Crouzeix's Conjecture

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Joint work with
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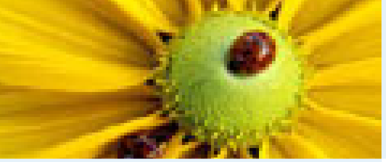

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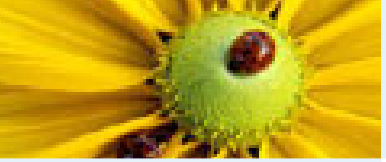

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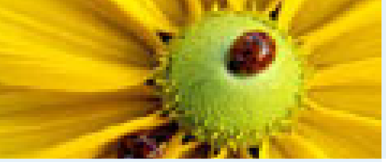

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Toeplitz-Hausdorff Theorem: $W(A)$ is convex for all $A \in \mathbb{C}^{n\times n}$.

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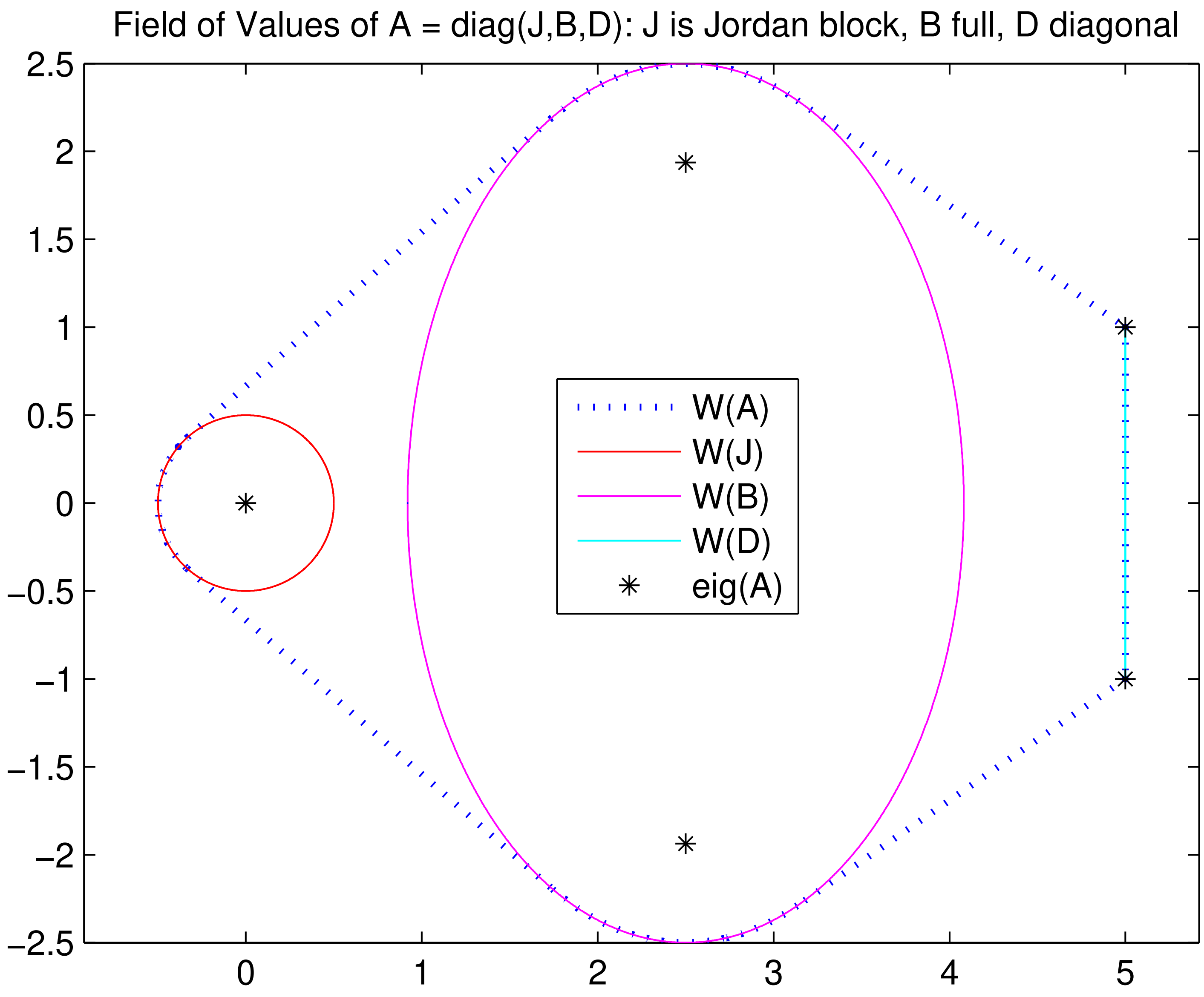
$$A = \text{diag}(J, B, D) : \quad W(A) = \text{conv}\,(W(J), W(B), W(D))$$

Example, continued



Complex plane plot of field of values of $A = \mathrm{diag}(J, B, D)$

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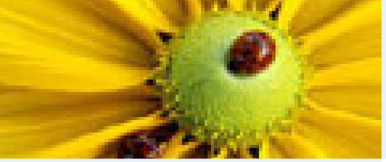

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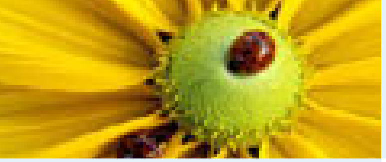

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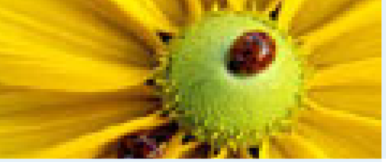
If $p = \chi(A)$, the characteristic polynomial (or minimal polynomial) of A, then $\|p(A)\|_2 = 0$ by Cayley-Hamilton, but $\|p\|_{W(A)} = 0$ only if $A = \lambda I$ for $\lambda \in \mathbb{C}$, so that $W(A) = \{\lambda\}$.



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Conjecture extends to analytic functions and to Hilbert space

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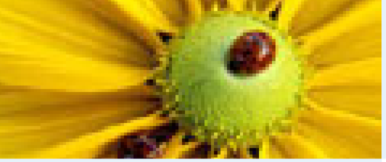

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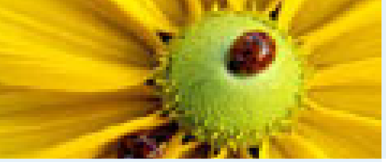

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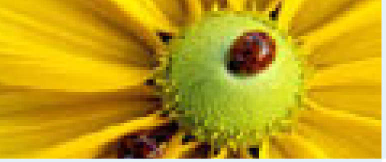
If $0 \in W(A)$, then this is useless, but one can instead consider $B = A^{1/q}$ where q is chosen large enough that $0 \notin W(B)$.

Crouzeix's Theorem

$$\|p(A)\|_2 \leq 11.08\,\|p\|_{W(A)}$$

i.e., the conjecture is true if we replace 2 by 11.08.



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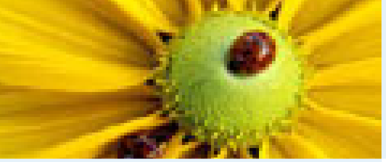

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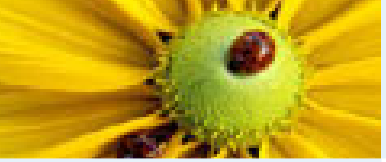
Remarkably broad impact: the norm of an analytic function of a matrix A is bounded by a modest constant times its norm on the field of values $W(A)$.



Palencia's Theorem

$$\|p(A)\|_2 \leq \left(1+\sqrt{2}\right)\|p\|_{W(A)}$$

i.e., the conjecture is true if we replace 2 by $1+\sqrt{2}$

Presented at a conference in Greece, 2016

Published in a SIMAX paper with Crouzeix, 2017

Much cleaner proof than that of Crouzeix's previous bound. Uses novel but relatively brief arguments based on complex analysis.

Special Cases

The conjecture is known to hold for certain restricted classes of polynomials $p \in P^m$ or matrices $A \in \mathbb{C}^{n \times n}$.

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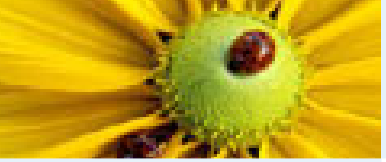

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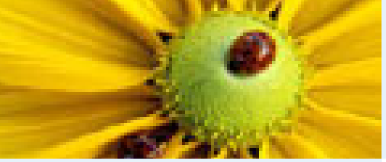

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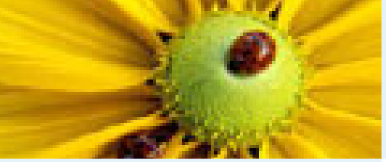
- $AA^* = A^*A$ (then the constant 2 can be improved to 1).



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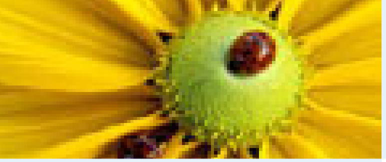
The proof uses a supporting hyperplane argument.



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The extreme points of a convex set are those that cannot be expressed as a convex combination of two other points in the set.

Based on R. Kippenhahn (1951), C.R. Johnson (1978) observed that the extreme points of $W(A)$ can be characterized as

$$\text{ext } W(A) = \{z_\theta = v_\theta^* A v_\theta : \theta \in [0, 2\pi)\}$$

where v_θ is a normalized eigenvector corresponding to the largest eigenvalue of the Hermitian matrix

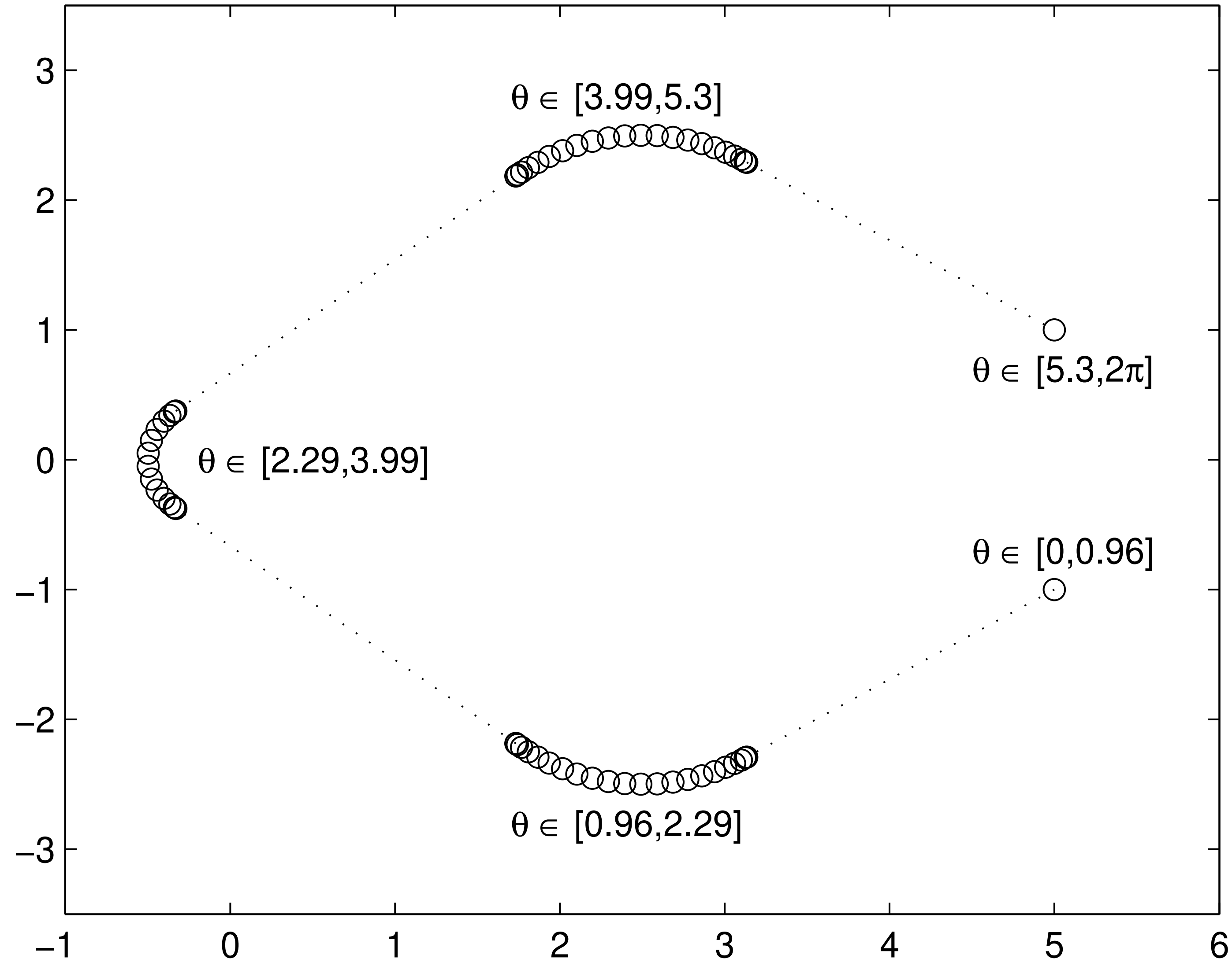
$$H_\theta = \frac{1}{2}\left(e^{i\theta}A + e^{-i\theta}A^*\right).$$

The proof uses a supporting hyperplane argument.

Thus, we can compute as many extreme points as we like.

Continuing with the previous example...

Johnson's Algorithm Finds the Extreme Points



The extreme points of W(A) lie in the union of 5 connected sets

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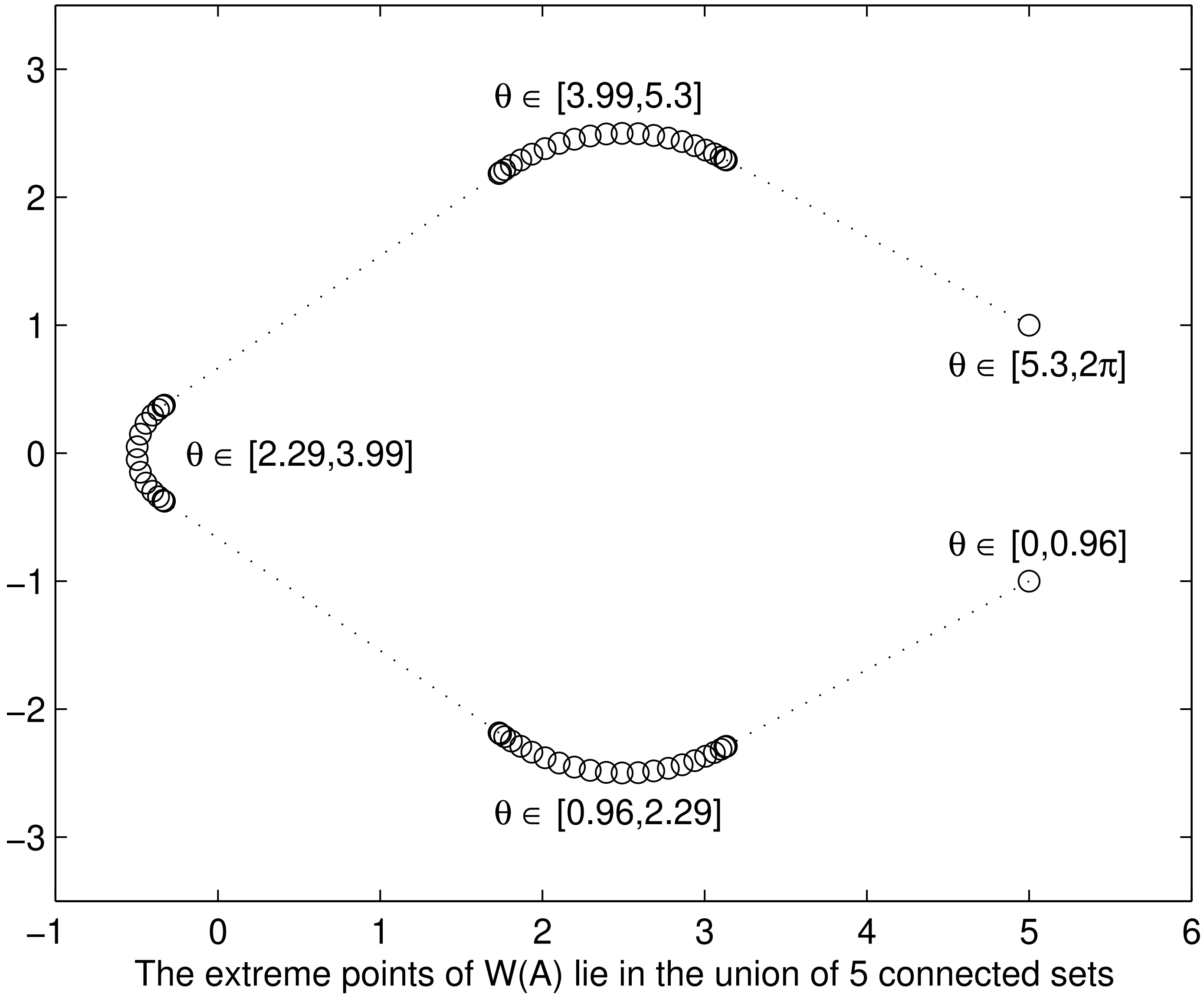


But how can we do this accurately, automatically and efficiently?

Chebfun

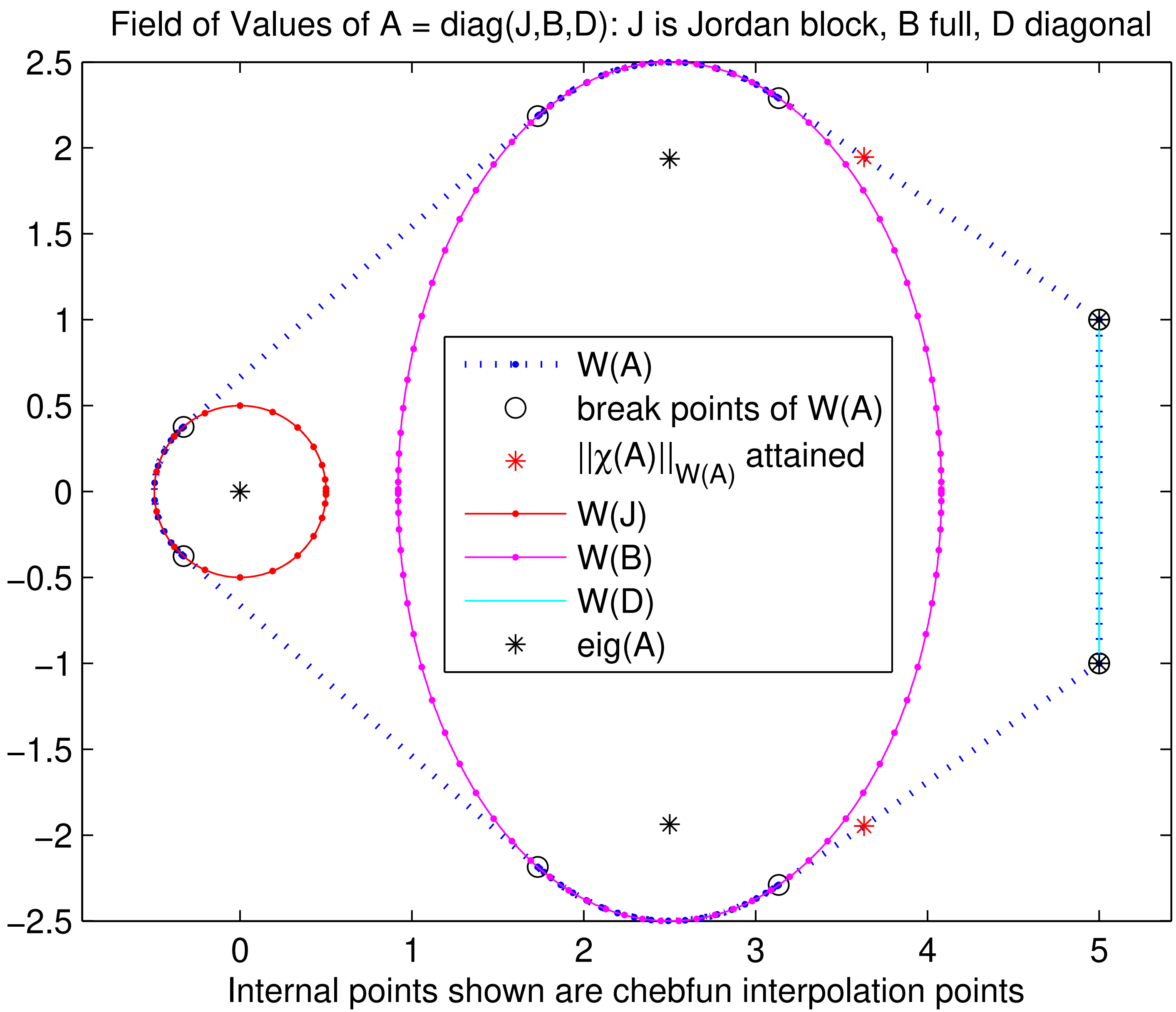
Chebfun (Trefethen et al, 2004–present) represents real- or complex-valued functions on real intervals to machine precision accuracy using Chebyshev interpolation.

The necessary degree of the polynomial is determined automatically. For example, representing $\sin(\pi x)$ on $[-1, 1]$ to machine precision requires degree 19.

Most MATLAB functions are overloaded to work with chebfun's.

Applying Chebfun's **fov** to compute the boundary of $W(A)$ for the previous example...

Example, continued



The Crouzeix Ratio

Define the Crouzeix ratio

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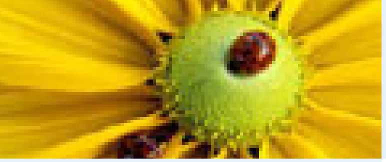

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- A mapping from $\mathbb{C}^{m+1} \times \mathbb{C}^{n \times n}$ to $\mathbb{R}$ (associating polynomials $p \in P^m$ with their vectors of coefficients $c \in \mathbb{C}^{m+1}$ using the monomial basis)

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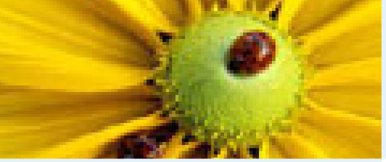

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- Semialgebraic (its graph is a finite union of sets, each of which is defined by a finite system of polynomial inequalities)

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Numerator: use Chebfun's **fov** (modified to return any line segments in the boundary) combined with its overloaded **polyval** and **norm(·,inf)**.

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The main cost is the construction of the chebfun defining the field of values.

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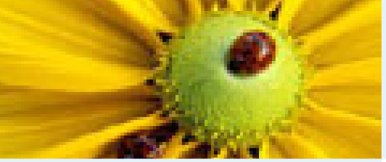

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In all of these cases the gradient of f is not defined.
But in practice, none of these cases ever occur, except the first one *in the limit*.

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Although its global convergence theory is limited to the convex case (Powell, 1976), it generally finds local minimizers efficiently in the nonconvex case too, although there are pathological counterexamples.

Remarkably, this property seems to extend to nonsmooth functions too, with a linear rate of local convergence, although the convergence theory is extremely limited (Lewis and Overton, 2013). It builds a very ill conditioned "Hessian" approximation, with "infinitely large" curvature in some directions and finite curvature in other directions.

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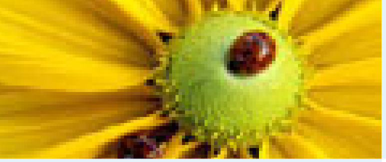

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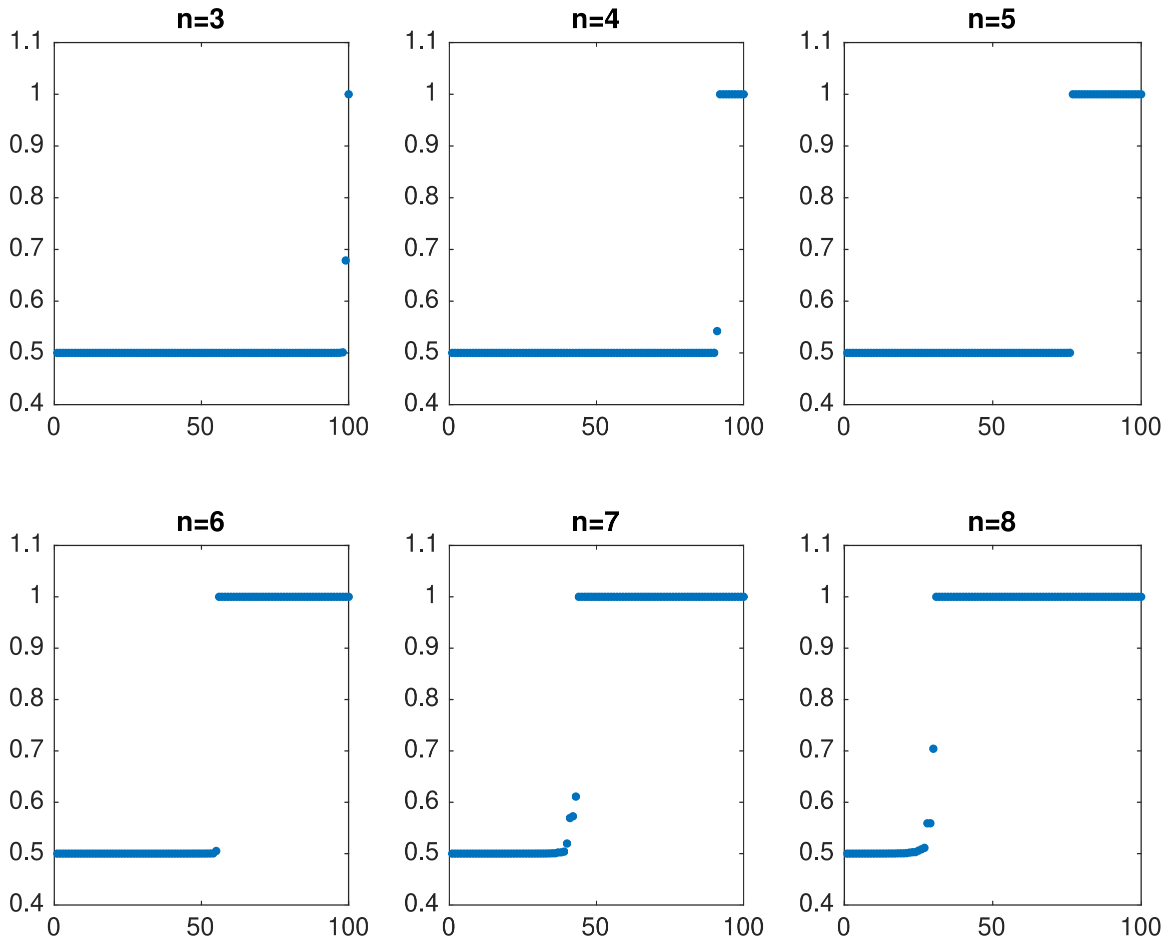
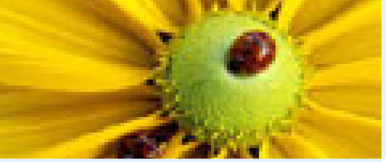
We have obtained similar results for p with complex coefficients and complex A (then can take A to be triangular).

Optimizing over A (order n) and p (deg $\leq n-1$)



Sorted final values of the Crouzeix ratio f found starting from 100 randomly generated initial points.



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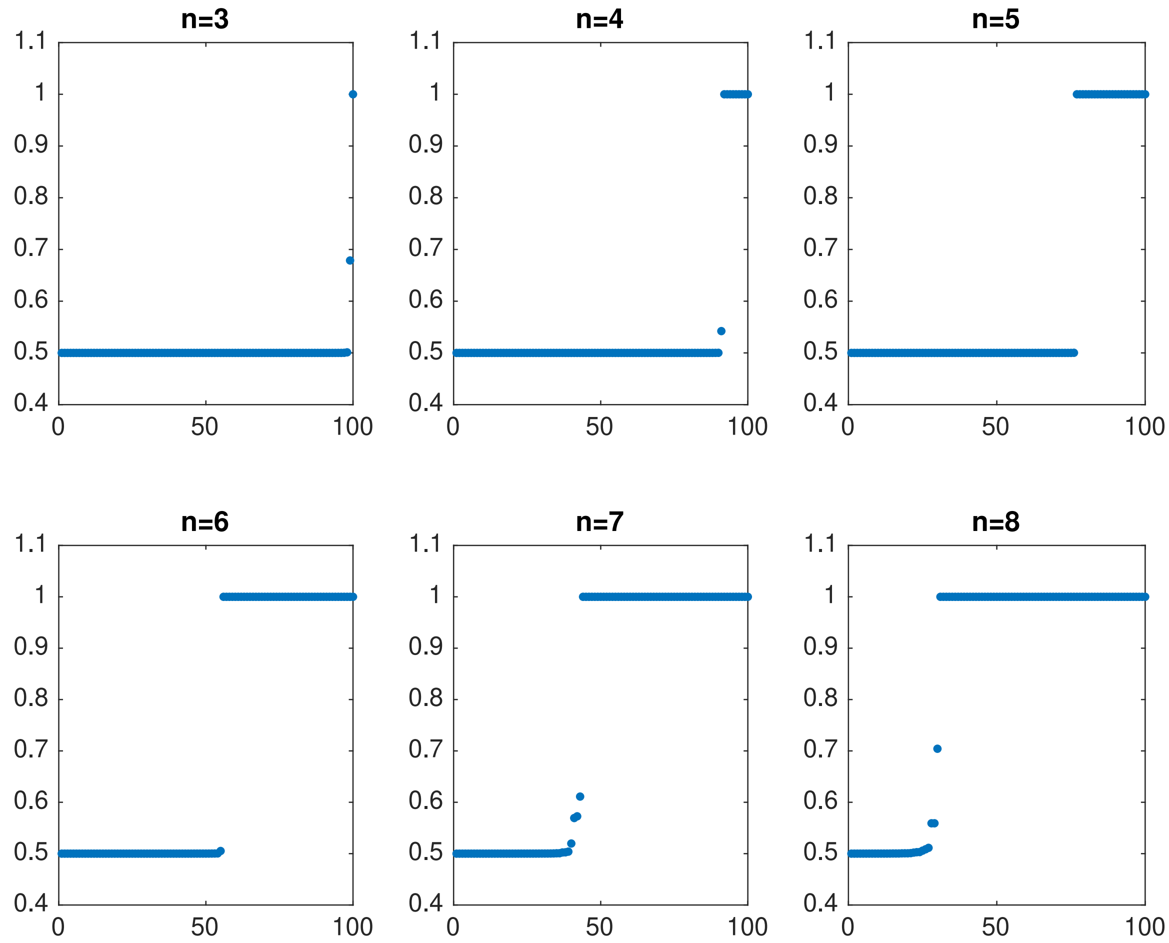

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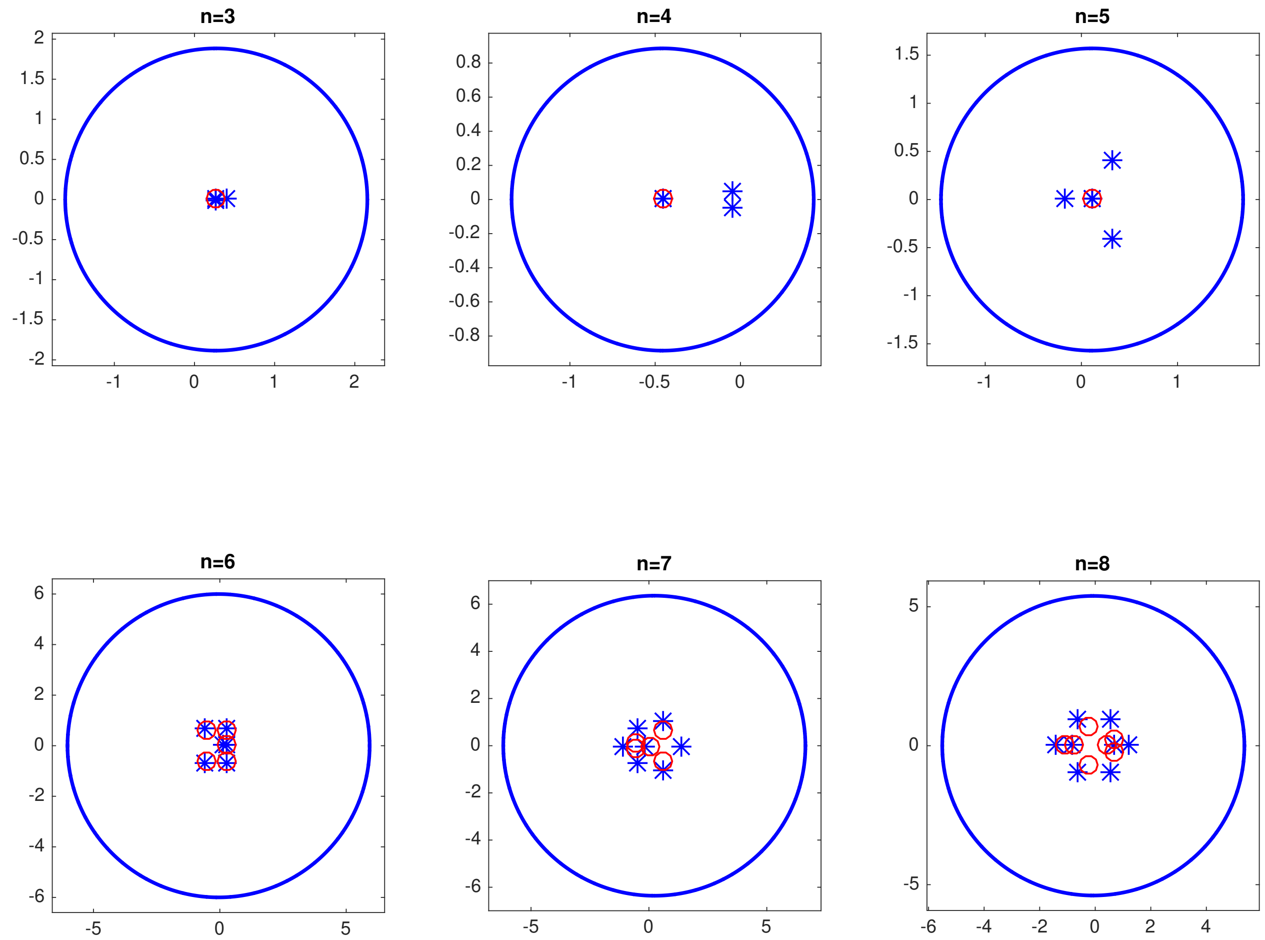
Are 0.5 and 1 the only locally optimal values of f?

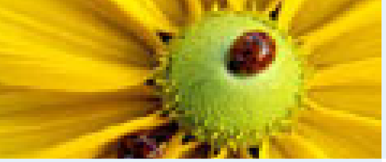
Final Fields of Values for Lowest Computed f



Solid blue curve is boundary of field of values of final computed A

Blue asterisks are eigenvalues of final computed A

Small red circles are roots of final computed p



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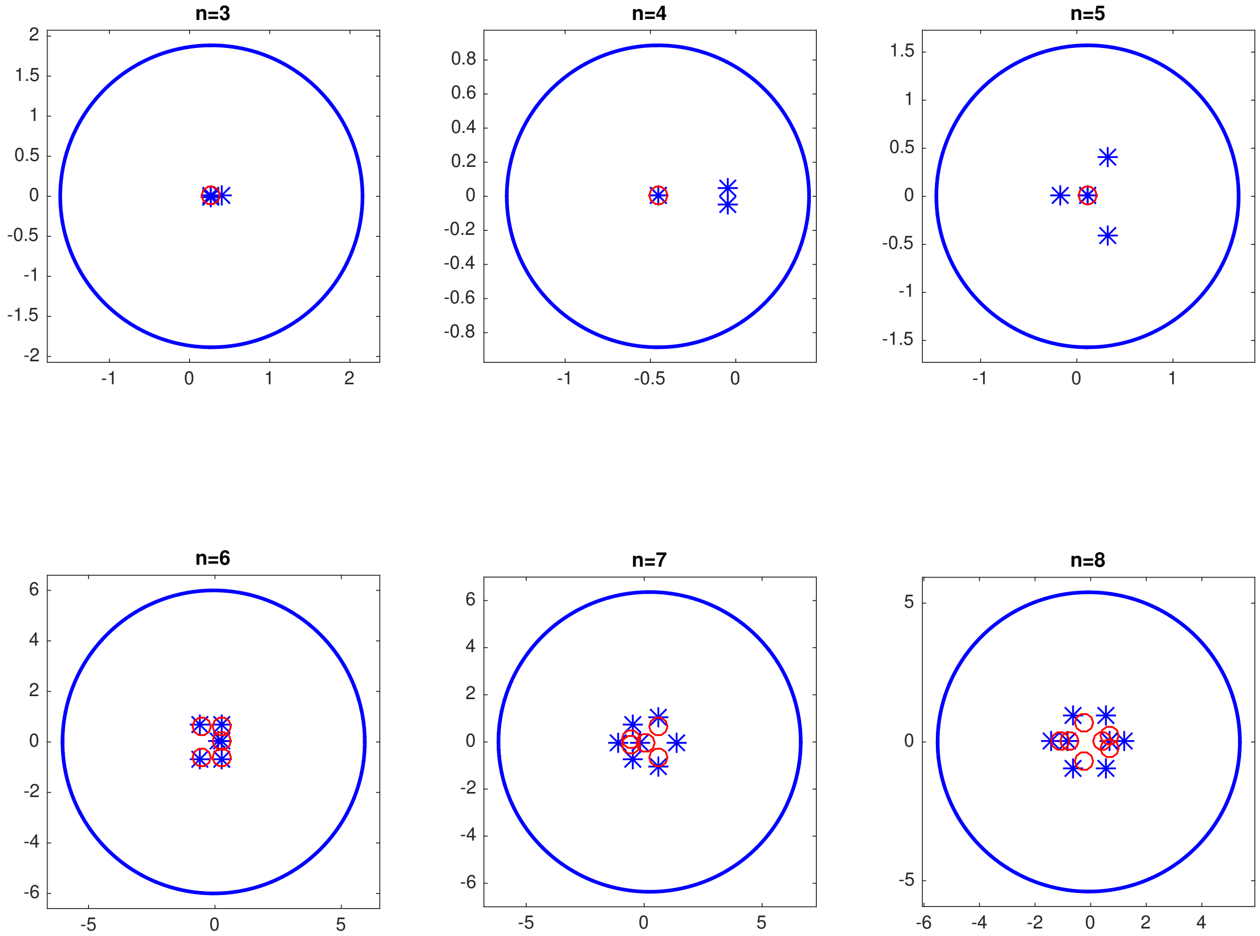

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$n = 3, 4, 5$: two eigenvalues of A and one root of p nearly coincident

Optimizing over both p and A: Final $f(p, A)$

n	f
3	0.500000000000000
4	0.500000000000000
5	0.500000000000014
6	0.500000017156953
7	0.500000746246673
8	0.500002065638131

f is the lowest value $f(p, A)$ found over 100 runs

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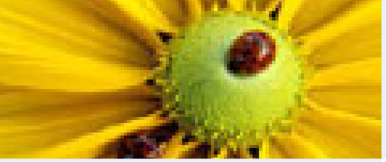

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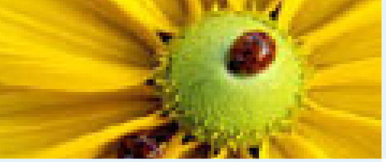

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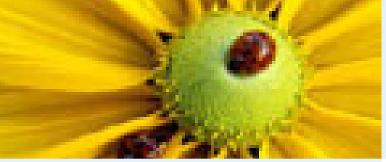

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Note: f is *nonsmooth* at these pairs (p, A) because $|p|$ is constant on the boundary of $W(A)$.

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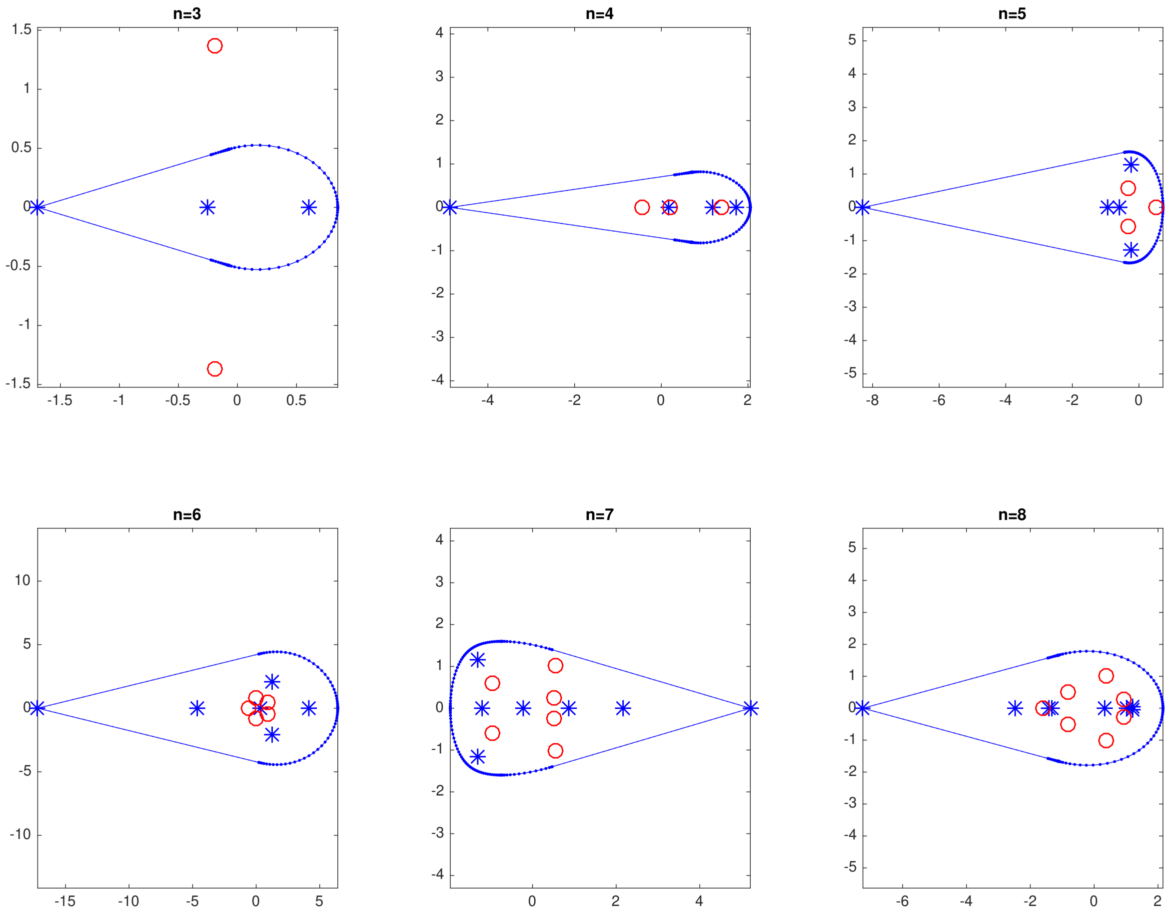
We find that computed minimizers have the form

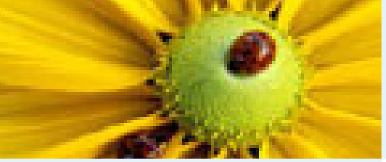
$$A = \lambda I + \alpha\, U \mathrm{diag}(\Xi_k, B) U^T + E,$$

$$p(\zeta) = c_{n-1}(\zeta - \lambda)^{n-1} + \ldots + c_1(\zeta - \lambda) + c_0$$

where $k \geq 2$ (usually $k = 2$), $\alpha \neq 0$, U is orthogonal, $W(B) \subset \overline{\mathcal{D}}$, $\|E\|$ is small and $|c_j|$ is small for $j \geq k$.

Final Fields of Values for f Closest to 1





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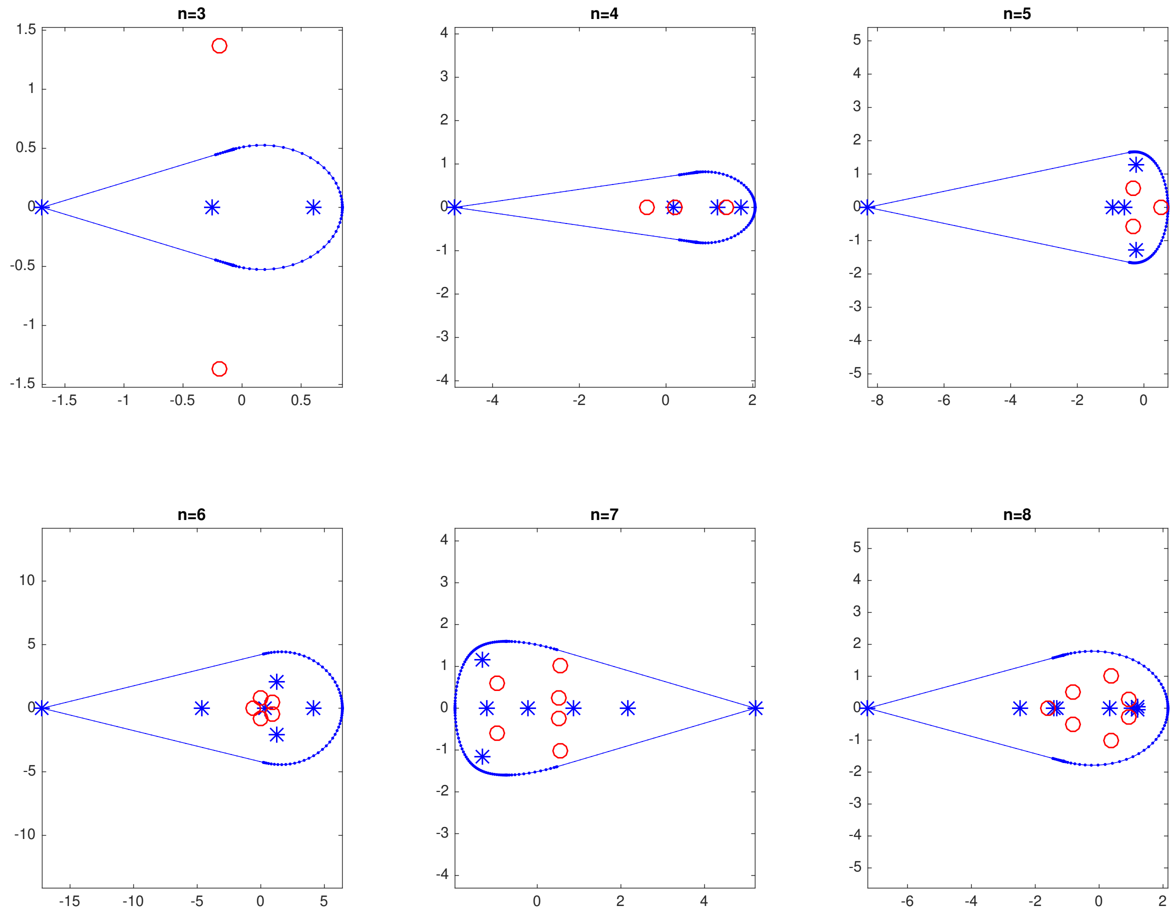

Ice cream cone shape:
exactly one eigenvalue at a vertex of the field of values

Solid blue curve is boundary of field of values of final computed A
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Because for this computed local minimizer, A is nearly unitarily similar to a block diagonal matrix

$$\mathrm{diag}(\lambda, B), \quad \lambda \in \mathbb{R}$$

so

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So, $\|p\|_{W(A)} = |p(\lambda)| = \|p(A)\|_2$ and hence $f(p, A) = 1$.

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with λ *active* and the block B *inactive*, that is:

- $\|p\|_{W(A)}$ is attained only at λ
- $|p(\lambda)| > \|p(B)\|_2$

So, $\|p\|_{W(A)} = |p(\lambda)| = \|p(A)\|_2$ and hence $f(p, A) = 1$.

Furthermore, f is differentiable at this pair (p, A), with zero gradient. Thus, such (p, A) is a *smooth* stationary point of f.

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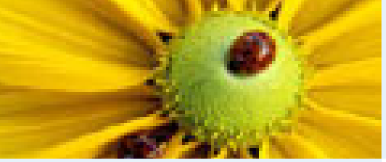

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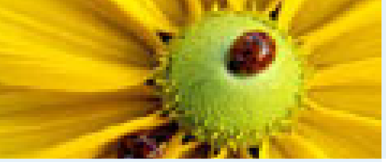
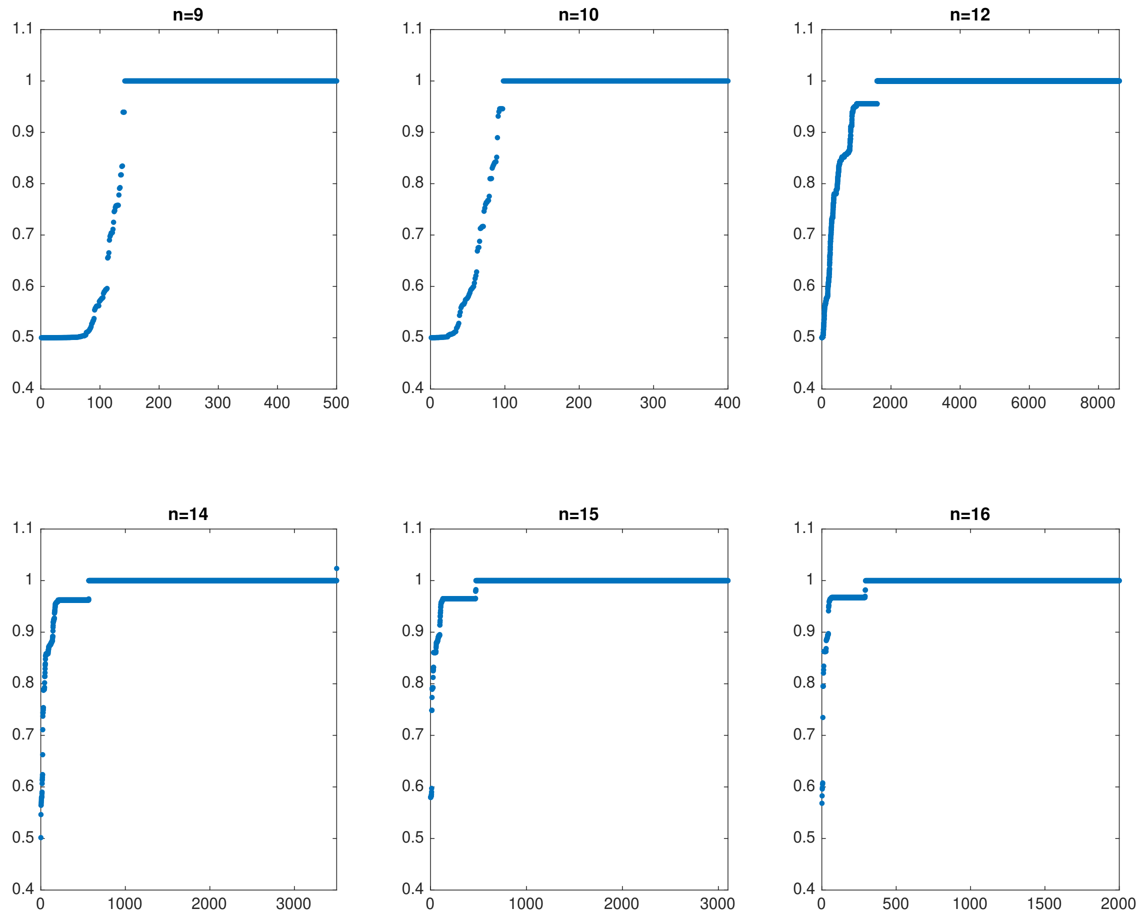
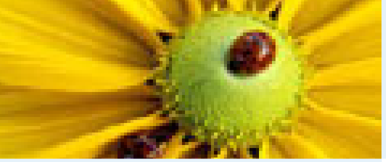
As n increases, ice cream cone stationary points become increasingly common and it becomes very difficult to reduce f below 1.



Results for Larger Dimension n and Degree $n-1$



Sorted final values of the Crouzeix ratio f
found starting from **many** randomly generated initial points.



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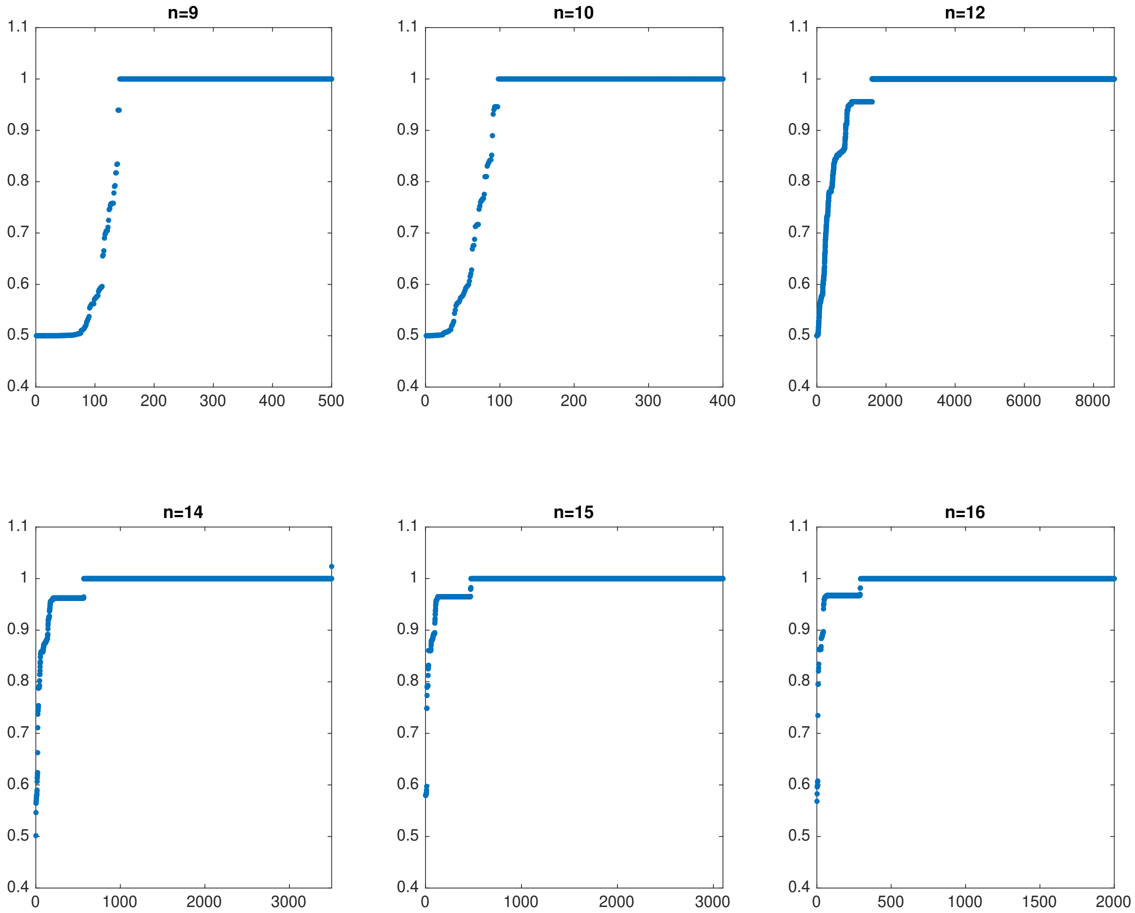

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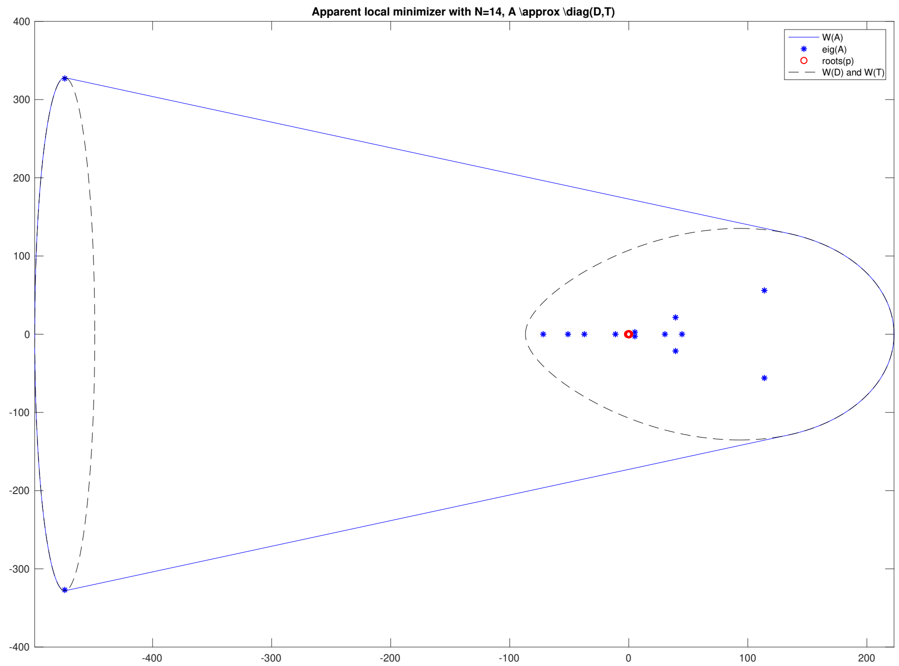
There **are** other locally optimal values of f between 0.5 and 1 !

An Apparent Local Minimum: $N = 14$



The apparently locally optimal matrix A is nearly unitarily similar to a block diagonal matrix with a 2×2 block A_{11} and a 14×14 block A_{22}.

Black dashed curves show boundaries of field of values of final computed A_{11} and A_{22}

Solid blue curve is boundary of field of values of final computed A

Blue asterisks are eigenvalues of final computed A

Small red circles are roots of final computed p

What if we Fix p and Optimize over A?

Experiments fixing p with degree m and optimizing over A with order $\geq m+1$ led us to:

Theorem 1. For any fixed polynomial p of degree $m \geq 1$, there exists a divergent sequence $\{A^{(k)}\}$ of order $n = m+1$ for which $f(p, A^{(k)}) \to 0.5$ as $k \to \infty$. Furthermore, we can choose $A^{(k)}$ so $\{W(A^{(k)})\}$ is a sequence of disks with radius $\to \infty$.

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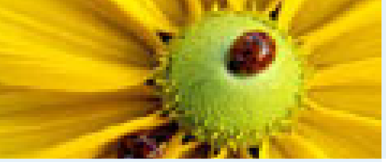

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Nonsmooth Analysis of the Crouzeix Ratio

Assume $h : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz, and let $D = \{x \in \mathbb{R}^n \ : \ h \text{ is differentiable at } x\}$.

The Clarke Subdifferential

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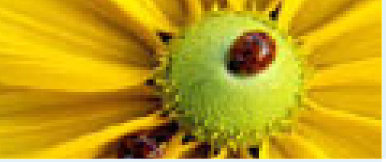
Rademacher's Theorem: $\mathbb{R}^n \backslash D$ has measure zero.



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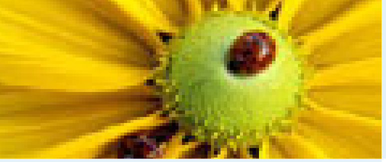

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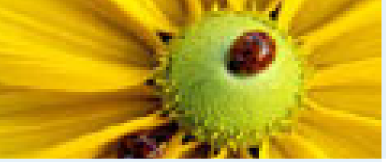

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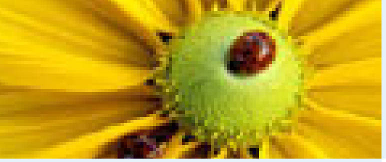

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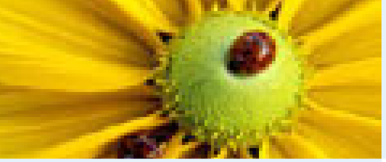

The Gradient or Subgradients of the Crouzeix Ratio

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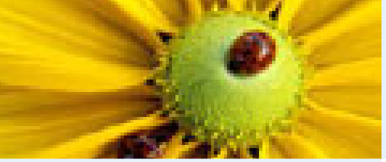

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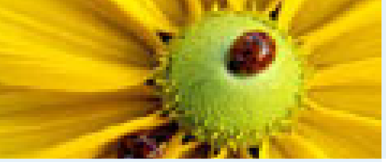

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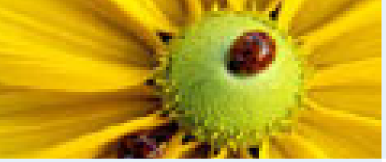

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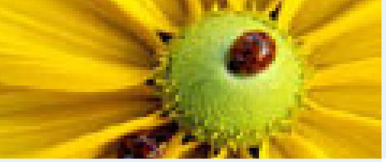
Finally, use the quotient rule.



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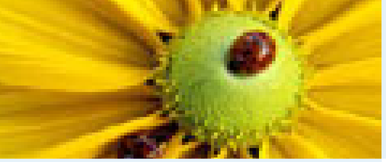

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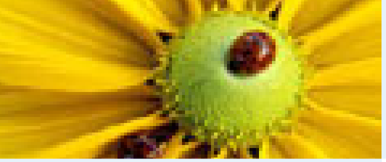

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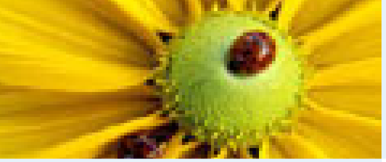

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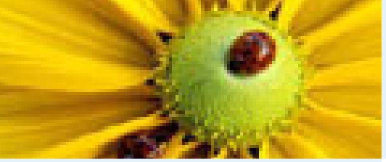
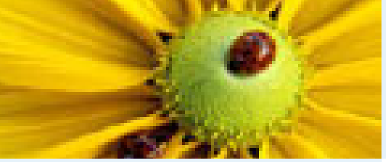
- Nonsmooth concave functions, e.g. $h(x) = -|x|$, are not regular.



Simplest Case where Crouzeix Ratio is Nonsmooth

Optimize over complex monic linear polynomials $p(z) \equiv c + z$ and complex matrices with order $n = 2$. Let $f(p, A) \equiv f(c, A)$, where now $f : \mathbb{C} \times \mathbb{C}^{2\times 2} \to \mathbb{R}$.



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Let $\hat{c} = 0$ ($\hat{p}(z) = z$) and $\hat{A} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$, so $W(\hat{A}) = \overline{\mathcal{D}}$, the unit disk, and hence $|p(z)|$ is maximized everywhere on the unit circle, with f nonsmooth at $(\hat{c}, \hat{A})$ and $f(\hat{c}, \hat{A}) = 1/2$.

Simplest Case where Crouzeix Ratio is Nonsmooth

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Theorem 3. The Crouzeix ratio f is regular at $(\hat{c}, \hat{A})$, with

$$\partial f(\hat{c}, \hat{A}) = \mathrm{conv}_{\theta \in [0, 2\pi)} \left\{ \left(\frac{1}{2} e^{-i\theta}, \frac{1}{4} \begin{bmatrix} e^{-i\theta} & 0 \\ e^{-2i\theta} & e^{-i\theta} \end{bmatrix} \right) \right\}$$

Corollary.

$$0 \in \partial f(\hat{c}, \hat{A})$$

$(\hat{c}, \hat{A})$ is a Nonsmooth Stationary Point of $f(\cdot,\cdot)$

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Actually, we knew this must be true as Crouzeix's conjecture is known to hold for $n = 2$, and hence $(\hat{c}, \hat{A})$ is a global minimizer of $f(\cdot,\cdot)$, but we can extend the result to larger values of m, n, for which we don't know whether the conjecture holds.

The General Case

Optimize over complex polynomials $p(z) \equiv c_0 + \cdots + c_m z^m$ and complex matrices with order n. Let $f(p, A) \equiv f(c, A)$, where $f : \mathbb{C}^{m+1} \times \mathbb{C}^{n \times n} \to \mathbb{R}$.

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Let $\hat{c} = [0, 0, \ldots, 1]$, corresponding to the polynomial z^m, and $\hat{A}$ equal the C-matrix of order $n = m + 1$ so $W(\hat{A}) = \overline{\mathcal{D}}$, the closed unit disk, and hence $f(\hat{c}, \hat{A}) = 1/2$.

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Theorem 4. The Crouzeix ratio on $(c, A) \in \mathbb{C}^{m+1} \times \mathbb{C}^{n\times n}$ is regular at $(\hat{c}, \hat{A})$ with

$$\partial f(\hat{c}, \hat{A}) = \mathrm{conv}_{\theta\in[0,2\pi)} \Big\{ \big(y_\theta, Y_\theta \big) \Big\}$$

where

$$y_\theta = \frac{1}{2} \left[z^m, z^{m-1}, \ldots, z, 0 \right]^T$$

and Y_θ is the $n \times n$ matrix

$$Y_\theta = \frac{1}{4} \begin{bmatrix} z & 0 & \sqrt{2}z^{-1} & \sqrt{2}z^{-2} & \cdots & \sqrt{2}z^{3-n} & z^{2-n} \\ \sqrt{2}z^2 & 2z & 0 & 2z^{-1} & \cdots & 2z^{4-n} & \sqrt{2}z^{3-n} \\ \vdots & & & & & & \vdots \\ \sqrt{2}z^{n-2} & 2z^{n-3} & 2z^{n-4} & 2z^{n-5} & \cdots & 0 & \sqrt{2}z \\ \sqrt{2}z^{n-1} & 2z^{n-2} & 2z^{n-3} & 2z^{n-4} & \cdots & 2z & 0 \\ z^n & \sqrt{2}z^{n-1} & \sqrt{2}z^{n-2} & \sqrt{2}z^{n-3} & \cdots & \sqrt{2}z^2 & z \end{bmatrix}$$

with $z = e^{-i\theta}$.

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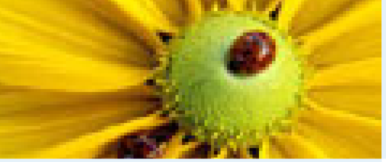
is zero.



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$$\tilde{A}(t) = \begin{bmatrix} 0 & \sqrt{2} & 2t \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}$$

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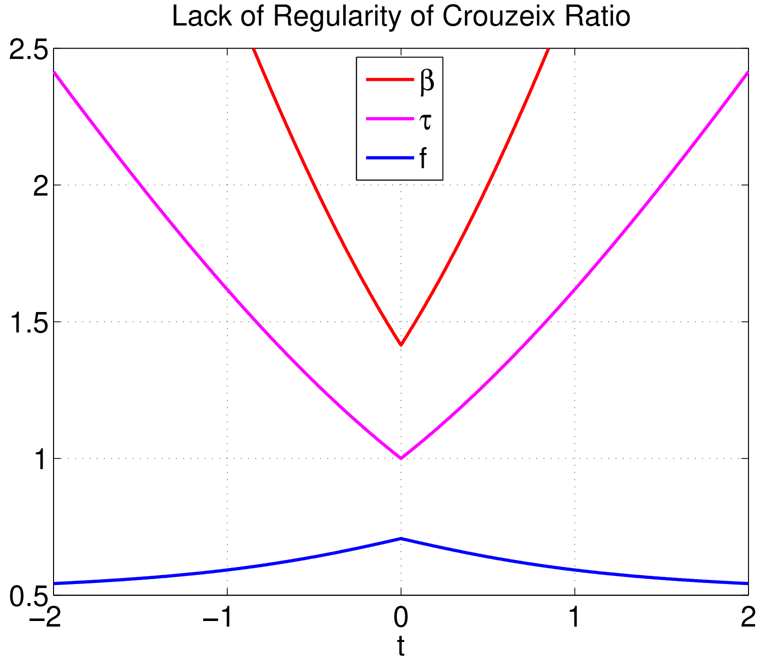
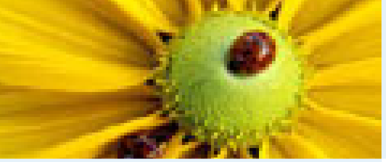
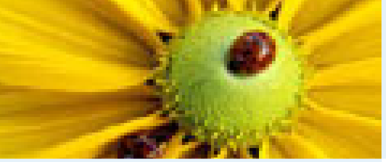


Plot of the denominator β, the numerator τ and the Crouzeix ratio f evaluated at $(\tilde{p}, \tilde{A}(t))$, $t \in [-2, 2]$.



Concluding Remarks

Summary



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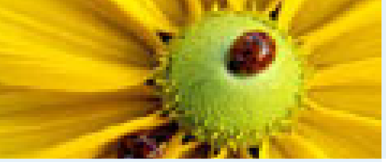

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The results strongly support Crouzeix's conjecture: the globally minimal value of the Crouzeix ratio $f(p, A)$ is 0.5.

Our Papers

A. Greenbaum and M.L. Overton
Numerical Investigation of Crouzeix's Conjecture
Linear Alg. Appl., 2017

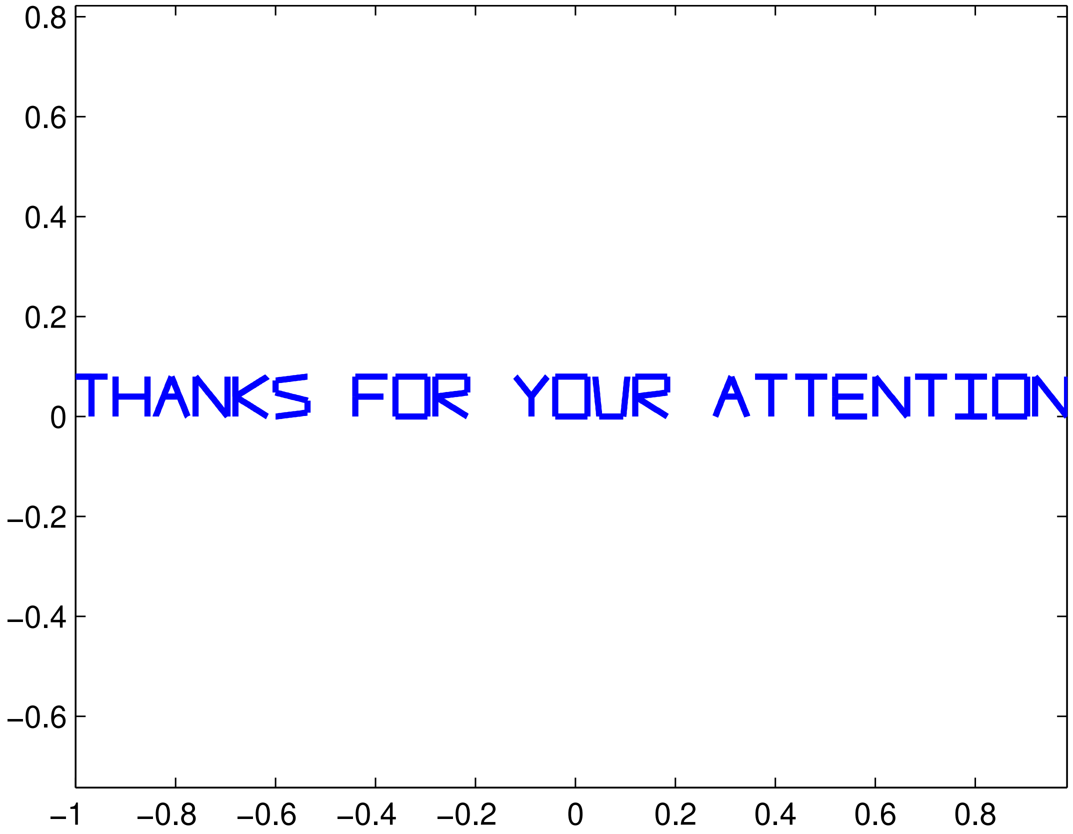
A. Greenbaum, A.S. Lewis and M.L. Overton
Variational Analysis of the Crouzeix Ratio
Math. Programming, 2016

M.L. Overton
Local Minimizers of the Crouzeix Ratio: A Nonsmooth Optimization Case Study
Calcolo, 2022

All available at `www.cs.nyu.edu/overton`

Saying Thanks in Chebfun


```
% define and plot a chebfun with 87 pieces
%s=scribble('Thanks for your attention');
%plot(s,'b','LineWidth',2), axis equal
```



Or, More Circularly


```
plot(exp(3i*s),'m','LineWidth',2), axis equal
```