^2_{H^{k,-\beta}(K)}, \quad \|u\|^2_{\tilde{H}^{k,\beta}(\mathcal{T})} = \sum_{K\in\mathcal{T}} \|\tilde{u}\|^2_{\tilde{H}^{1,\beta}(K)}.$$

In the framework of Jacobi-weighted spaces $H^{1,-\beta}(\mathcal{T})$ and $\tilde{H}^{1,\beta}(\mathcal{T})$ with $\beta \in [0,1)$ we reformulate the elliptic problem (3.1) and its variational equation : find $u \in \tilde{H}_0^{1,\beta}(\mathcal{T})$ such that

$$B(u,v) = F(v) \quad \forall v \in H_0^{1,-\beta}(\mathcal{T}).$$

Here $B(u,v)$ is a bilinear forms on $\tilde{H}^{1,\beta}(\mathcal{T}) \times H^{1,-\beta}(\mathcal{T})$, and $F(v)$ is a linear functional on $L^2_{-\beta}(\mathcal{T})$ if $f \in L^2_\beta(\mathcal{T})$. We introduce a new norm for the error e denoted by $|||e|||$ and $|||e|||_K$ by

$$|||e|||_K = \sup_{\|v\|_{H^{1,-\beta}(K)}=1} |B(e,v)_K| \leq \|e\|_{\tilde{H}^{1,\beta}(K)} \tag{4.1}$$

and

$$|||e||| = \sup_{\|v\|_{H^{1,-\beta}(\mathcal{T})}=1} |B(e,v)| \leq \|e\|_{\tilde{H}^{1,\beta}(\mathcal{T})}. \tag{4.2}$$

Remark 4.1 If $u \in W^{1,q}(\Omega)$ with $q > 2/\beta$ or $u \in H^{1+s}(\Omega)$ with $s > 1-\beta$, $0 < \beta < 1$, then $e \in \tilde{H}^{1,\beta}(K)$ and $\tilde{H}^{1,\beta}(\mathcal{T})$, which implies existence of $|||e|||_K$ and $|||e|||$, and there hold

$$\|e\|_{E(K)} \leq |||e|||_K \leq \|e\|_{\tilde{H}^{1,\beta}(K)}$$

and

$$\|e\|_{E(\Omega)} \leq |||e||| \leq \|e\|_{\tilde{H}^{1,\beta}(\mathcal{T})}.$$

Let $\beta_1 = \beta_2 = \beta \in (0,1)$. We define the error indicator on element K

$$\eta_K = p^{-\beta}\|r\|_{L^2_\beta(K)} \quad \forall K \in \mathcal{T}, \tag{4.3}$$

and the error indicator on internal edge γ_ℓ

$$\eta_\gamma = p^{-\beta}\|R\|_{L^2_\beta(\gamma)} \quad \forall \gamma \in \partial\mathcal{T}. \tag{4.4}$$

The error estimator is the sum of these indicators on elements and internal edges

$$\eta = \{\sum_{K\in\mathcal{T}} \eta_K^2 + \sum_{\gamma\in\partial\mathcal{T}} \eta_\gamma^2\}^{1/2}. \tag{4.5}$$

In the framework of the Jacobi-weighted Sobolev spaces we are able to prove the results on the error estimators and indicators.

THEOREM 4.1. *For $\beta \in (1/2, 1)$, there holds*

$$|||e||| \leq C(\epsilon,\beta)\, p^{\epsilon}\, \eta \tag{4.6}$$

with $C(\epsilon,\beta) = C_0\epsilon^{-1/2}\Gamma(1-\beta)$, $\epsilon > 0$, arbitrary.

THEOREM 4.2. *If $f \in S^p(\Omega, \mathcal{T})$ and $\beta \in (0,1)$, then there holds*

$$|||e|||_K \geq Cp^{\beta-1}\ \eta_K. \tag{4.7}$$

Furthermore, if $R\,|_{\gamma} \in \mathcal{P}_p(\gamma)$, there holds

$$\sum_{K\in Q_\gamma} |||e|||_K \geq \frac{1}{D(\beta)}\ \eta_\gamma \tag{4.8}$$

with $D(\beta) = C_0\Gamma^{1/2}(1-\beta)$, Q_γ is a pair of elements sharing γ.

In general $f \notin S^p(\Omega, \mathcal{T})$ and $R\,|_{\gamma} \notin \mathcal{P}_p(\gamma)$, we have to modify our error indicators on interior of K and at internal edges γ by

$$\tilde{\eta}_K = \|r_p\|_{L^2_\beta(K)} \tag{4.9}$$

with

$$r_p = f_p + \triangle u_p - u_p \tag{4.10}$$

where $f_p = \prod_K^\beta f$ is $L^2_\beta(K)$-projection of f on $\mathcal{P}_p(K)$, and

$$\tilde{\eta}_\gamma = \|R_p\|_{L^2_\beta(\gamma)} \tag{4.11}$$

where $R_p = \prod_\gamma^\beta R$ is $L^2_\beta(\gamma)$-projection of R on $\mathcal{P}_p(\gamma)$. Consequently, the error estimator is modified as

$$\tilde{\eta} = \{\sum_{K\in\mathcal{T}} \tilde{\eta}_K^2 + \sum_{\gamma\in\partial\mathcal{T}} \tilde{\eta}_\gamma^2\}^{1/2}.$$

Then Theorem 4.2 is modified to the following one.

THEOREM 4.3. *If $f \notin S^p(\Omega, \mathcal{T})$, then*

$$\tilde{\eta}_K \leq Cp^{1-\beta}|||e|||_K + p^{-\beta}\|f - f_p\|_{L^2_\beta(K)}. \tag{4.12}$$

Also, if $R\,|_{\gamma} \notin \mathcal{P}_p(\gamma)$, then

$$\tilde{\eta}_\gamma \leq D(\beta) \sum_{K\in Q_\gamma} |||e|||_K + p^{-\beta}\|R - R_p\|_{L^2_\beta(\gamma)}. \tag{4.13}$$

For the details of the proof of the above theorems we refer to [21]. The following corollary is a combination of the above theorems.

COROLLARY 4.1. *If choosing $\beta = 1-\epsilon, \epsilon \in (0,1)$ and adopting the modified indicators, there hold*

$$\tilde{\eta} \leq D(\epsilon)p^{\epsilon}\, |||e||| + p^{-(1-\epsilon)}\big(\|R - R_p\|_{L^2_\beta(\partial\mathcal{T})} + \|f - f_p\|_{L^2_\beta(\mathcal{T})}\big) \tag{4.14}$$

and

$$|||e||| \leq C(\epsilon, 1-\epsilon)\, p^{\epsilon}\, \{\eta + p^{-(1-\epsilon)}(\|R - R_p\|_{L^2_\beta(\partial\mathcal{T})} + \|f - f_p\|_{L^2_\beta(\mathcal{T})})\} \tag{4.15}$$

where $D(\epsilon) = C_0\Gamma^{1/2}(\epsilon)$ and $C(\epsilon, 1-\epsilon) = \tilde{C}_0\epsilon^{-1/2}\Gamma(\epsilon)$.

Remark 4.2 To evaluate the quality of the error estimator and indicator we assume again that $f \in S^p(\Omega, \mathcal{T})$ and $R\mid_\gamma \in \mathcal{P}_p(\gamma)$. Then $\eta = \tilde{\eta}$, and (4.14) $\sim$ (4.15) lead to

$$C_1(\epsilon, p) \le \frac{|||e|||}{\eta} \le C_2(\epsilon, p) \tag{4.16}$$

with $C_1(\epsilon, p) = \tilde{C}_1 \Gamma^{-1/2}(\epsilon)\, p^{-\epsilon}$ and $C_2(\epsilon) = \tilde{C}_2 \epsilon^{-1/2}\Gamma(\epsilon) p^{\epsilon}$, which implies

$$\tau = \frac{C_2(\epsilon, p)}{C_1(\epsilon, p)} = \epsilon^{-1/2}\Gamma^{3/2}(\epsilon) O(p^{2\epsilon}). \tag{4.17}$$

Therefore the above error estimator η is quasi optimal.

Remark 4.3 From the discussion in previous section, the operator of Clément interpolation type fails to bring us to the optimal error estimator. Hence we need to construct an operator $\Pi_{\mathcal{T}}^{-\beta} : H^{1,-\beta}(\mathcal{T}) \to S_0^{p,1}(\Omega, \mathcal{T})$, which is based on Jacobi projection on $\mathcal{P}_p(K)$ associate with the weight $-\beta \in (-1, -1/2)$ and is modified for the continuity in Ω. We call this operator quasi Jacobi projection. To prove Theorem 4.1 the quasi Jacobi projection $\Pi_{\mathcal{T}}^{-\beta}$ have to possess two desired properties:

$$\|v - \Pi_{\mathcal{T}}^{-\beta} v\|_{L^2_{-\beta}(K)} \le C(\epsilon, \beta) p^{-\beta+\epsilon} \|v\|_{H^{1,-\beta}(K)} \tag{4.18}$$

and

$$\|v - \Pi_{\mathcal{T}}^{-\beta} v\|_{L^2_{-\beta}(\gamma)} \le C(\epsilon, \beta) p^{-\beta+\epsilon} \|v\|_{H^{1,-\beta}(K)}. \tag{4.19}$$

For the construction of $\Pi_{\mathcal{T}}^{-\beta}$ and the proof of the above desired properties we refer to the forth coming paper [21]. To outline the proof of Theorem 4.1, we start with the variation equation

$$\begin{aligned} B(e, v) &= B(e, v - \Pi_{\mathcal{T}}^{-\beta} v) \\ &= \sum_{K \in \mathcal{T}} (r, v - \Pi_{\mathcal{T}}^{-\beta} v)_K + \sum_{\gamma \in \partial\mathcal{T}} (R, v - \Pi_{\mathcal{T}}^{-\beta} v)_\gamma \end{aligned}$$

which implies

$$|B(e, v)| \le \sum_{K \in \mathcal{T}} \|r\|_{L^2_\beta(K)} \|v - \Pi_{\mathcal{T}}^{-\beta} v\|_{L^2_{-\beta}(K)} + \sum_{\gamma \in \partial\mathcal{T}} \|R\|_{L^2_\beta(\gamma)}\, \|v - \Pi_{\mathcal{T}}^{-\beta} v\|_{L^2_{-\beta}(\gamma)}.$$

This, together with (4.18) and (4.19), leads easily to (4.6).

Remark 4.4 In order to prove (4.8) of Theorem 4.2

$$\eta_\gamma \le D(\beta) \sum_{K \in Q_\gamma} |||e|||_K,$$

we need a well-designed extension v_γ of R_γ to Ω with a support Q_γ, which leads to $v_\gamma \in S_0^{p,1}(\Omega, \mathcal{T})$,

$$B(e, v_\gamma) = \sum_{K \in Q_\gamma} (r, v_\gamma)_K + (R, v_\gamma)_\gamma. \tag{4.20}$$

Such an extension v_γ needs the following desired properties :

$$(R, v_\gamma)_\gamma = \|R\|^2_{L^2_\beta(\gamma)}, \tag{4.21}$$

$$\|v_\gamma\|_{L^2_{-\beta}(K)} \le D(\beta) p^{\beta-1} \|R\|_{L^2_\beta(\gamma)} \text{ for } K \in Q_\gamma, \tag{4.22}$$

$$|v_\gamma|_{H^{1,-\beta}(K)} \le D(\beta) p^{\beta} \|R\|_{L^2_\beta(\gamma)} \text{ for } K \in Q_\gamma. \tag{4.23}$$

Applying above three properties to (4.20) we have

$$\begin{aligned}\|R\|^2_{L^2_\beta(\gamma)} &\le \sum_{K\in Q_\gamma}\Big(\|r\|_{L^2_\beta(K)}\|v_\gamma\|_{L^2_{-\beta}(K)} + |||e|||_K\|v_\gamma\|_{H^{1,-\beta}(K)}\Big)\\ &\le D(\beta)p^\beta\|R\|_{L^2_\beta(\gamma)}\sum_{K\in Q_\gamma}|||e|||_K\end{aligned}$$

which leads immediately to (4.8).

For the construction of the extension v_γ and the proof of its properties we refer to forthcoming paper [21].

5. Concluding Remarks

After analyzing the results, methodology and techniques for residual based a-posteriori error for high-order FEM in one and two dimensions, we address the problem in new framework of Jacobi-weighted spaces. We introduce new norm $|||e|||$ for the error in this framework and proved the quasi optimality of the error estimator in the sense of (4.17), i.e:

$$C_1(\epsilon)\ p^{-\epsilon} \le \frac{|||e|||}{\eta} \le C_2(\epsilon)\ p^\epsilon$$

with $C_1(\epsilon) = C\Gamma^{-1/2}(\epsilon)$ and $C_1(\epsilon) = \tilde{C}\epsilon^{-1/2}\Gamma(\epsilon)$. This is the best one comparing with others in current literature.

The key for the quasi optimality is construction of quasi Jacobi projection based on local Jacobi projection. The exploration of its desired properties involves approximation, imbedding, extension and trace in Jacobi-weighted spaces. Various interpolation or quasi interpolation operator of Clément-type have been tried for desired results in two dimensions, but most of them, if not all, have failed to lead to the quasi optimal error estimator. The success in one-dimensional case relies largely on the operator Π_p^{-1} based on integration of local Legendre projection, but it can not be utilized simply for two-dimensional case. Instead, we should realize that Legendre projection works very well for both a-priori and a-posteriori error analysis because such projection operator reflects the nature of high-order polynomials, but only for one-dimensional case. For the high-order FEM in two- or three-dimensional cases we should build the theory, methods and techniques in Jacobi projection and Jacobi-weighted norms.

We next compare the approaches and results for cases in one dimension and in two dimensions.

(i) Projection $\Pi_{\mathcal{T}}^{-\beta}$:

In one dimension : Integral of local Legendre projection, and satisfies

$$(v - \Pi_{\mathcal{T}}^{-\beta} v)(x_i) = 0;$$

This projection is equivalent to the quasi Jacobi projection with weight $-\beta = -1$;

In two dimension : Quasi Jacobi projection $\Pi_{\mathcal{T}}^{-\beta}$ with $0 < \beta < 1$.

(ii) Error indicator η_K

In one dimension :

$$\eta_{K_i} = \|rW_i^{1/2}\|_{L^2(K_i)}, \quad W_i(x) = |(x - x_{i-1})(x_i - x)|^\beta,\ \beta = 1$$

In two dimension :

$$\eta_{K_i} = \|rW_i^{1/2}\|_{L^2(K_i)}, \quad W_i(x) = \prod_{\ell=1}^{L_i} |dist(x, \gamma_\ell^i)|^\beta, \ 0 < \beta < 1$$

(iii) Optimality of error estimators

One dimension :

$$C_1 \le \frac{|||e|||}{\eta} \le C_2 = 1$$

Two dimension :

$$C_1(\epsilon)\ p^{-\epsilon} \le \frac{|||e|||}{\eta} \le C_2(\epsilon)\ p^{\epsilon}.$$

After making the above comparison one may ask whether we can make the results optimal for two-dimensional case as those for one-dimensional case, e.g. by selecting $\beta = 1$. The answer is NO. First of all, it requires for $\beta = 1$

$$v - \prod_{\mathcal{T}}^{-\beta} v = 0 \text{ on the boundary of elements .}$$

It can be done ONLY for one dimensional case by subtracting a piecewise linear function, and but it is impossible in two-dimensional case unless the solution u is polynomials on the internal edges. Secondly, the constant $C_i(\epsilon), i = 1, 2$ will blow up as $\beta \to 1$. At third, there is another factor to prevent the optimality, which is the imbedding inequality in trace theorem. It causes a loss of $O(p^\epsilon)$ in addition to $O(p^{(1-\beta)})$, which never happens in one dimension. Hence, there is an essential difference between one-dimensional case and two-dimensional case, which prevent us from achieving the optimality. We have a conjecture that the quasi optimality may be the best result one can expected for error estimator based on residual in two and three dimensions.

The methodology and techniques for the p-version of FEM can be easily used for the spectral method and p-version of boundary element method(BEM). By a proper scaling argument, the above results are expected to be generalized to the h-p of FEM and BEM without substantial difficulties.

The current work for quadrilateral elements can be generalized to triangular elements if the Jacobi weights and Jacobi projection in triangular elements and quasi Jacobi projection on triangular partition $\mathcal{T}$ are properly designed. To this direction, one may combine the results, approach and techniques of following papers,

- Braess & Schwab(1999) : Jacobi weight in the simplices [15]
- Dubiner(1991) and B.Y. Guo & Wang (2004): Jacobi projection and orthogonal polynomial base for triangular elements [16, 23].
- B.Q. Guo & Ainsworth : A-posterior error analysis for the p-version of FEM for quadrilateral elements [21].

It is expected that the results on triangular meshes may not be as sharp as those on quadrilateral meshes and the proof will be much complicated because of the obvious difficulties. It is also expected that substantial losses could be avoided.

The approximation theory in the framework of the Jacobi-weighted Besov and Sobolev spaces has been developed as the mathematical foundation for the approximation theory of the p and h-p (with quasi uniform meshes) version of FEM and BEM , spectral and spectral element methods in one, two, and three dimensions. It is expected to be very helpful to a-posteriori error analysis for the p- and h-p version of FEM and BEM, spectral method and spectral element methods. The work presented in previous section is just the beginning, and a comprehensive study on a posteriori error analysis based on both residuals and solutions of local auxiliary problems is needed. Nevertheless, it certainly present a new way of thinking, a new approach, and a new direction, which seem encouraging and promising.

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