Control Charts Based on Likelihood Ratio For Preliminary Analysis of Linear Profile

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(Joint work with Changliang Zou & Yi Dai)


Jackie Chan's
WHO AM I ?
Fight now. Ask questions later.
DVD

Outline

- Background & Motivation
  (Phase I & Phase II)
- Control Charts for Linear Profile
- Our Proposed CUSUM Charts
- Performance Comparison
- Conclusion & Consideration
- And then...

- Background & Motivation
  - Phase II control charts
    - Definition

      The distribution of the quality characteristic is completely known.
    - Goal

      To detect the shifts in the process mean and/or variance, quickly.

∗ Methods

· Shewhart charts (Shewhart 1924)

$$LCL = \mu - 3\frac{\sigma}{\sqrt{n}}, \; CL = \mu, \; UCL = \mu + 3\frac{\sigma}{\sqrt{n}}$$

· Shewhart chart with Runs Rules (Champ & Woodall 1987): $T(k, m, a, b)$ (k of the last m standardized sample means fall in the interval (a, b))

· Exponentially Weighted Moving Average (EWMA, Robert 1959)

$$Y_n = (1 - \lambda)Y_{n-1} + \lambda \bar{X}_n, Y_0 = \mu.$$

- Cumulative Sum (CUSUM, Page 1954)

$$S_n^+ = \max\{0, S_{n-1}^+ + \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} - k\}, \;\; S_0^+ = 0,$$

$$S_n^- = \min\{0, S_{n-1}^- + \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} + k\}, \;\; S_0^- = 0.$$

- Others (Adaptive EWMA, Double EWMA, Dual EWMA, Weighted CUSUM,...)

– Phase I control charts

* Definition

  The historical data is used to decide if the process is statistically in control and to estimate the parameters of the process.

  · Stage 1 (retrospective): To test if the process is statistically in control.

  · Stage 2 (prospective): To estimate the unknown parameters.

* Methods

· X and MR charts (Nelson 1982)

Shewhart Chart: 3σ principle

· LRT chart (Sullivan & Woodall 1996)

$$lr(n_1, n) = -2(l_0 - l_1) = n \log[\widehat{\sigma}_n^2 (\widehat{\sigma}_{n_1}^2)^{-\frac{n_1}{n}} (\widehat{\sigma}_{n_2}^2)^{-\frac{n_2}{n}}]$$
$$= V_{lrt} + M_{lrt}.$$

where $l_i = \max_{H_i} \log L(\mu, \sigma^2 | x), i = 0, 1.$

- CUSUM chart (Koning & Does 2000)
  The model is considered to be

$$X_i = \mu + i\theta + \epsilon_i.$$

The CUSUM statistics are given by

$$S_i^+ = \max\{0, S_{i-1}^+ + \sqrt{i(i-1)}(Y_i - f\sqrt{i(i-1)})\}$$

$$S_i^- = \max\{0, S_{i-1}^- + \sqrt{i(i-1)}(-Y_i - f\sqrt{i(i-1)})\}$$

where $Y_i = \sqrt{\frac{i-1}{i}}(X_i - \bar{X}_{i-1})$ is the standardized residual.

(It's a UMP test for this trend model)

- Others (Brown, Durbin, & Evans 1975, Quesenberry 1991, Wavelet Method)
  BDE: $\frac{1}{S_n}\Sigma_{j=1}^{i} Y_j$
  Queseberry: $Q_i = \Phi^{-1}\left(G_i - 2(\frac{Y_i}{S_{i-1}})\right)$

- Structural Model for AR(1) (Boyles 2000)

- Control Charts for Linear Profile
  - The Model of Linear Profile

    The process quality is characterized by a relationship between a response and one explanatory variable as follows:

$$\begin{cases} Y_{ij} = A_{0j} + A_{1j}X_i + \epsilon_{ij}, \\ \epsilon_{ij} \sim N(0, \sigma_j^2),\ i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, k. \end{cases}$$

* In-control:

$$A_{0j} = A_0, A_{1j} = A_1, \sigma_j = \sigma,\ j = 1, \cdots, k$$

* Out-of-control:

$$A_{0j} = A_0, A_{1j} = A_1, \sigma_j = \sigma,\ j = 1, \cdots, k_1,$$

$$A_{0j} = A_0', A_{1j} = A_1', \sigma_j = \sigma',\ j = k_1+1, \cdots, k_1+k_2.$$

– The Methods for the Linear profile (Phase I)

  * T^2 charts (Stover & Brill 1998, Kang & Albin 2000)
  * EWMA charts (Kim, Mahmoud, & Woodall 2003)
    · Three EWMA charts.
    · R Chart.
  * Shewhart charts (Mahmoud & Woodall 2004)
    · Three Shewhart charts.
    · F-test based on a multiple regression model.

  For details: Woodall et al (2004) (JQT: Journal of Quality Technology)

- Our Proposed CUSUM Charts

  Outline

  – Some notations
  – The limit distribution of $lr(k_1 n, kn)$
  – the expectation & variance of $lr(k_1 n, \infty)$
  – Our proposed CUSUM chart
  – The design of our proposed CUSUM chart
  – The detecting ability for intercept, slope, and variance

– Some notations

Historical Data:

$$\{(x_i, y_{ij}),\ i = 1, \cdots, n,\ j = 1, \cdots, k\}$$

Mean & Variance:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad S_{xx} = \sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$\bar{y}_{kn} = \frac{1}{kn}\sum_{j=1}^{k}\sum_{i=1}^{n} y_{ij}, \qquad S_{xy(kn)} = \sum_{j=1}^{k}\sum_{i=1}^{n}(x_i - \bar{x})y_{ij}$$

$$\bar{y}_{k_1 n} = \frac{1}{k_1 n}\sum_{j=1}^{k_1}\sum_{i=1}^{n} y_{ij}, \qquad S_{xy(k_1 n)} = \sum_{j=1}^{k_1}\sum_{i=1}^{n}(x_i - \bar{x})y_{ij}$$

$$\bar{y}_{k_2 n} = \frac{1}{k_2 n}\sum_{j=k_1+1}^{k}\sum_{i=1}^{n} y_{ij}, \qquad S_{xy(k_2 n)} = \sum_{j=k_1+1}^{k}\sum_{i=1}^{n}(x_i - \bar{x})y_{ij}$$

Estimations (MLE):

$$\widehat{\sigma}^2_{kn} = \frac{1}{kn} \sum_{j=1}^{k} \sum_{i=1}^{n} (y_{ij} - \widehat{A}_{0(kn)} - \widehat{A}_{1(kn)} x_i)^2$$

$$\widehat{A}_{1(kn)} = \frac{S_{xy(kn)}}{kS_{xx}}, \ \widehat{A}_{0(kn)} = \bar{y}_{kn} - \widehat{A}_{1(kn)} \bar{x}$$

Maximized Likelihood:

$$l_0 = -\frac{kn}{2} \log(2\pi) - \frac{kn}{2} \log(\widehat{\sigma}^2_{kn}) - \frac{kn}{2},$$

$$l_1 = -\frac{kn}{2} \log(2\pi) - \frac{k_1 n}{2} \log(\widehat{\sigma}^2_{k_1 n}) - \frac{k_2 n}{2} \log(\widehat{\sigma}^2_{k_2 n}) - \frac{kn}{2}.$$

Likelihood Ratio Statistic:

$$lr(k_1n, kn) = -2(l_0 - l_1) = kn \log[\widehat{\sigma}^2_{kn}(\widehat{\sigma}^2_{k_1n})^{-\frac{k_1}{k}}(\widehat{\sigma}^2_{k_2n})^{-\frac{k_2}{k}}]$$

– The limit distribution of $lr(k_1n, kn)$

$lr(k_1n, kn) \xrightarrow{\mathcal{L}} \chi^2(3)$, as $k_1, k_2 \longrightarrow \infty$. ($n$ is fixed)

As $k_2 \longrightarrow \infty$ (n and k_1 is fixed)

$lr(k_1n, kn) \xrightarrow{\mathcal{P}} z_1 - k_1 n \log \frac{z_1}{k_1 n} + z_2 + z_3 - k_1 n,$

where $z_1 \sim \chi^2(k_1 n - 2)$, $z_2, z_3 \sim \chi^2(1)$, z_1, z_2 and z_3 are independent.

– The expectation & variance of $lr(k_1 n, \infty)$

$$E[lr(k_1 n, \infty)] = k_1 n[\log(\frac{k_1 n}{2}) - \psi_0(\frac{k_1 n - 2}{2})],$$

$$Var[lr(k_1 n, \infty)] = (k_1 n)^2 \psi_1(\frac{k_1 n - 2}{2}) - 2k_1 n,$$

where $\psi_0(\cdot)$ and $\psi_1(\cdot)$ are, respectively, the digamma and trigamma function.

$$\psi_0(z+1) = \psi_0(z) + \frac{1}{z},\ \psi_0(1) = -\gamma,\ \psi_0(\frac{1}{2}) = -\gamma - 2\log 2,$$

$$\psi_1(z+1) = \psi_1(z) - \frac{1}{z^2},\ \psi_1(1) = \frac{\pi^2}{6},\ \psi_1(\frac{1}{2}) = \frac{\pi^2}{2},$$

Table 1 The expectation and variance of $lr(jn, kn)$ and $lr(jn, \infty)$ for $n = 10, k = 20$

j	$E[lr(jn, kn)]$	$Var[lr(jn, kn)]$	$E[lr(jn, \infty)]$	$Var[lr(jn, \infty)]$
1	3.54	8.41	3.53	8.38
2	3.24	7.02	3.24	7.00
3	3.16	6.66	3.15	6.64
4	3.12	6.49	3.11	6.47
5	3.09	6.39	3.09	6.37
6	3.08	6.34	3.07	6.30
7	3.07	6.32	3.06	6.26
8	3.06	6.30	3.06	6.22
9	3.06	6.27	3.05	6.20
10	3.05	6.25	3.04	6.18

– Out Proposed CUSUM Chart

The standardized likelihood ratio is defined by

$$slr(jn, kn) = \frac{lr(jn, kn) - E[lr(jn, \infty)]}{\sqrt{Var[lr(jn, \infty)]}}.$$

Define the CUSUM statistic based on $slr(jn, kn)$ as

$$S_j = \max\{0, S_{j-1} + slr(jn, kn)\}, \ j = 1, 2, \cdots, k-1.$$

where the initial value $S_0 = 0$.

– The design of our proposed CUSUM Chart

For given false alarm probability (FAP) α, the decision interval h_α is tabulated at Table 2.

Table 2 Simulated h_α of our proposed CUSUM chart

k	$h_{0.05}$	$h_{0.04}$	$h_{0.03}$	$h_{0.02}$	$h_{0.01}$	$h_{0.0075}$	$h_{0.005}$	$h_{0.0025}$	$h_{0.001}$
10	11.04	11.97	13.27	14.98	17.81	18.85	20.65	23.27	26.19
20	22.60	24.63	26.95	30.49	36.30	38.90	41.67	48.89	56.57
30	33.38	36.28	39.77	45.20	54.53	57.89	63.57	72.52	83.94
40	44.62	48.68	53.78	60.62	71.62	77.22	83.49	94.92	109.4
50	55.92	60.48	66.23	74.87	91.26	98.17	106.3	121.0	139.7

The approximated decision interval h_α are given by:

$$h_\alpha = -(0.4343 \log \alpha + 0.1843) \cdot k.$$

– The detecting ability for intercept, slope, and variance $lr(k_1 n, kn)$ can be partitioned into three terms:

$$I_{lr} = kn \log \left[1 + \frac{k_1 k_2 (\bar{y}_{k_1 n} - \bar{y}_{k_2 n})^2}{k(k_1 \widehat{\sigma}^2_{k_1 n} + k_2 \widehat{\sigma}^2_{k_2 n})} \right],$$

$$V_{lr} = kn \log \left[\frac{k_1 \widehat{\sigma}^2_{k_1 n} + k_2 \widehat{\sigma}^2_{k_2 n}}{k} (\widehat{\sigma}^2_{k_1 n})^{-\frac{k_1}{k}} (\widehat{\sigma}^2_{k_2 n})^{-\frac{k_2}{k}} \right],$$

$$S_{lr} = kn \log \left[1 + \left(\frac{k_1 k_2 (\frac{1}{k_1} S_{xy(k_1 n)} - \frac{1}{k_2} S_{xy(k_2 n)})^2}{k^2 n S_{xx}} / (1 + \frac{k_1 k_2 (\bar{y}_{k_1 n} - \bar{y}_{k_2 n})^2}{k(k_1 \widehat{\sigma}^2_{k_1 n} + k_2 \widehat{\sigma}^2_{k_2 n})}) \right) \right],$$

- Performance Comparison
  - Criterion of comparison

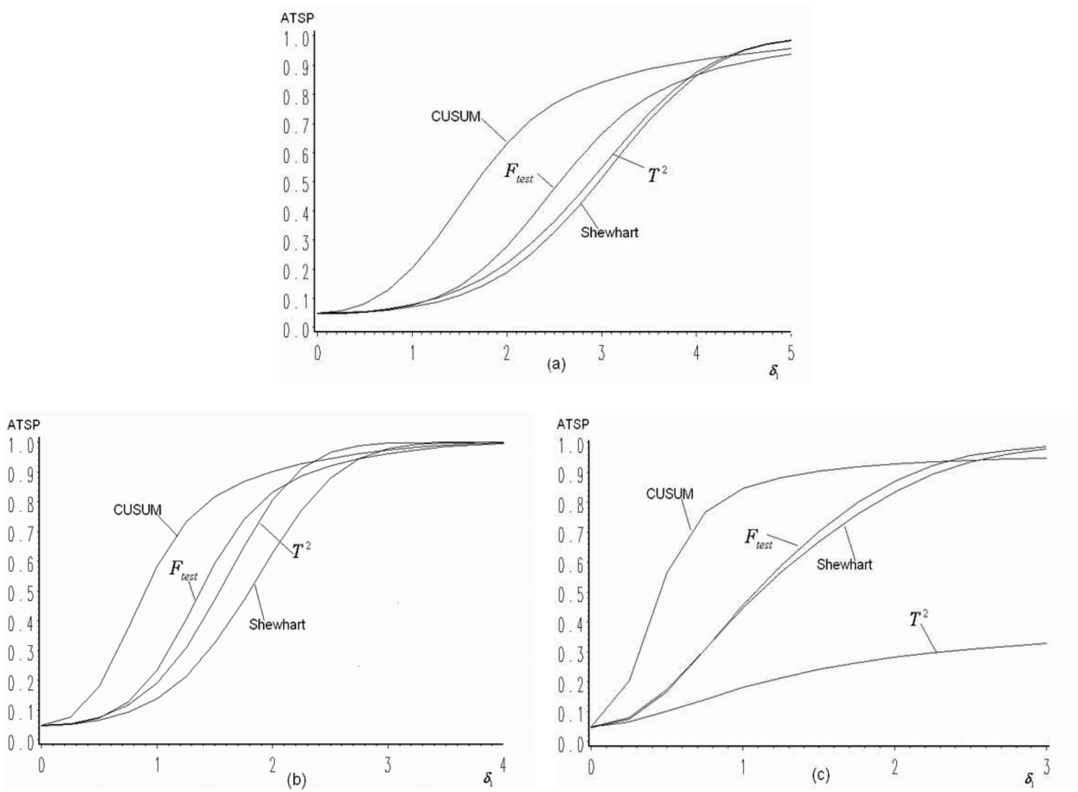
$$\text{ATSP} = \sum_{k=1}^{n-1} F(k)\text{TSP}_k,$$

  - The Parameters
    - $\alpha = 0.05$, $k = 20, n = 10$
    - $A_0 = 0, A_1 = 1, \sigma^2 = 1$, $x = 0(0.2)0.8$
  - Competition
    - T^2 (Kang & Albin 2000)
    - Shewhart & F-test (Mahmoud & Woodall 2000)

– Comparisons

* Scenario 1: One step shift in the intercept, slope, and standard deviation (Figure 1)
* Scenario 2: One trend shift in the intercept and slope (Figure 2)
* Scenario 3: Three step shifts: $\delta_1, \delta_2 = 1$, and $\delta_3 \in [-4, 4]$ in slope, deviation, intercept (Figure 3a)
* Scenario 4: Three step shifts: $\delta_1, \delta_2 = 1$, and $\delta_3 \in [-4, 4]$ in intercept, slope, deviation (Figure 3b)

Figure 1 ATSP's for one step shift in intercept, slope, and standard deviation.



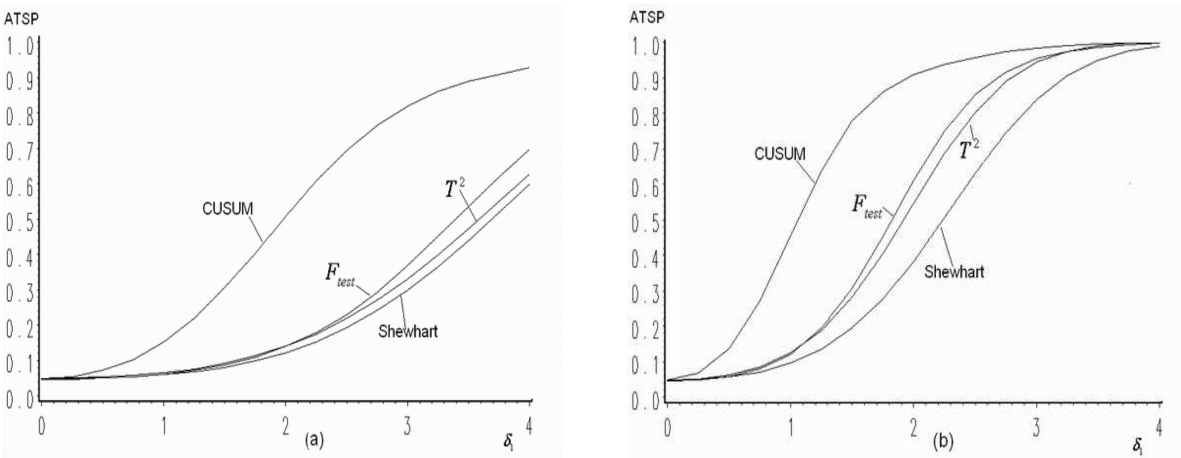


Figure 2 ATSP's for a trend shift in intercept and slope.

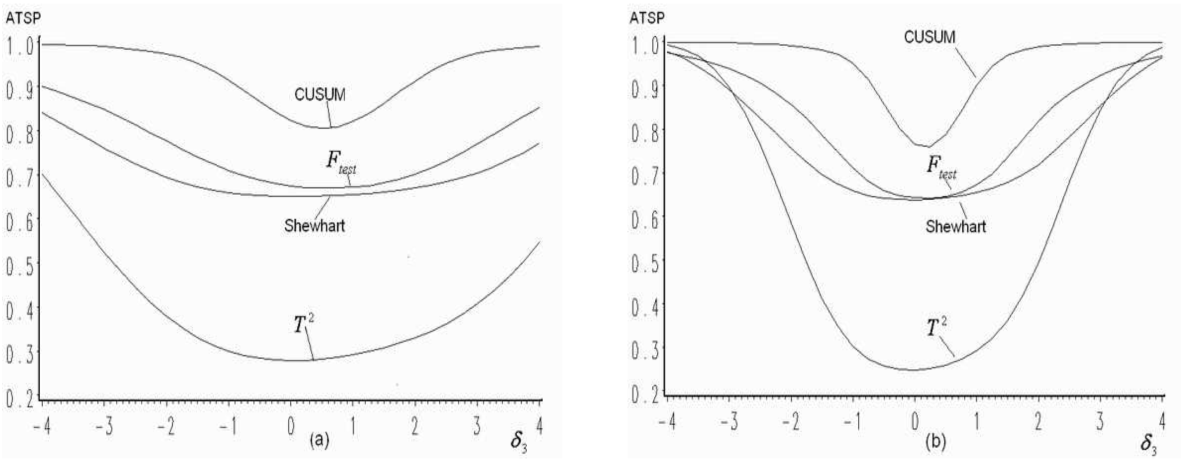


Figure 3 The ATSP's for Scenario 3 and Scenario 4.

- Conclusion & Consideration
  - Conclusion
    - ∗ Our proposed CUSUM chart has better performance than the others in terms of ATSP.
    - ∗ Our proposed CUSUM chart is more robust than others.
    - ∗ The criterion for comparison.

– Consideration

* How to generalize it to Phase II
* How to generalize it to the multivariate case
* How about EWMA based on the LRT
* How about the sensitivity of them

- And then...

Thank you for coming!

Thank HKBU and Mathematical Certer for supporting!!

Welcome to Nankai University!!!