A Posteriori Error Estimates of Mixed Methods for Two Phase Flow Problems

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Abstract. Two phase flow problem in a porous medium is governed by a system of nonlinear equations. One is an elliptic equation for the pressure and the other is a parabolic equation for the concentration of one of the fluids. Since the pressure appears in the concentration only through its velocity field, we choose a mixed finite element to approximate the pressure equation and use the standard Galerkin method to treat the concentration equation. We shall obtain an explicit a posteriori error estimator in $L^2(L^2)$ for the semi-discrete scheme of the nonlinear coupled system.

1. Introduction

As a generalized mathematical model problem for single and two phase flow problems in porous media, the following nonlinear coupled system of partial differential equations is also called the miscible displacement problem. We look for a pressure $p(x,t)$, a Darcy velocity $\boldsymbol{u}$ and a concentration of one of the fluids $c(x,t)$ (see [**5**]):

$$\phi\frac{\partial c}{\partial t} + \boldsymbol{u}\cdot\nabla c - \nabla\cdot(D(\boldsymbol{u})\nabla c) = (1-c)q^+, \tag{1.1}$$

$$\nabla\cdot\boldsymbol{u} = q, \tag{1.2}$$

$$\boldsymbol{u} = -a(c)\mathrm{grad}p, \tag{1.3}$$

for $x \in \Omega \in \mathbb{R}^2$ and $t \in J = [0,T]$, with $a(c) = a(x,c) = k(x)/\mu(c)$, where $k(x)$ is the permeability of the porous rock, $\mu(c)$ is the viscosity of the fluid mixture,

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$\boldsymbol{u}(x,t)$ is the Darcy velocity of the mixture, $q(x,t)$ represents the flow rate at wells and q^+ its positive part, $\phi(x)$ is the porosity of the rock. The above system is subject to boundary conditions

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0, \qquad x\in\partial\Omega \quad t\in J, \tag{1.4}$$

$$D(\boldsymbol{u})\nabla c\cdot\boldsymbol{n} = 0, \qquad x\in\partial\Omega \quad t\in J, \tag{1.5}$$

and an initial condition

$$c(x,0) = c_0(x), \qquad x\in\Omega. \tag{1.6}$$

In (1.1) and (1.5), $D(\boldsymbol{u})$ is the coefficient of molecular diffusion and mechanical dispersion of one fluid into the other, i.e. D is a 2×2 matrix,

$$D = \phi(x)\left[d_m I + |\boldsymbol{u}|\left(d_l E(\boldsymbol{u}) + d_t E^{\perp}(\boldsymbol{u})\right)\right],$$

where E is the projection along the direction of flow given by

$$E(\boldsymbol{u}) = \left(u_i u_j/|\boldsymbol{u}|^2\right),$$

$E^{\perp} = I - E$, d_m is the molecular diffusion coefficient, and d_l and d_t are, respectively, the longitudinal and transverse dispersion coefficients.

We assume that the functions $a(c)$ and $\phi(x)$ are bounded above and below by positive constants and the matrix D is uniformly positive-definite:

$$\sum_{i,j=1}^{2} D_{ij}(\boldsymbol{u})\xi_i\xi_j \geq D_*|\xi|^2, \qquad \xi\in\mathbb{R}^2, \tag{1.7}$$

with D_* being independent of x and $\boldsymbol{u}$.

Two phase flow and transport of fluids in porous media is important in many applications. For example, petroleum engineers have been interested in efficient recovery of energy resources, and hydrologists have been concerned with improvement of groundwater resource utilization for a long time. Recently, Chen and Ewing [**4**] have studied the mathematical theory for these models. On the numerical side, the study of the miscible displacement problem (1.1)-(1.6) has been extensively carried out using finite element methods (see, e.g., [**5**]).

It is known that a posteriori error estimates are computable quantities that measure the actual error without having the knowledge of the exact solution. They are essential in designing algorithms for mesh and time-step modification. During the last decade, there has been increasing interest in the development of reliable and efficient adaptive algorithms for various linear and nonlinear partial differential equations. However, it is observed that there is little work on deriving a posteriori error estimates for the nonlinear coupled system (1.1)-(1.6). Ohlberger has designed an adaptive mesh refinement method for single and two phase flow problems in porous media in [**11**] and [**1**]. However, he only obtained a posteriori error estimates separately for the pressure equation and the concentration equation which are used to implement an adaptive scheme. In [**13**], Sun and Wheeler proposed and analyzed an explicit a posteriori error estimator of discontinuous Galerkin approximations for the transport equation with the Darcy velocity field $\boldsymbol{u}$ being given.

In this paper, we adopt the method introduced by Chen and Dai [**3**] to derive a posteriori error estimates for the nonlinear coupled system (1.1)-(1.6), which provides the necessary information to modify the mesh and time-step according to the varying external flow rate. The estimates are based on the analysis of a dual problem. As the variable of physical interest is the concentration $c(x,t)$ and only the Darcy velocity $\boldsymbol{u}$ appears in the concentration equation (1.1), it is natural to use a mixed finite element method directly to approximate $\boldsymbol{u}$ and p. This avoids the differentiation of p and the multiplication by the coefficient k/μ that can be rough. For simplicity, we use the standard Galerkin method to treat the concentration equation, which is convection dominated.

Now, we state some standard notations which will be used in this article. We denote $W^{m,p}(\Omega)$ the Sobolev spaces on Ω with a norm $||\cdot||_{m,p}$ given by $||\phi||^p_{m,p} = \sum_{|\alpha|\leq m} ||D^\alpha \phi||^p_{L^p(\Omega)}$. We set $W^{m,p}_0(\Omega) = \{\phi \in W^{m,p}(\Omega) : \phi|_{\partial\Omega} = 0\}$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H^m_0(\Omega) = W^{m,2}_0(\Omega)$, $||\cdot||_m = ||\cdot||_{m,2}$ and $||\cdot|| = ||\cdot||_{0,2}$. In addition C denotes a general positive constant independent of h.

The paper is organized as follows. In Section 2 we first propose the weak formulation and the corresponding discrete finite element scheme for the miscible displacement problem, and then we state some preliminary results. In Section 3 we introduce some intermediate variables. Next we derive some intermediate error estimators of residual type for the pressure and velocity in Section 4 and residual type intermediate error estimates for the concentration in Section 5. Finally, in Section 6 we obtain some a posteriori error estimators for the coupled problem.

2. Formulation of the method and preliminaries

Let the spaces $\boldsymbol{V}$ and W be defined by

$$\begin{aligned} \boldsymbol{V} &= \{\boldsymbol{v} \in H(\mathrm{div};\Omega), \qquad \boldsymbol{v}\cdot\boldsymbol{n} = 0 \;\; x \in \partial\Omega\}, \\ W &= \{w \in L^2(\Omega), \qquad (w,1) = 0\}, \end{aligned}$$

which are endowed with the norm

$$||\boldsymbol{v}||_V = ||\boldsymbol{v}||_{H(\mathrm{div};\Omega)} = \left(||\boldsymbol{v}||^2_{0,\Omega} + ||\mathrm{div}\boldsymbol{v}||^2_{0,\Omega}\right)^{1/2}$$

and $||w||$, respectively. It is clear that $\nabla\cdot\boldsymbol{V} = W$. Then a weak form of (1.1)-(1.6) is given by the determination of a map $\{c,\boldsymbol{u},p\}$: $J \to H^1(\Omega)\times\boldsymbol{V}\times W$ such that $c(\cdot,0) = c_0$ and for $t \in J$,

$$\left(\phi\frac{\partial c}{\partial t}, z\right) + (\boldsymbol{u}\cdot\nabla c, z) + (D(\boldsymbol{u})\nabla c, \nabla z) = ((1-c)q^+, z), \;\; z \in H^1(\Omega), \tag{2.1}$$

$$(\alpha(c)\boldsymbol{u},\boldsymbol{v}) - (\nabla\cdot\boldsymbol{v}, p) = 0, \qquad \boldsymbol{v} \in \boldsymbol{V}, \tag{2.2}$$

$$(\nabla\cdot\boldsymbol{u}, w) = (q, w), \qquad w \in W, \tag{2.3}$$

where $\alpha(c) = a(c)^{-1}$.

Let $\mathcal{T}_h$ be a family of regular finite element partitions of Ω and $\mathcal{E}_h$ be the union of all edges of the elements on a given mesh. Let $\mathcal{E}_{I,h} \subset \mathcal{E}_h$ be the set of all interior edges. For an integer $l \geq 1$, let $M_h \subset H^1(\Omega)$ be the standard finite element space of degree l. We approximate the concentration equation (1.1) by a standard Galerkin procedure based on the space M_h. For approximating the pressure and Darcy velocity equations (1.2)-(1.3), we choose a Raviart-Thomas mixed finite element space $\boldsymbol{V}_h \times W_h \subset \boldsymbol{V} \times W$ of index $k \geq 0$ over the partition $\mathcal{T}_h$. Thus, we preserve the relation $\nabla \cdot \boldsymbol{V}_h = W_h$. The continuous-time finite element procedure of (2.1)-(2.3) is given by the finding of $\{c_h, \boldsymbol{u}_h, p_h\}$: $J \to M_h \times \boldsymbol{V}_h \times W_h$ such that $c_h(x,0) = c_h^0 \in M_h$, $(\phi c_h^0, z) = (\phi c_0, z)$, $\forall\ z \in M_h$ and for $t \in J$,

$$\left(\phi \frac{\partial c_h}{\partial t}, z\right) + (\boldsymbol{u}_h \cdot \nabla c_h, z) + (D(\boldsymbol{u}_h)\nabla c_h, \nabla z) = ((1-c_h)q^+, z),\ z \in M_h, \tag{2.4}$$

$$(\alpha(c_h)\boldsymbol{u}_h, \boldsymbol{v}) - (\nabla \cdot \boldsymbol{v}, p_h) = 0, \qquad \boldsymbol{v} \in \boldsymbol{V}_h, \tag{2.5}$$

$$(\nabla \cdot \boldsymbol{u}_h, w) = (q, w), \qquad w \in W_h. \tag{2.6}$$

We will make use of the L^2-projection P_h onto W_h, defined as

$$(w - P_h w, w_h) = 0, \qquad \forall\ w_h \in W_h, \tag{2.7}$$

and the Raviart-Thomas projection Π_h onto $\boldsymbol{V}_h$, defined as

$$(\nabla \cdot (\boldsymbol{v} - \Pi_h \boldsymbol{v}), w_h) = 0, \qquad \forall\ w_h \in W_h. \tag{2.8}$$

For example, on a rectangular element τ of index k, for any sufficiently smooth vector-valued function $\boldsymbol{v}$ one can locally define RT projection Π_h by solving the following system of linear equations:

$$\begin{aligned} &((\boldsymbol{v} - \Pi_h \boldsymbol{v}) \cdot \boldsymbol{n}_e, p)_e = 0, && \forall\ \ p \in P_k(e),\quad e \in \partial\tau,\quad \tau \in \mathcal{T}_h,\\ &(\boldsymbol{v} - \Pi_h \boldsymbol{v}, \boldsymbol{q})_\tau = 0, && \forall\ \ \boldsymbol{q} \in Q_{k-1,k}(\tau) \times Q_{k,k-1}(\tau),\quad \tau \in \mathcal{T}_h. \end{aligned} \tag{2.9}$$

From [**2**] and standard interpolation results, the following approximation properties hold. For all $\tau \in \mathcal{T}_h$ and smooth enough functions $\boldsymbol{v}$ and w,

$$||w - P_h w||_\tau \leq C h_\tau ||w||_{1,\tau}, \tag{2.10}$$

$$||\boldsymbol{v} - \Pi_h \boldsymbol{v}||_\tau \leq C h_\tau ||\boldsymbol{v}||_{1,\tau}. \tag{2.11}$$

The following lemmas are important in driving a posteriori error estimates of residual type for the concentration.

Lemma 2.1. *Let $\hat{\pi}_h$ be the average interpolation operator defined in* [**12**]. *For* $m = 0$ *or* 1, $1 \leq r \leq \infty$ *and* $\forall z \in W^{1,r}(\Omega)$,

$$|z - \hat{\pi}_h z|_{W^{m,r}(\tau)} \leq C h_\tau^{1-m} \sum_{\tau' \cap \tau \neq \emptyset} |z|_{W^{1,r}(\tau')}, \quad \forall\ \tau \in \mathcal{T}_h. \tag{2.12}$$

From the standard functional analysis theory (see [**8**]), we know that

LEMMA 2.2. *For all* $z \in W^{1,r}(\Omega)$, $1 \leq r \leq \infty$,

$$|z|_{W^{0,r}(\partial\tau)} \leq C\left(h_\tau^{-\frac{1}{r}}|z|_{W^{0,r}(\tau)} + h_\tau^{1-\frac{1}{r}}|z|_{W^{1,r}(\tau)}\right), \quad \forall\, \tau \in \mathcal{T}_h. \tag{2.13}$$

3. Some intermediate variables

Consider the following auxiliary problem to find $(\tilde{\boldsymbol{u}}, \tilde{p}) \in \boldsymbol{V} \times W$ such that

$$(\alpha(c_h)\tilde{\boldsymbol{u}}, \boldsymbol{v}) - (\nabla \cdot \boldsymbol{v}, \tilde{p}) = 0, \qquad \boldsymbol{v} \in \boldsymbol{V}, \tag{3.1}$$

$$(\nabla \cdot \tilde{\boldsymbol{u}}, w) = (q, w), \qquad w \in W. \tag{3.2}$$

Subtract (2.5)-(2.6) from (3.1)-(3.2) to derive the following error equations

$$(\alpha(c_h)(\tilde{\boldsymbol{u}} - \boldsymbol{u}_h), \boldsymbol{v}) - (\nabla \cdot \boldsymbol{v}, \tilde{p} - p_h) = 0, \quad \boldsymbol{v} \in \boldsymbol{V}_h, \tag{3.3}$$

$$(\nabla \cdot (\tilde{\boldsymbol{u}} - \boldsymbol{u}_h), w) = 0, \qquad w \in W_h. \tag{3.4}$$

Similarly, by subtracting (3.1)-(3.2) from (2.2)-(2.3) we have

$$(\alpha(c)(\boldsymbol{u} - \tilde{\boldsymbol{u}}), \boldsymbol{v}) - (\nabla \cdot \boldsymbol{v}, p - \tilde{p}) = -((\alpha(c) - \alpha(c_h))\tilde{\boldsymbol{u}}, \boldsymbol{v}), \quad \boldsymbol{v} \in \boldsymbol{V}, \tag{3.5}$$

$$(\nabla \cdot (\boldsymbol{u} - \tilde{\boldsymbol{u}}), w) = 0, \qquad w \in W. \tag{3.6}$$

By using stability of mixed weak form, we have

$$||\boldsymbol{u} - \tilde{\boldsymbol{u}}||_V + ||p - \tilde{p}||_W \leq C||\tilde{\boldsymbol{u}}||_{0,\infty}||c - c_h||. \tag{3.7}$$

Introduce another intermediate variable $\tilde{c} \in H^1(\Omega)$ satisfying the initial condition $\tilde{c}(x, 0) = c_0(x)$ for $x \in \Omega$ and

$$\left(\phi\frac{\partial}{\partial t}\tilde{c}, z\right) + (\boldsymbol{u}_h \cdot \nabla\tilde{c}, z) + (D(\boldsymbol{u}_h)\nabla\tilde{c}, \nabla z) = ((1 - \tilde{c})q^+, z), \quad z \in H^1(\Omega). \tag{3.8}$$

We subtract (3.8) from (2.1) to obtain that

$$\begin{aligned}&\left(\phi\frac{\partial}{\partial t}(c - \tilde{c}), z\right) + (\boldsymbol{u}_h \cdot \nabla(c - \tilde{c}), z) + (D(\boldsymbol{u}_h)\nabla(c - \tilde{c}), \nabla z) + ((c - \tilde{c})q^+, z) \\ &\quad = -((\boldsymbol{u} - \boldsymbol{u}_h) \cdot \nabla c, z) - ([D(\boldsymbol{u}) - D(\boldsymbol{u}_h)]\nabla c, \nabla z), \quad z \in H^1(\Omega),\end{aligned} \tag{3.9}$$

and subtract (2.4) from (3.8) to get that

$$\begin{aligned}&\left(\phi\frac{\partial}{\partial t}(\tilde{c} - c_h), z\right) + (\boldsymbol{u}_h \cdot \nabla(\tilde{c} - c_h), z) + (D(\boldsymbol{u}_h)\nabla(\tilde{c} - c_h), \nabla z) \\ &\qquad + (q^+(\tilde{c} - c_h), z) = 0, \qquad z \in M_h.\end{aligned} \tag{3.10}$$

4. Residual based error estimators for the pressure and velocity

First, we bound the intermediate error of the pressure $||\tilde{p} - p_h||$, by using a duality argument (see [**7**] and [**14**]). For any $t \in J$, consider the auxiliary problem

$$\begin{aligned}\nabla \cdot \boldsymbol{v} &= \tilde{p} - p_h & x \in \Omega, \\ \boldsymbol{v} &= -a(c_h)\nabla w & x \in \Omega, \\ \boldsymbol{v} \cdot \boldsymbol{n} &= 0, & x \in \partial\Omega.\end{aligned} \tag{4.1}$$

We assume the above problem possesses the elliptic regularity

$$||w||_2 \le C||\tilde{p} - p_h||. \tag{4.2}$$

These functions satisfy the following variational problem

$$\begin{aligned} &(\alpha(c_h)\boldsymbol{v}, \boldsymbol{\psi}) - (w, \nabla\cdot\boldsymbol{\psi}) = 0, \quad && \boldsymbol{\psi} \in \boldsymbol{V}, \\ &(\nabla\cdot v, \varphi) = (\tilde{p} - p_h, \varphi), && \varphi \in W. \end{aligned} \tag{4.3}$$

Now, let $\varphi = \tilde{p} - p_h$ and $\boldsymbol{\psi} = \tilde{\boldsymbol{u}} - \boldsymbol{u}_h$ in (4.3). By subtracting the second equation from the first one in (4.3) and using the orthogonality relation (3.3)-(3.4), we can prove that

$$\begin{aligned} ||\tilde{p} - p_h||^2 &\le C \sum_{\tau\in\mathcal{T}_h} \Big(||q - \nabla\cdot\boldsymbol{u}_h||_\tau \cdot h_\tau ||w||_{1,\tau} + ||\alpha(c_h)\boldsymbol{u}_h + \nabla p_h||_\tau \cdot h_\tau ||\boldsymbol{v}||_{1,\tau} \Big) \\ &\le C \left\{ \sum_{\tau\in\mathcal{T}_h} \Big(h_\tau ||q - \nabla\cdot\boldsymbol{u}_h||_\tau + h_\tau ||\alpha(c_h)\boldsymbol{u}_h + \nabla p_h||_\tau \Big)^2 \right\}^{1/2} \cdot ||w||_2. \end{aligned} \tag{4.4}$$

Set

$$\omega_\tau \equiv h_\tau ||q - \nabla\cdot\boldsymbol{u}_h||_\tau + h_\tau ||\alpha(c_h)\boldsymbol{u}_h + \nabla p_h||_\tau \tag{4.5}$$

and the elliptic regularity (4.2) implies that

$$||\tilde{p} - p_h|| \le C \left(\sum_{\tau\in\mathcal{T}_h} \omega_\tau^2 \right)^{1/2} := \eta_1. \tag{4.6}$$

Next, we bound the intermediate error of the velocity $||\nabla\cdot(\tilde{\boldsymbol{u}} - \boldsymbol{u}_h)||$ and $||\tilde{\boldsymbol{u}} - \boldsymbol{u}_h||$. From (3.2), it is easy to see that

$$||\nabla\cdot(\tilde{\boldsymbol{u}} - \boldsymbol{u}_h)||^2 = (\nabla\cdot(\tilde{\boldsymbol{u}} - \boldsymbol{u}_h), \nabla\cdot(\tilde{\boldsymbol{u}} - \boldsymbol{u}_h)) = (q - \nabla\cdot\boldsymbol{u}_h, \nabla\cdot(\tilde{\boldsymbol{u}} - \boldsymbol{u}_h)),$$

hence

$$||\nabla\cdot(\tilde{\boldsymbol{u}} - \boldsymbol{u}_h)|| \le ||q - \nabla\cdot\boldsymbol{u}_h||. \tag{4.7}$$

To derive a bound on $||\tilde{\boldsymbol{u}} - \boldsymbol{u}_h||$ we need a saturation assumption (see the argument in [**14**]). Let $\boldsymbol{V}_h'$ and W_h' be the finite element spaces of one order higher than $\boldsymbol{V}_h$ and W_h, respectively. Let $\boldsymbol{u}_h' \in \boldsymbol{V}_h'$ and $p_h' \in W_h'$ be the mixed finite element solution in the higher-order spaces (see (2.5)-(2.6)). The a priori error estimates from [**6**] motivate the following
Saturation assumption: There exist constants $\beta < 1$ and β' such that

$$||\tilde{\boldsymbol{u}} - \boldsymbol{u}_h'|| \le \beta ||\tilde{\boldsymbol{u}} - \boldsymbol{u}_h||, \tag{4.8}$$

$$||\tilde{p} - p_h'|| \le \beta' ||\tilde{p} - p_h||. \tag{4.9}$$

Similar to (3.3)-(3.4), we have that $(\boldsymbol{u}'_h-\boldsymbol{u}_h, p'_h-p_h) \in \boldsymbol{V}'_h \times W'_h$ satisfy the residual equations

$$(4.10)\ (\alpha(c_h)(\boldsymbol{u}'_h-\boldsymbol{u}_h),\boldsymbol{v})-(\nabla\cdot\boldsymbol{v},p'_h-p_h) = -(\alpha(c_h)\boldsymbol{u}_h,\boldsymbol{v})+(\nabla\cdot\boldsymbol{v},p_h),\ \ \boldsymbol{v}\in\boldsymbol{V}'_h,$$
$$(4.11)\ (\nabla\cdot(\boldsymbol{u}'_h-\boldsymbol{u}_h),w) = (q-\nabla\cdot\boldsymbol{u}_h,w), \qquad w\in W'_h,$$

and the orthogonality condition

$$(\alpha(c_h)(\boldsymbol{u}'_h-\boldsymbol{u}_h),\boldsymbol{v})-(\nabla\cdot\boldsymbol{v},p'_h-p_h)=0,\quad \boldsymbol{v}\in\boldsymbol{V}_h, \tag{4.12}$$
$$(\nabla\cdot(\boldsymbol{u}'_h-\boldsymbol{u}_h),w)=0, \qquad w\in W_h. \tag{4.13}$$

Set $\boldsymbol{v}=(\boldsymbol{u}'_h-\boldsymbol{u}_h)-\Pi_h(\boldsymbol{u}'_h-\boldsymbol{u}_h)$ and $w=p'_h-p_h$ in (4.10) and (4.11), respectively, add the two relations using the lower bound of $\alpha(c_h)$, (4.12)-(4.13), the Green's formula and (2.9), we derive that

$$||\tilde{\boldsymbol{u}}-\boldsymbol{u}_h|| \le \frac{1}{1-\beta}||\boldsymbol{u}'_h-\boldsymbol{u}_h|| \le \frac{C}{1-\beta}\left(\sum_{\tau\in\mathcal{T}_h}\omega_\tau^2+\tilde{\omega}_\tau^2\right)^{1/2} := \eta_2, \tag{4.14}$$

where

$$\tilde{\omega}_\tau \equiv ||q-\nabla\cdot\boldsymbol{u}_h||_\tau + ||\alpha(c_h)\boldsymbol{u}_h+\nabla p_h||_\tau. \tag{4.15}$$

By using (4.7) we derive that

$$||\tilde{\boldsymbol{u}}-\boldsymbol{u}_h||_V \le \eta_2. \tag{4.16}$$

5. Residual based intermediate error estimators for the concentration

The following analysis is based on the arguments in [**10**] and [**13**]. We first recall a theorem from [**9**]:

THEOREM 5.1. *Consider the parabolic equation*

$$\begin{aligned}
\phi(x)\frac{\partial s}{\partial t}-\nabla\cdot(\boldsymbol{\beta}(x,t)s+K(x,t)\nabla s)+\lambda(x,t)s &= f(x,t), \quad x\in\Omega \quad t\in J,\\
K(x,t)\nabla s\cdot\boldsymbol{n} &= 0, \quad x\in\partial\Omega \quad t\in J,\\
s(x,0) &= 0, \quad x\in\Omega \quad t=0,
\end{aligned}$$

where f, K, $\boldsymbol{\beta}$, λ and ϕ are known real-valued functions in $L^\infty(\Omega)$, of which K is a matrix function and $\boldsymbol{\beta}$ is a vector function. It is assumed that ϕ and eigenvalues of K are bounded below and above by positive constants and λ is a nonnegative function. Then there exists a unique solution s satisfying the above equation and the regularity bounds given by

$$||s||_{L^\infty(J;H^1(\Omega))}+||s||_{L^2(J;H^2(\Omega))} \le C||f||_{L^2(J;L^2(\Omega))},$$

where C is a constant independent of the input data f.

We need the condition

$$\|\boldsymbol{u}_h\|_{L^\infty(\Omega)} \le C. \tag{5.1}$$

Now we consider the "backward" parabolic equation

$$\begin{aligned}
-\phi(x)\tfrac{\partial s}{\partial t} - \nabla\cdot(\boldsymbol{u}_h s + D(\boldsymbol{u}_h)\nabla s) + q^+ s &= \tilde{c} - c_h, && x\in\Omega && t\in J,\\
D(\boldsymbol{u}_h)\nabla s\cdot \boldsymbol{n} &= 0, && x\in\partial\Omega && t\in J,\\
s(x,0) &= 0, && x\in\Omega && t=T.
\end{aligned} \tag{5.2}$$

From Theorem 5.1, we know there exists a unique solution s for the above equation, which satisfies

$$\|s\|_{L^\infty(J;H^1(\Omega))} + \|s\|_{L^2(J;H^2(\Omega))} \le C\|\tilde{c} - c_h\|_{L^2(J;L^2(\Omega))}. \tag{5.3}$$

We then use the integrating by parts and apply the orthogonality (3.10) for $\tilde{c}-c_h$. It follows from (5.2), (3.8), the Green's formula, and Lemmas 2.1 and 2.2 that

$$\begin{aligned}
\|\tilde{c} - c_h\|^2_{L^2(J;L^2(\Omega))} \quad &\le \\
C\sum_{\tau\in\mathcal{T}_h}\Big(h_\tau^2 &\left\|\phi(x)\tfrac{\partial c_h}{\partial t} + \boldsymbol{u}_h\cdot\nabla c_h - \nabla\cdot(D(\boldsymbol{u}_h)\nabla c_h) - q^+(1-c_h)\right\|^2_{L^2(J;L^2(\tau))}\\
&+ h_\tau\|D(\boldsymbol{u}_h)\nabla c_h\cdot\boldsymbol{n}_\tau\|^2_{L^2(J;L^2(\partial\tau))} + h_\tau^2\|c_0 - c_h^0\|^2_\tau\Big)^{\frac{1}{2}}\\
&\cdot\left(\|s\|^2_{L^2(J;H^1(\Omega))} + \|s\|^2_{L^\infty(J;H^1(\Omega))}\right)^{\frac{1}{2}}.
\end{aligned}$$

Using the regularity (5.3) of the dual parabolic equation (5.2), we obtain that

$$\begin{aligned}
\|\tilde{c} - c_h\|^2_{L^2(J;L^2(\Omega))} \quad &\le \\
C\sum_{\tau\in\mathcal{T}_h}\Big(h_\tau^2 &\left\|\phi(x)\tfrac{\partial c_h}{\partial t} + \boldsymbol{u}_h\cdot\nabla c_h - \nabla\cdot(D(\boldsymbol{u}_h)\nabla c_h) - q^+(1-c_h)\right\|^2_{L^2(J;L^2(\tau))}\\
&+ h_\tau\|D(\boldsymbol{u}_h)\nabla c_h\cdot\boldsymbol{n}_\tau\|^2_{L^2(J;L^2(\partial\tau))} + h_\tau^2\|c_0 - c_h^0\|^2_\tau\Big)^{\frac{1}{2}} := \eta_3.
\end{aligned} \tag{5.4}$$

6. A posteriori error estimators for the coupled problem

Let $\xi = c - \tilde{c}$. Then we choose $z = \xi$ in (3.9) to get

$$\begin{aligned}
\left(\phi\frac{\partial\xi}{\partial t},\xi\right) &+ (\boldsymbol{u}_h\cdot\nabla\xi,\xi) + (D(\boldsymbol{u}_h)\nabla\xi,\nabla\xi) + (q^+\xi,\xi)\\
&= -((\boldsymbol{u}-\boldsymbol{u}_h)\cdot\nabla c,\xi) - ([D(\boldsymbol{u}) - D(\boldsymbol{u}_h)]\nabla c,\nabla\xi).
\end{aligned} \tag{6.1}$$

It follows from §5 and §7 in [**5**] that

$$|([D(\boldsymbol{u}) - D(\boldsymbol{u}_h)]\nabla c,\nabla\xi)| \le C\|\nabla c\|_{0,\infty}\|\boldsymbol{u}-\boldsymbol{u}_h\|\cdot\|\nabla\xi\|, \tag{6.2}$$

$$|((\boldsymbol{u}-\boldsymbol{u}_h)\cdot\nabla c,\xi)| \le \|\nabla c\|_{0,\infty}\|\boldsymbol{u}-\boldsymbol{u}_h\|\cdot\|\nabla\xi\|, \tag{6.3}$$

and the left-hand side of (6.1) is bounded below by

$$\frac{1}{2}\frac{d}{dt}(\phi\xi,\xi)+(\phi(d_m+d_t|\boldsymbol{u}|)\nabla\xi,\nabla\xi)-\big(q^-\xi,\xi\big)-C(1+||\xi||^2)||\xi||^2. \tag{6.4}$$

The application of (6.2)-(6.4) to (6.1) leads to the evolution inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\phi\xi,\xi)+(\phi(d_m+d_t|\boldsymbol{u}|)\nabla\xi,\nabla\xi) &\le C\left[(1+||\xi||^2)||\xi||^2\right.\\ &\quad \left.+||\nabla c||_{0,\infty}||\boldsymbol{u}-\boldsymbol{u}_h||\right],\end{aligned} \tag{6.5}$$

where we used the ε-Cauchy's inequality. From (3.7) and (4.14), we note that

$$\begin{aligned}||\boldsymbol{u}-\boldsymbol{u}_h|| &\le ||\boldsymbol{u}-\tilde{\boldsymbol{u}}||+||\tilde{\boldsymbol{u}}-\boldsymbol{u}_h||\\ &\le C||\tilde{\boldsymbol{u}}||_{0,\infty}||c-c_h||+\eta_2\\ &\le C||\tilde{\boldsymbol{u}}||_{0,\infty}\left(||\xi||+||\tilde{c}-c_h||\right)+\eta_2.\end{aligned} \tag{6.6}$$

Let us make the induction hypothesis as in [**5**] that

$$||\xi||_{L^\infty(J;L^2(\Omega))}\le 1. \tag{6.7}$$

Certainly, the initial condition $\xi(0)=0$ holds since we have made the choice of $\tilde{c}(x,0)=c_0(x)$ in (3.8). Furthermore, we make some regularity assumption that there exists a positive constant C such that

$$||\nabla c||_{0,\infty}\le C, \qquad ||\tilde{\boldsymbol{u}}||_{0,\infty}\le C. \tag{6.8}$$

Substitute (6.6) into (6.5), and use the assumptions (6.7)-(6.8) and (5.4). Using the Gronwall lemma and noting $\xi(0)=0$ give

$$||c-\tilde{c}||_{L^\infty(J;L^2(\Omega))}\le \eta_2+\eta_3. \tag{6.9}$$

Finally, combining (3.7), (4.6)-(4.7), (4.14), (5.4), (6.9), and note that the estimator η_1 is of higher order than that for the estimator η_2, we can obtain the following main result:

THEOREM 6.1. *Let $\{c,\boldsymbol{u},p\}\in H^1(\Omega)\times\boldsymbol{V}\times W$ and $\{c_h,\boldsymbol{u}_h,p_h\}\in M_h\times\boldsymbol{V}_h\times W_h$ be the solutions of (2.1)-(2.3) and (2.4)-(2.6) respectively. Then, we have*

$$||c-c_h||_{L^2(J;L^2(\Omega))}+||p-p_h||_{L^\infty(J;L^2(\Omega))}+||\boldsymbol{u}-\boldsymbol{u}_h||_{L^\infty(J;L^2(\Omega))}\le\eta_2+\eta_3. \tag{6.10}$$

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