Spreading speed and traveling waves for the diffusive logistic equation with a sedentary compartment

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Outline

1. Introduction

2. Preliminaries

3. The spreading speed

4. Traveling wave solutions

5. Conclusion

1. Introduction

It is well known that the diffusive logistic or Verhulst equation is a scalar reaction diffusion equation with a simple hump nonlinearity (quadratic nonlinearity in the classical case). This equation describes the immigration of a species into a territory or the advance of an advantageous gene into a population. The equation provides the classical example for traveling fronts in parabolic equations, and it forms the nucleus of more complex multi-species models in ecology, pattern formation and epidemiology. In order to consider the case where the population individuals switch between mobile and stationary states during their lifetime, Lewis and Schmitz (1996) presented and analysed the following reaction-diffusion model

$$\begin{cases} \partial_t v = D\Delta v - \mu v - \gamma_2 v + \gamma_1 w, \\ \partial_t w = rw(1 - w/K) - \gamma_1 w + \gamma_2 v, \end{cases} \tag{1}$$

where $v(t,x)$ and $w(t,x)$ are spatial densities of migrating and sedentary subpopulations, respectively, D is diffusion coefficient of migrating subpopulation, γ_1 and γ_2 are transition rates between two states. In model (1), the migrants have a positive mortality μ while the sedentary subpopulation reproduces (with the intrinsic growth rate r) and is subject to a finite carrying capacity K. They determined the minimal speed for traveling waves under the assumption that the emigration rate is less than the intrinsic growth rate for the sedentary class ($\gamma_1 < r$). Recently, Hadeler and Lewis (2002) studied, among others, the spread rate for the system (1) in the general case. We note that the existence and nonexistence of monotone traveling wave, and hence the existence of minimal wave speed, for system (1) need to be investigated further.

The purpose of this work is to use the theory developed in a number of papers for nonlinear integral equations to study the asymptotic speed of spread and monotone traveling waves of system (1). For convenience and other possible applications, we then consider the following general diffusive logistic equation with a sedentary compartment

$$\begin{cases} \partial_t v(t,x) = D\Delta v(t,x) - rv(t,x) + f(w(t,x)), \\ \partial_t w(t,x) = g(w(t,x)) + \beta v(t,x), \end{cases} \tag{2}$$

with initial conditions

$$v(0,x) = \phi_1(x) \geq 0,\ w(0,x) = \phi_2(x) \geq 0, x \in \mathbb{R}^n, \tag{3}$$

where D, r and β are positive constants, and the conditions on functions f and g are to be specified in section 3.

2. Preliminaries

In this section, based on the paper [H. R. Thieme, X.-Q. Zhao, *J. Differential Equations*, (2003)], we present the preliminary results that will be used in the subsequent sections.

Consider nonlinear integral equations

$$u(t,x) = u_0(t,x) + \int_0^t \int_{\mathbb{R}^n} F(u(t-s, x-y), s, y)\, dyds, \tag{4}$$

where $F : \mathbb{R}_+^2 \times \mathbb{R}^n \to \mathbb{R}$ is continuous in u and Borel measurable in (s, y), and $u_0 : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is Borel measurable and bounded. Assume that

(A) There exists a function $k : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ such that

(A1) $k^* := \int_0^\infty \int_{\mathbb{R}^n} k(s, x)\, dxds < \infty$.

(A2) $0 \leq F(u,s,x) \leq uk(s,x), \forall u,s \geq 0, x \in \mathbb{R}^n$.

(A3) For every compact interval I in $(0,\infty)$, there exists some $\varepsilon > 0$ such that

$$F(u,s,x) \geq \varepsilon k(s,x), \quad \forall u \in I, s \geq 0, x \in \mathbb{R}^n.$$

(A4) For every $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$F(u,s,x) \geq (1-\varepsilon)uk(s,x), \ \ \forall u \in [0,\delta], s \geq 0, x \in \mathbb{R}^n.$$

(A5) For every $w > 0$, there exists some $\Lambda > 0$ such that

$$|F(u,s,x) - F(v,s,x)| \leq \Lambda |u-v| k(s,x), \\ \forall u,v \in [0,w], s \geq 0, x \in \mathbb{R}^n.$$

To obtain asymptotic properties of the solutions of equation (4), we make a couple of assumptions concerning k.

(B) $k : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is a Borel measurable function such that

(B1) $k^* := \int_0^\infty \int_{\mathbb{R}^n} k(s,y)\, dyds \in (1,\infty)$.

(B2) There exists some $\lambda^\diamondsuit > 0$ such that

$$\int_0^\infty \int_{\mathbb{R}^n} e^{\lambda^\diamondsuit y_1} k(s,y)\, dyds < \infty,$$

where y_1 is the first coordinate of y.

(B3) There exist numbers $\sigma_2 > \sigma_1 > 0, \rho > 0$ such that

$$k(s,x) > 0, \quad \forall s \in (\sigma_1, \sigma_2), |x| \in [0,\rho).$$

(B4) k is isotropic.
Here a function $k : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is said to be isotropic if for almost all $s > 0$, $k(s,x) = k(s,y)$

whenever $|x| = |y|$. For a fixed $z \in \mathbb{R}^n$ with $|z| = 1$, define

$$\mathcal{K}(c,\lambda) := \int_0^\infty \int_{\mathbb{R}^n} \mathrm{e}^{-\lambda(cs - z\cdot y)} k(s,y)\, dyds, \qquad \forall c \geq 0,\ \lambda \geq 0,$$

where $\cdot$ means the usual inner product on $\mathbb{R}^n$. Assume that k is isotropic, there holds

$$\mathcal{K}(c,\lambda) = \int_0^\infty \int_{\mathbb{R}^n} \mathrm{e}^{-\lambda(cs + y_1)} k(s,y)\, dyds,$$

where y_1 is the first coordinate of y. Define

$$c^* := \inf\{c \geq 0 : \mathcal{K}(c,\lambda) < 1 \text{ for some } \lambda > 0\}.$$

The following result is useful for the computation of c^*.

Proposition 2.1. *Let (B) hold and assume that* $\liminf_{\lambda \nearrow \lambda^{\#}(c)} \mathcal{K}(c,\lambda) \geq k^*$ *for every* $c > 0$*. Then there exists a unique* $\lambda^* \in (0, \lambda^{\#}(c^*))$ *such that* $\mathcal{K}(c^*,\lambda^*) = 1$ *and* $\mathcal{K}(c^*,\lambda) > 1$ *for*

$\lambda \neq \lambda^*$. Moreover, c^* and λ^* are uniquely determined as the solutions of the system

$$\mathcal{K}(c,\lambda) = 1, \ \frac{d}{d\lambda}\mathcal{K}(c,\lambda) = 0.$$

Definition 2.1. A number $c^* > 0$ is called the asymptotic speed of spread for a function $u : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ if $\lim_{t\to\infty, |x|\geq ct} u(t,x) = 0$ for every $c > c^*$, and there exists some $\bar{u} > 0$ such that $\lim_{t\to\infty, |x|\leq ct} u(t,x) = \bar{u}$ for every $c \in (0, c^*)$.

The following two results show that c^* defined above is the asymptotic speed of spread for solutions of (4).

Theorem 2.1. *Let (A) and (B) hold and let $u(t,x)$ be a solution of (4) with $u_0(t,x)$ being admissible. Then $\lim_{t\to\infty, |x|\geq ct} u(t,x) = 0$ for each $c > c^*$.*

Theorem 2.2. *Let (A) and (B) hold and let $u_0 : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ be a bounded and*

Borel measurable function with the property that $u_0(t,x) \geq \eta > 0, \forall t \in (t_1,t_2), |x| \leq \eta$, *for appropriate* $t_2 > t_1 \geq 0$ *and* $\eta > 0$. *Also, let* u *be a bounded solution of (4) and* $u^\infty := \limsup_{t\to\infty} \sup_{x\in\mathbb{R}^n} u(t,x)$. *Assume that* $F(\,\cdot\,,s,x)$ *is monotone increasing on* $[0,u^\infty]$ *for each* $(s,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ *and* $\lim_{t\to\infty} u_0(t,x) = 0$ *uniformly in* $x \in \mathbb{R}^n$. *Let* $u^* > 0$ *be such that* $\tilde{F}(u) := \int_0^\infty \int_{\mathbb{R}^n} F(u,s,y)\,dyds > u$ *whenever* $u \in (0,u^*)$ *and* $\tilde{F}(u) < u$ *whenever* $u \in (u^*,u^\infty]$. *Then we have* $\lim_{t\to\infty,|x|\leq ct} u(t,x) = u^*$, $\forall c \in (0,c^*)$.

Next we consider the limiting equation of (4) with $n = 1$

$$u(t,x) = \int_0^\infty \int_{\mathbb{R}} F(u(t-s,x-y),s,y)\,dyds. \quad (5)$$

A solution $u(t,x)$ of (5) is said to be a traveling wave solution if it is of the form $u(t,x) = U(x+ct)$. The parameter c is called the wave speed, and the function $U(\cdot)$ is called the wave profile.

Here, we require the following conditions on the wave profile:

$$U(\cdot) \text{ is positive and bounded on } \mathbb{R}, \text{ and } \lim_{\xi \to -\infty} U(\xi) = 0. \quad (6)$$

The following two results deal with the existence and nonexistence of traveling wave solutions of (5).

Theorem 2.3. *Let (A2) and (B) with $n = 1$ hold. Assume that there exists some $u^* > 0$ such that $\tilde{F}(u^*) = u^*$ and $\tilde{F}(u) > u$ for all $u \in (0, u^*)$, where $\tilde{F}(u) := \int_0^\infty \int_{\mathbb{R}} F(u, s, y)\, dyds$. Moreover, suppose that $F(\,\cdot\,, s, x)$ is increasing on $[0, u^*]$ for each $(s, x) \in \mathbb{R}_+ \times \mathbb{R}$, and $F(u, s, x) \geq (u - bu^\sigma)k(s, x), \forall u \in [0, \delta], (s, x) \in \mathbb{R}_+ \times \mathbb{R}$, for appropriate $\delta \in (0, u^*], \sigma > 1$ and $b > 0$. Then for each $c > c^*$, there exists a monotone traveling wave solution of (5) with speed c and connecting 0 and u^*.*

Theorem 2.4. *Let (A) and (B) hold. Then for each $c \in (0, c^*)$, there exists no traveling wave solution of (5) and (6) with speed c.*

Finally, we consider nonlinear integral equations

$$u(t,x) = u_0(t,x) + \int_0^t e^{-as} f_0(u(t-s,x))ds + \int_0^t \int_{\mathbb{R}^n} F_0\Big(u(t-s,x-y),s,y\Big)dyds \quad (7)$$

where $a > 0$, $f_0 \in C(\mathbb{R}_+, \mathbb{R})$, $F_0 : \mathbb{R}_+^2 \times \mathbb{R}^n \to \mathbb{R}$ is continuous in u and Borel measurable in (s,y), and $u_0 : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is Borel measurable and bounded. We assume that

(H1) $f_0 \in C^1(\mathbb{R}_+, \mathbb{R}_+)$, $f_0'(u) \geq 0$ and $f_0(u) \leq f_0'(0)u$ for all $u \geq 0$.

(H2) $F_0(u,s,x)$ satisfies (A1)-(A5), and the associated $k_0(s,x)$ satisfies (B2)-(B4).

Using the measure integral for Dirac function $\delta(x)$ on $\mathbb{R}^n$, we write equation (7) as

$$
\begin{aligned}
u(t,x) &= u_0(t,x) \\
&\quad + \int_0^t \int_{\mathbb{R}^n} e^{-as} f_0(u(t-s,x-y))\delta(y)dyds \\
&\quad + \int_0^t \int_{\mathbb{R}^n} F_0\Big(u(t-s,x-y),s,y\Big)dyds.
\end{aligned}
$$

It then follows that (7) can be written formally as the equation (4) with

$$
\begin{aligned}
F(u,s,x) &:= f_0(u)e^{-as}\delta(x) + F_0(u,s,x), \\
k(s,x) &:= f_0'(0)e^{-as}\delta(x) + k_0(s,x).
\end{aligned}
$$

Remark 1 *By modifying slightly the previous proofs, we see that Theorems 2.1–2.4 in this section remain valid for equation (7) provided that assumptions (H1), (H2) and (B1) hold. Note that in all integral computations it is understood that* $\int_{\mathbb{R}^n} \phi(x-y)\delta(y)dy = \phi(x)$.

3. The spreading speed

Motivated by the biological model (1), we impose the following conditions on equation (2).

(C1) $f : \mathbb{R}_+ \to \mathbb{R}_+$ is Lipschitz continuous and nondecreasing, differentiable at 0, $f(0) = 0$, $f(u) > 0, \forall u > 0$, and f is sublinear on $\mathbb{R}_+$ in the sense that $f(\theta w) \geq \theta f(w)$ for any $\theta \in (0,1), w \in \mathbb{R}_+$.

(C2) $g : \mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable, $g(0) = 0$, strictly sublinear on $\mathbb{R}_+$ in the sense that $g(\theta w) > \theta g(w)$ for any $\theta \in (0,1), w > 0$.

(C3) $\beta f'(0) + r g'(0) > 0$, and there exists $w^* > 0$ such that $r g(w^*) + \beta f(w^*) = 0$.

Consider the reaction system associated with (2)

$$\begin{cases} \frac{dv}{dt} = -rv + f(w), \\ \frac{dw}{dt} = g(w) + \beta v. \end{cases} \tag{8}$$

Because of assumptions (C1)–(C3) on f and g, system (8) is cooperative on $\mathbb{R}^2_+$, and admits a positive equilibrium $(\frac{f(w^*)}{r}, w^*)$. Also, two roots of the characteristic equation associated with the linearization at zero equilibrium of (8) are

$$\lambda_\pm = \frac{g'(0) - r \pm \sqrt{[g'(0) - r]^2 + 4[\beta f'(0) + rg'(0)]}}{2},$$

and hence, $\lambda_+ > 0$ and $\lambda_- < 0$. It is easy to see that every solution to (8) with nonnegative initial value remains nonnegative. Also, system (8) admits a unique steady state $(\frac{f(w^*)}{r}, w^*)$, which is globally asymptotically stable in $\mathbb{R}^2_+ \backslash \{0\}$. By the standard comparison arguments, it follows that solutions to (8) are uniformly bounded on $\mathbb{R}^2_+$.

Let $\mathbb{X} := BUC(\mathbb{R}^n, \mathbb{R}^2)$ be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}^n$ to $\mathbb{R}^2$ with the usual supreme norm, and

$$\mathbb{X}_+ = \{(\phi_1, \phi_2) \in \mathbb{X} : \phi_i(x) \geq 0,\ \forall x \in \mathbb{R}^n, i = 1, 2\}.$$

Then $\mathbb{X}_+$ is a positive cone of $\mathbb{X}$, and its induced partial ordering makes $\mathbb{X}$ into a Banach lattice.

Lemma 3.1. *Let (C1)–(C3) hold. For any $\phi \in \mathbb{X}_+$, system (2) has a unique, bounded and nonnegative mild solution $U(t,x,\phi) = (v(t,x,\phi), w(t,x,\phi))$ with $U(0,\cdot,\phi) = \phi$, and the solution semiflow associated with (2) is monotone on $\mathbb{X}_+$.*

In the rest of this section, we will find the spreading speed c^* for solutions of system (2). In order to use the theory prsented in Section

2, we need to reduce (2)–(3) into a scalar integral equation. Let $\Gamma(t, x-y)$ be the Green function associated with the parabolic equation

$$\begin{cases} \partial_t u = D\Delta u, \\ u(0,x) = \phi(x), \quad x \in \mathbb{R}^n, t > 0. \end{cases}$$

Then $\partial_t v = D\Delta v - rv$ generates a linear semigroup $T(t) : BUC(\mathbb{R}^n, \mathbb{R}) \to BUC(\mathbb{R}^n, \mathbb{R})$, which is defined by

$$(T(t)\phi)(x) = \mathrm{e}^{-rt} \int_{\mathbb{R}^n} \Gamma(t, x-y)\phi(y)\, dy, \quad \forall \phi \in BUC(\mathbb{R}^n, \mathbb{R}). \tag{9}$$

Integrating the first equation of system (2)

$$\partial_t v(t,x) = D\Delta v(t,x) - rv(t,x) + f(w(t,x)),$$

we have the following abstract integral form

$$v(t) = T(t)v(0) + \int_0^t T(t-s) f(w(s))\, ds,$$

that is,

$$v(t,x) = \mathrm{e}^{-rt}\int_{\mathbb{R}^n}\Gamma(t,x-y)\phi_1(y)\,dy + \int_0^t \mathrm{e}^{-r(t-s)}\int_{\mathbb{R}^n}\Gamma(t-s,x-y)f(w(s,y))\,dyds. \quad (10)$$

Given $\alpha > 0$, we define a nondecreasing function $g_\alpha(\,\cdot\,)$ on $\mathbb{R}_+$ by

$$g_\alpha(w) = \sup\{\alpha u + g(u) : 0 \le u \le w\}, \quad \forall w \ge 0.$$

Then, for any bounded solution of (2), we can choose sufficiently large $\alpha > 0$ such that the second equation in system (2)

$$\partial_t w(t,x) = g(w(t,x)) + \beta v(t,x)$$

takes the form

$$\partial_t w(t,x) = -\alpha w(t,x) + g_\alpha(w(t,x)) + \beta v(t,x). \quad (11)$$

It follows from (11) that

$$w(t,x) = \mathrm{e}^{-\alpha t}\phi_2(x) + \beta \int_0^t \mathrm{e}^{-\alpha(t-s)} v(s,x)\, ds$$
$$+ \int_0^t \mathrm{e}^{-\alpha(t-s)} \int_{\mathbb{R}^n} \delta(x-y) g_\alpha(w(s,y))\, dy ds, \tag{12}$$

where $\delta(x)$ is the Dirac function. After a substitution, we have

$$\int_0^t ds \mathrm{e}^{-\alpha(t-s)} \int_{\mathbb{R}^n} \delta(x-y) g_\alpha(w(s,y))\, dy$$
$$= \int_0^t ds \int_{\mathbb{R}^n} k_1(s, x-y) g_\alpha(w(t-s,y))\, dy, \tag{13}$$

where $k_1(s,x) = \mathrm{e}^{-\alpha s}\delta(x), \forall x \in \mathbb{R}^n$ and $\forall s \geq 0$. By (10), we obtain

$$\int_0^t \mathrm{e}^{-\alpha(t-s)} v(s,x)\, ds = G(t,x)$$
$$+ \int_0^t ds \mathrm{e}^{-\alpha(t-s)} \mathrm{e}^{-rs} \int_{\mathbb{R}^n} \Gamma(s, x-y) \phi_1(y)\, dy \tag{14}$$

with

$$G(t,x) = \int_0^t ds \int_{\mathbb{R}^n} k_2(s, x-y) f(w(t-s,y))\, dy, \tag{15}$$

where $k_2(s,x) = \mathrm{e}^{-\alpha s} \int_0^s \mathrm{e}^{(\alpha - r)s_1} \Gamma(s_1, x)\, ds_1, \forall x \in \mathbb{R}^n$ and $\forall s \geq 0$.

Inserting (13)–(15) into (12), we obtain

$$\begin{aligned} w(t,x) &= w_0(t,x) \\ &+ \int_0^t \int_{\mathbb{R}^n} F_\alpha(w(t-s, x-y), s, y)\, dy ds, \end{aligned} \tag{16}$$

where

$$\begin{aligned} w_0(t,x) &= \mathrm{e}^{-\alpha t} \phi_2(x) \\ &+ \beta \int_0^t ds \mathrm{e}^{-\alpha(t-s)} \mathrm{e}^{-rs} \int_{\mathbb{R}^n} \Gamma(s, x-y) \phi_1(y)\, dy \end{aligned}$$

and

$$F_\alpha(w,s,y) = g_\alpha(w) k_1(s,y) + \beta f(w) k_2(s,y). \tag{17}$$

Let $\alpha + g'(0) > 0$. In view of (17), we define

$$k(s,y) := g'_\alpha(0) k_1(s,y) + \beta f'(0) k_2(s,y). \tag{18}$$

It follows that assumption (A) holds for (16).

Next, we need to compute some Laplace-like transforms of integral kernels. For any function $\phi : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$, let

$$\mathcal{K}_\phi(c,\lambda) := \int_0^\infty \int_{\mathbb{R}^n} \mathrm{e}^{-\lambda(cs+y_1)} \phi(s,y)\, dyds, \;\; c,\lambda \geq 0,$$

where y_1 is the first coordinate of y. It follows that

$$\mathcal{K}_k(c,\lambda) = \frac{1}{\lambda c + \alpha}\left(g'_\alpha(0) - \frac{\beta f'(0)}{\lambda^2 D - \lambda c - r}\right). \tag{19}$$

We define

$$c^* := \inf\{c \geq 0 : \mathcal{K}_k(c,\lambda) < 1 \text{ for some } \lambda > 0\}.$$

According to Proposition 2.1, c^* can be uniquely determined as the positive solution of the system

$$\mathcal{K}_k(c,\lambda) = 1, \quad \frac{d}{d\lambda}\mathcal{K}_k(c,\lambda) = 0.$$

That is, (c^*, λ^*) is the unique positive solution of the system

$$\begin{cases} (g'(0) - \lambda c)(\lambda^2 D - \lambda c - r) = \beta f'(0), \\ c(\lambda^2 D - \lambda c - r)^2 = \beta f'(0)(2\lambda D - c). \end{cases} \tag{20}$$

Let

$$P(c, \lambda) := a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0, \tag{21}$$

where the coefficients a_i $(i = 0, \cdots, 3)$ are given in terms of the original parameters as

$$a_0 = -[\beta f'(0) + r g'(0)], \quad a_1 = -c[g'(0) - r], \\ a_2 = c^2 + D g'(0), \quad a_3 = -cD.$$

A direct computation shows that (20) is equivalent to

$$P(c, \lambda) = 0 \quad \text{and} \quad \frac{\partial P}{\partial \lambda}(c, \lambda) = 0. \tag{22}$$

It follows that $P(c, \lambda)$ has two positive roots for $c > c^*$, one positive double root for $c = c^*$, and two complex roots for $0 < c < c^*$.

We now transform (22) so that it is expressed in terms of parameter c. Set

$$P(c,\lambda) = P_1(c,\lambda)Q_1(c,\lambda) + R_1(c,\lambda),$$
$$P_1(c,\lambda) = R_1(c,\lambda)Q_2(c,\lambda) + R_2(c),$$

where $P_1(c,\lambda) = \frac{\partial P}{\partial \lambda}(c,\lambda)$, $Q_1(c,\lambda)$ and $R_1(c,\lambda)$ are the quotient and remainder of $P(c,\lambda)$ divided by $P_1(c,\lambda)$, and $Q_2(c,\lambda)$ and $R_2(c)$ are the quotient and remainder of $P_1(c,\lambda)$ divided by $R_1(c,\lambda)$, respectively. Clearly, we must have $R_2(c^*) = 0$. By direct calculations, we see that $R_2(c) = 0$ is equivalent to

$$18a_0a_1a_2a_3 - 4a_2^3a_0 + a_2^2a_1^2 - 27a_3^2a_0^2 - 4a_1^3a_3 = 0,$$

that is,

$$\begin{aligned}\psi(c^2) := {} & 18Dc^2[c^2 + Dg'(0)][g'(0) - r]a_0 \\ & - 4[c^2 + Dg'(0)]^3a_0 \\ & + c^2[c^2 + Dg'(0)]^2[g'(0) - r]^2 \\ & - 27D^2c^2a_0^2 - 4Dc^4[g'(0) - r]^3 = 0.\end{aligned}$$

Sorting out terms with respect to c, we have

$$\psi(c^2) = c^6\{[g'(0) - r]^2 - 4a_0\} - 4D^3g'^3(0)a_0 + c^4D\{18[g'(0) - r]a_0 - 4[g'(0) - r]^3 - 12g'(0)a_0 + 2g'(0)[g'(0) - r]^2\} + c^2D^2\{18g'(0)[g'(0) - r]a_0 - 12g'^2(0)a_0 + g'^2(0)[g'(0) - r]^2 - 27a_0^2\}.$$

Thus, $\psi(c^{*2}) = 0$ and c^* is the positive square root of the largest zero of the cubic $\psi(x)$.

The subsequent result shows that c^* is the asymptotic speed of spread for solutions of (2) with initial functions having compact supports. In order to obtain the convergence for $0 < c < c^*$, we need the following additional condition:

(C4) $\beta f(w)+rg(w) > 0$, $\forall w \in (0, w^*)$, and $\beta f(w)+rg(w) < 0$, $\forall w > w^*$.

Theorem 3.1. *Let (C1)–(C3) hold and c^* be the positive square root of the largest zero of the cubic $\psi(x)$. Assume that $\phi = (\phi_1, \phi_2) \in \mathbb{X}_+$ has the property that $\phi_1(\cdot) + \phi_2(\cdot) \not\equiv 0$, and that for every $\kappa_1 > 0$, there exists $\kappa_2 > 0$ such that $\phi_1(y) + \phi_2(y) \le \kappa_2 e^{-\kappa_1 |y|}, \forall y \in \mathbb{R}^n$. Then the unique solution $u(t,x) = (v(t,x), w(t,x))$ of system (2)-(3) satisfies*

(i) $\lim_{t\to\infty,\, |x|\ge ct} u(t,x) = (0,0),\ \forall c > c^*$.

(ii) If, in addition, (C4) holds, then $\lim_{t\to\infty, |x|\le ct} u(t,x) = (v^*, w^*)$, $\forall c \in (0, c^*)$, *where w^* is the unique positive solution of $rg(w) + \beta f(w) = 0$, and $v^* = \frac{f(w^*)}{r}$.*

As an application, let us consider system (1), where $D, \mu, \gamma_1, \gamma_2, r$ and K are positive constants. It is easy to verify that system (1) satisfies conditions (C1)–(C4) provided $r > \frac{\mu\gamma_1}{\mu+\gamma_2}$.

Setting

$$
\begin{aligned}
\psi_0(x) := x^3[&(r-\gamma_1-\mu-\gamma_2)^2 \\
&+4(\mu r+r\gamma_2-\mu\gamma_1)] \\
&+x^2 D[-18(r-\gamma_1-\mu-\gamma_2) \\
&\times(\mu r+r\gamma_2-\mu\gamma_1) \\
&+12(r-\gamma_1)(\mu r+r\gamma_2-\mu\gamma_1) \\
&+2(r-\gamma_1)(r-\gamma_1-\mu-\gamma_2)^2 \\
&-4(r-\gamma_1-\mu-\gamma_2)^3] \\
&+xD^2[-18(r-\gamma_1) \\
&\times(r-\gamma_1-\mu-\gamma_2)(\mu r+r\gamma_2-\mu\gamma_1) \\
&+12(r-\gamma_1)^2(\mu r+r\gamma_2-\mu\gamma_1) \\
&+(r-\gamma_1)^2(r-\gamma_1-\mu-\gamma_2)^2 \\
&-27(\mu r+r\gamma_2-\mu\gamma_1)^2] \\
&+4D^3(r-\gamma_1)^3(\mu r+r\gamma_2-\mu\gamma_1),
\end{aligned}
$$

we then have the following result.

Proposition 3.1. *Let $r > \frac{\mu\gamma_1}{\mu+\gamma_2}$ hold, and c^* be the positive square root of the largest zero of the cubic $\psi_0(x)$. Assume that $\phi = (\phi_1, \phi_2) \in$*

$\mathbb{X}_+$ *has the property that* $\phi_1(\cdot)+\phi_2(\cdot) \not\equiv 0$*, and that for every* $\kappa_1 > 0$*, there exists* $\kappa_2 > 0$ *such that* $\phi_1(y) + \phi_2(y) \leq \kappa_2 e^{-\kappa_1|y|}, \forall y \in \mathbb{R}^n$*. Then the unique solution* $u(t,x) = (v(t,x), w(t,x))$ *of system (1) with (3) satisfies*

(i) $\lim_{t\to\infty,\, |x|\geq ct} u(t,x) = (0,0),\ \forall c > c^*.$

(ii) $\lim_{t\to\infty, |x|\leq ct} u(t,x) = (v^*, w^*)$*,* $\forall c \in (0, c^*)$*, where* $w^* = K\left(1 - \frac{\mu\gamma_1}{r(\mu+\gamma_2)}\right)$ *and* $v^* = \frac{\gamma_1 K}{\mu+\gamma_2}\left(1 - \frac{\mu\gamma_1}{r(\mu+\gamma_2)}\right)$.

Remark 3.1. (22) implies that the spreading speed c^* of (2) can also be obtained as the largest value c such that the polynomial $P(c,\lambda)$ defined by (21) has a real positive double root. For system (1), c^* defined in Proposition 3.1 coincides with the spreading rate $\overline{c}$ in [3, Theorem 1].

4. Traveling wave solutions

In this section, we consider the existence and nonexistence of traveling wave solutions of system (2) with $n = 1$. We will show that there is a minimal wave speed for monotone traveling waves and it coincides with the spreading speed c^* obtained in section 3.

Theorem 4.1. *Let (C1)–(C3) hold, and let c^*, v^*, w^* be defined as in Theorem 3.1. Then the following statements are valid:*

(i) System (2) with $n = 1$ admits no traveling wave solution with wave speed $c \in (0, c^*)$.

(ii) Assume in addition that (C4) holds, $f''(0)$ exists, and there exist $\delta, b, \theta > 0$ such that $g'(u) - g'(0) \geq -bu^{\theta}, \forall u \in [0, \delta]$. Then for every $c \geq c^*$, system (2) with $n = 1$ has a

monotone traveling wave connecting $(0,0)$ *and* (v^*, w^*) *with speed* c.

Returning to system (1), we have the following result.

Proposition 4.1. *Let* $r > \frac{\mu\gamma_1}{\mu+\gamma_2}$ *hold, and let* c^*, v^*, w^* *be defined as in Proposition 3.1. Then the following statements are valid:*

(i) System (1) with $n = 1$ *subject to (**??**) admits no traveling wave solution with wave speed* $c \in (0, c^*)$.

(ii) For every $c \geq c^*$*, system (1) with* $n = 1$ *has a monotone traveling wave connecting* $(0,0)$ *and* (v^*, w^*) *with speed* c.

5. Conclusion

In this talk, by applying the theory of asymptotic speeds of spread and traveling waves to the diffusive logistic equation with a sedentary compartment, we establish the existence of minimal wave speed for monotone traveling waves and show that it coincides with the spreading speed for solutions with initial functions having compact supports.

Thank you!

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